Monday, January 7, 2013

1. (a) Suppose $X \times Y$ is not connected $\Rightarrow$

$X \times Y = U \sqcup V$ with U, V open. Let $U = U_1 \times U_2$,

$V = V_1 \times V_2$. Then U_1, U_2 open in X, V_1, V_2 open i Y

since projection maps are ~~continuous~~ opens. Note,

$U \cap V = \emptyset \Rightarrow U_1 \cap V_1 = \emptyset$ or $U_2 \cap V_2 = \emptyset \Rightarrow$

either $X = U_1 \sqcup V_1$ or $Y = U_2 \sqcup V_2 \Rightarrow \times$

$\Rightarrow X \times Y$ is connected.

(b) Let $x, y \in \mathbb{R}^\infty$. Let

$\gamma(t) : [0,1] \longrightarrow \mathbb{R}^\omega$ be given by $\gamma(t) = x + (y - x)t$.

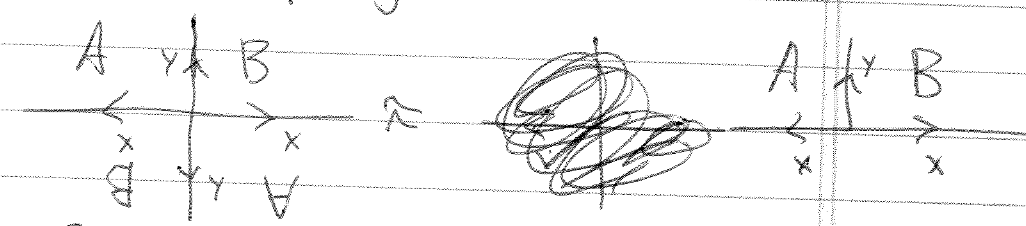
This is a path and therefore $\mathbb{R}^\infty$ is path connected

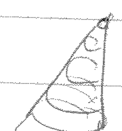
$\Rightarrow$ connected.

2. Suppose X is a manifold. Clear $\exists$ nbd $\approx \mathbb{R}^n$ for any p.t. in interior of quadrant under quotient map $\Rightarrow$ n-manifold.

Let U_0 be a nbd around $0/\sim \approx \mathbb{R}^n$. Then we have that $U_0 \setminus \{0\} \approx \mathbb{RP}^{n-1} \not\approx S^{n-1}$ for $n \geq 3$. $\Rightarrow$ X is not a topological manifold for $n \geq 3$. Note $\mathbb{R}/\sim$ $=$ •———> Which is not a topological manifold.

Now, $\mathbb{R}^2/\sim$ $=$



$\approx$  cone $\approx \mathbb{R}^2 \Rightarrow X$ is a topological manifold $\Longleftrightarrow n=2$. //

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3. Let $f: \mathbb{R}^3 \mapsto \mathbb{R}$ be defined by

$$f(x,y,z) = x^2 + 2x + 3y^2 - 6y + z^2 + 4z$$

$$= (x+1)^2 - 1 + 3(y^2 - 2y) + (z+2)^2 - 4$$

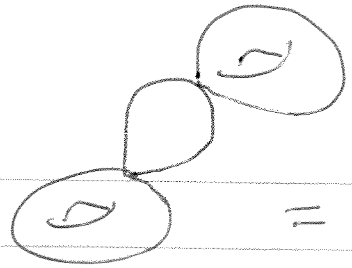
$$= (x+1)^2 - 1 + 3(y-1)^2 - 3 + (z+2)^2 - 4$$

$$= (x+1)^2 + 3(y-1)^2 + (z+2)^2 - 8$$

Let $g: \mathbb{R}^3 \mapsto \mathbb{R}$ be given by

$g(x,y,z) = (x+1)^2 + 3(y-1)^2 + (z+2)^2$. Want to figure out when $f^{-1}(c+8)$ is a 2-manifold.

$dg_{(x,y,z)} = \big(2(x+1), 6(y-1), 2(z+2)\big) \Rightarrow$ Worry some points are $(-1,1,-2)$. If $c > -8$, then $c+8$ is a regular value $\Rightarrow$ smooth 2-manifold. If $c < -8$ $\Rightarrow$ it is empty $\Rightarrow$ Not 2-manifold. If $c = 1 \Rightarrow$ single point $\Rightarrow$ 0-manifold and not 2-manifold. //

4. $Y \simeq$  $= T \vee T \vee S^1$

$\Rightarrow \pi_1(Y, \bar{x}) = (\mathbb{Z} \times \mathbb{Z}) * (\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z}$.

5. We will show that f is a closed mapping.

Let $K \subset X$ be closed. Let y be a limit point of $f(K)$.

Let U_y be a compact nbd of y in $\mathbb{R}^n$. Then, y is a limit point of $f(K) \Leftrightarrow y$ is a limit point of $f(K) \cap U_y$.

Since f is a proper map, we have that $f^{-1}(U_y)$ is compact.

So, $K \cap f^{-1}(U_y)$ is compact in X.

But, $f(K \cap f^{-1}(U_y)) = f(K) \cap U_y$ is compact $\Rightarrow$ closed $\Rightarrow y \in f(K) \cap U_y \Rightarrow y \in f(K) \Rightarrow f(K)$ contains all of its limit points $\Rightarrow f(K)$ is closed. $\Rightarrow$

$f : X \mapsto f(X)$ is a continuous bijection of closed sets $\Rightarrow$ homeomorphism. //

Afternoon.

