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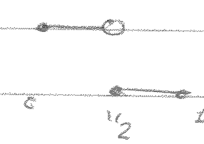
11 May 2013

1. (a) Let $C \subset A$ be closed, $A \subset X$ compact $\Rightarrow C \subset A$ compact $\Rightarrow$ every closed subset of A is compact. So, $q_{|A}$ takes closed sets to compact sets. But Y is Hausdorff and compact in Hausdorff $\Rightarrow$ closed. So $q_{|A}$ is closed $\Rightarrow$ quotient map.

(b) Let $X = [0,1] \times [0,1]$, $Y = [0,1]$, $q : X \mapsto Y$ be projection. Then X, Y, q satisfy the hypothesis. Let $B =$



Then $q^{-1}(U)$ where $U = [1/2, 1]$ is open in B but not in $Y \Rightarrow$ ~~[illegible]~~ q not qoutreat. /

2. If there did, then $\chi(S^n) = k\chi(X) \Rightarrow$

$2 = k\chi(X) \Rightarrow 2/k = \chi(X)$. But $2/k \in (0,1)$

$\Rightarrow \Rightarrow\!\Leftarrow$.

3. (a) $dg_{(x,y,z)} = (8x, 2y, 2z) = (0,0,0) \Leftrightarrow$

$x = y = z$. But $4x^2 + y^2 + z^2 = 1 \Rightarrow x, y, z$ not all 0

$\Rightarrow 1$ is a regular value of g.

(b) Critical points occur whenever

$(2x, 0, -1)$ and $(8x, 2y, 2z)$ are linearly dependent.

We can't have $x = y = z = 0 \Rightarrow z \neq 0$

$\Rightarrow_{-2z}(2x, 0, -1) = (8x, 2y, 2z)$

$\Rightarrow -4xz = 8x, \quad y = 0$

$\Rightarrow x = 0$ or $z = -2 \quad \Rightarrow$ Critical Points

are $M \cap \{(0,0,z) \mid z \in \mathbb{R}\} \cup M \cap \{(x, 0, -2) \mid x \in \mathbb{R}\}$

But $z^2 = 18 \Rightarrow$ $\quad 4x^2 + 4 = 18 \Rightarrow x^2 = 7/2$

Critical Points are $(0, 0, \pm 3\sqrt{2}) \cup (\pm\sqrt{7/2}, 0, 0)$

4. $X = D \amalg_{\sim} S^3$. We use MV and we let

$U = D$, $V = S^3$, $U \cap V = S^1$.

We have

$$0 \to \overset{0}{\tilde{H}_3(S^1)} \to \overset{0}{\tilde{H}_3(D)} \oplus \overset{\mathbb{Z}}{\tilde{H}_3(S^3)} \to \tilde{H}_3(X) \to \overset{0}{\tilde{H}_2(S^1)} \to \overset{0}{\tilde{H}_2(D)} \oplus \overset{0}{\tilde{H}_2(S^3)} \to$$

$$\tilde{H}_2(X) \to \overset{\mathbb{Z}}{\tilde{H}_1(S^1)} \to \overset{0}{\tilde{H}_1(D)} \oplus \overset{0}{\tilde{H}_1(S^3)} \to \tilde{H}_1(X) \to 0$$

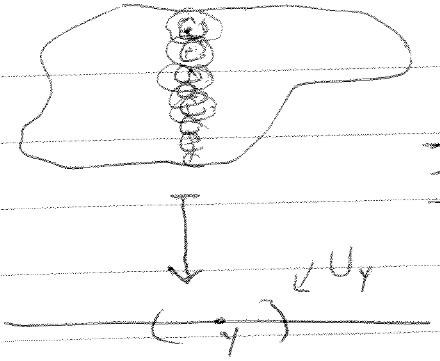
$\Rightarrow \tilde{H}_3(X) \cong \mathbb{Z}$, $\tilde{H}_2(X) \cong \mathbb{Z}$, $\tilde{H}_1(X) \cong 0$

$\Rightarrow H_n(X) = 0$ if $n \geq 4$, $H_3(X) = H_2(X) = H_0(X) = \mathbb{Z}$,

$H_1(X) = 0$. //

$f: X \longmapsto Y$

5.



$f^{-1}(y)$ compact. For every $x \in f^{-1}(y)$, $\exists\, V_x \subset C_x \subset f^{-1}(U_y)$

Now finitely many of these V_x cover $f^{-1}(y)$, say $V_1, \ldots, V_N$.