1. We can equivalently think of X as a torus T formed by identifying a 2-disk in T with boundary C to $M \amalg_f D$ where $M \amalg_f D$ is a mobius band with disk glued along boundary and this disk and boundary identified to 2-disk in T. But $M \amalg_f D \simeq \mathbb{R}P^2$ and a 2-disk contracts to a point $\Rightarrow X \simeq T \vee \mathbb{R}P^2 \Rightarrow$

$H_n(X) \cong 0$ for $n \geq 3$, $H_2(X) \cong \mathbb{Z}$,

$H_1(X) \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2$, $H_0(X) \cong \mathbb{Z}$. //

2. Let $y \in Y$. Then for each $x \in f^{-1}(y)$ $\exists$ U_x open $x \in U_x$, $f: U_x \mapsto f(U_x)$ homomorphism. Thus $f^{-1}(y)$ must be discrete but X compact Hausdorff $\Rightarrow$ finite.

Let $f^{-1}(y) = \{x_1, x_2, \dots, x_n\}$. Let $U_1, \dots, U_n$ be open sets around these points with $f: U_i \mapsto f(U_i)$ homeomorphism.

Let $V = \bigcap_{i=1}^{n} f(U_i)$. Then $f: U_i \cap f^{-1}(V) \mapsto V$ is a homeomorphism $\forall$ U_i $\Rightarrow$ f is a cover as long as f is onto. $f(X)$ compact in Hausdorff space $\Rightarrow$ closed. Also open since f is a local homeomorphism $\Rightarrow$ $f(X)$ open + closed. Since Y connected $\Rightarrow$ $f(X) = Y$ $\Rightarrow$ f is a covering. //

3. (a) $\pi_1(S^3) = 0 \Rightarrow$ Does not.

(b) S^1 nontrivially covers its. For example we have

$f: S^1 \to S^1$ given by $f(z) = z^3$ is a 3-fold cover.

$\Rightarrow S^1 \times S^1$ nontrivially covers itself.

(c) Since S^1 nontrivial covers itself $\Rightarrow$

$S^1 \times S^2$ does.

(d) $\chi(\Sigma_2) = 1 - 4 + 1 = -2$

$\Rightarrow -2 = K(-2) \Rightarrow K = 1 \Rightarrow$ cannot nontrivially

cover itself. //

4.

First note that there is a homeomorphism $f_{xy}: \overline{B_1}(0) \mapsto \overline{B_1}(0)$ where $\overline{B_1}(0)$ denotes the closed ball of radius 1 that is the identity on the boundary and which switches the interior points x + y. Now, M connected manifold $\Rightarrow$ also locally path connected $\Rightarrow$ path connected.

Now, let $x, y \in M$. Let $p: [0,1] \mapsto M$ be a path from x to y. Then $p([0,1])$ is compact in M. Consider the open cover $\mathcal{O}$ of $p([0,1])$ given by $\mathcal{O} = \{U_z \mid z \in p([0,1]) \text{ and } U_z \approx B_1(0)\}$

Then $\mathcal{O}$ has a subcover, say $\{U_1, \ldots, U_n\}$. After reordering, we may assume $x \in U_1$, $y \in U_n$, $U_i \cap U_{i+1} \neq \emptyset$.

Let $x_i \in U_i \cap U_{i+1}$. Produce a sequence $x, x_1, x_2, \ldots, x_{n-1}, y$

Then we have homeomorphisms $M \setminus \{x\} \mapsto M \setminus \{x_1\} \mapsto \cdots \mapsto M \setminus \{y\}$

$\Rightarrow M \setminus \{x\} \approx M \setminus \{y\}$. //

(a)

5. "$\Rightarrow$" Let $x, y \in CX$. if $x, y \in X \times (0,1]/\sim$

Then ~~comp~~ Hausdorffness is clear since

$X \times (0,1]$ is Hausdorff. Suppose $x \neq y$, $x = X \times \{0\}$,

Then $y = (x, a)$ $a \neq 0$ So, we can choose the

open set $X \times [0, 1/8 a)$ around x and $X \times (1/4 a, 1]$

around y. "$\Leftarrow$" Since CX is Hausdorff $\Rightarrow$

there are disjoint neighborhoods around $\underset{U}{x \times \{1/2\}}$ and $\underset{V}{y \times \{1/2\}}$ for

$x \neq y$. ~~~~ Then

~~Then~~ $q: X \times [0,1] \mapsto CX \Rightarrow q^{-1}(U), q^{-1}(V)$ disjoint and

open in $X \times [0,1]$. Since projection maps are open

$\Rightarrow \pi_X(q^{-1}(U))$ and $\pi_X(q^{-1}(V))$ are disjoint open

nbhds of x and y in X. //

"$\Rightarrow$"

5. (b) Suppose X is ~~Hausdorff~~ Connected. Then CX is connected since CX is the image of X under a quotient map, and quotient maps are continuous and the continuous image of a connected space is connected. ~~Now, suppose CX is connected.~~

Actually this statement is false "$\Leftarrow$" doesn't work.

Let ~~$X = \mathbb{R} \times [0,1]$~~ $X = \{0,1\}$. Then $X \times [0,1] = [0,1] \sqcup [0,1]$ and $CX = <$ identifies them at a point $\Rightarrow$ CX connected but X is not. //