Then $\bigcup_{i=1}^{N} V_i$ open, $\Rightarrow X \setminus \bigcup_{i=1}^{N} V_i$ closed

$f\left(X \setminus \bigcup_{i=1}^{N} V_i\right) = f(X) \setminus \bigcup_{i=1}^{N} f(V_i)$ = Closed in Y

$\Rightarrow \bigcup_{i=1}^{N} f(V_i) = U$ open and by construction $f\left(\bigcup_{i=1}^{N} C_i\right) = C$

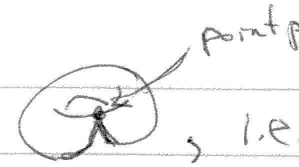
closed and $U \subset C \subset U_y \Rightarrow$ Y is locally compact. //

Afternoon

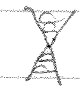
1. Define $g: S^1 \mapsto S^2$ by $g(x) = \frac{df_x}{\|df_x\|}$.

That is, for $x \in S^1$, g takes x to the unit tangent vector at f(x). This is a smooth map and $\exists$ plane C (through origin) s.t. $p \circ f: S^1 \mapsto C$ has $d(p \circ f)_x \neq 0$ $\forall x \in S^1 \iff$ it does for $p \circ g$. Note that g is smooth so it misses two points $v, -v$ of S^2 since $\dim(S^1) < \dim(S^2)$ let C be the plane orthogonal to v. This does the trick. //

2. Note that subgroups of F_5 of index 3 are in bijective cooorespondence with 3 fold coverings of $\bigvee_{i=1}^{5} S^1 = X$. Let $f: Y \mapsto X$ be a 3-fold covering. Then we have $\chi(X) = 1-5 = -4 \Rightarrow$ $\chi(Y) = -12 = |H_0(Y)| - |H_1(Y)| \Rightarrow H_1(Y) = 13$ $\Rightarrow Y \simeq \bigvee_{i=1}^{13} S^1 \Rightarrow$ Only subgroups of F_5 of index 3 are those $\ni$ $\pi_1(\bigvee_{i=1}^{13} S^1) = F_{13}$. //

3. Note $X \approx$ 

, i.e. a pinch torus.

Then any neighborhood of p contains , i.e. a double cone $\Rightarrow$ no nbd of $p \approx \mathbb{R}^2$. We can also use homology.

$0 \to \tilde{H}_2(X \setminus \{x\}) \to \tilde{H}_2(X) \to \tilde{H}_2(X, X \setminus \{x\}) \to$

$\tilde{H}_1(X \setminus \{x\}) \to \tilde{H}_1(X) \to \tilde{H}_1(X, X \setminus \{x\}) \to 0$

$X \setminus \{x\} \approx S^1$, $\tilde{H}_1(X) \approx \mathbb{Z} \oplus \mathbb{Z}$ if manifold, $\tilde{H}_1(X, X \setminus \{x\}) = 0$

$\Rightarrow$ We have $\cdots \xrightarrow{a} \mathbb{Z} \xrightarrow{b} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{c} 0$

$\Rightarrow \operatorname{im} b = \ker c = \mathbb{Z} \oplus \mathbb{Z} \Rightarrow \Rightarrow\Leftarrow \Rightarrow$

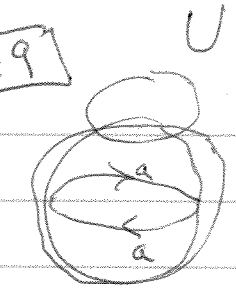
$\tilde{H}_1(X, X \setminus \{o\}) \neq 0 \Rightarrow X$ is not a topological manifold. //

1. (a) $\mathbb{R}$. Clear

(b) Comb Space $X = [0,1] \times \{0\} \cup \{\{1/n\} \times [0,1] \mid n \in \mathbb{Z}^{>0}\} \cup \{0\} \times [0,1]$.

It is clear this is path connected $\Rightarrow$ connected. Not locally path connected. Take any nbd of $(0,1)$ of radius $\leq 1/10$. Then the nbd is a disjoint union of ∞ vertical line pieces.

(c) Take a complete metric space which is not compact. For example we can take $\mathbb{R}^2$. Let d be the standard metric on $\mathbb{R}^2$. define $\bar{d}(x,y) = \min\{d(x,y), 1\}$. Then its bounded. //

5. 

Use Van-Kompen. Then we have that $U \cap V = S^1$, $U = *$, $V = \mathbb{R}P^2$.

$$\begin{array}{ccc} \pi_1(S^1) & \xrightarrow{g_*} & 0 \\ f_* \downarrow & & \downarrow \\ \pi_1(\mathbb{R}P^2) & \longrightarrow & \pi_1(X) \end{array}$$

$\Rightarrow \pi_1(X) \cong \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$ since f_* must send any generator to 0 since any nonzero element in $\pi_1(S^1)$ has order ∞. //