Analysis QR January 11, 2014

(1) This is false, Let $E = \bigcup_{n=1}^{\infty} (n-2^{-(n+1)}, n+2^{-(n+1)})$

Then, $m(E) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$.

E is unbounded $\Rightarrow$ No finite union of bounded intervals cover $E \Rightarrow m(F) = +\infty$

$\Rightarrow$ F does not necessarily exist.

(2) Equivalently, we will show that $g(p) = \|f\|_p^p$ is a continuous function.

Note, $g(p) = \int |f|^p = \int_{|f|<1} |f|^p + \int_{|f|\geq 1} |f|^p$

$= g_1(p) + g_2(p)$.

We show that g_1 is continuous.

For g_1

We have that for $|\varepsilon|$ small,

$|f|^{p+\varepsilon}$ is dominated by $|f|$

$\Rightarrow \lim_{\varepsilon \to 0} g_1(p+\varepsilon) = g_1(p)$

Now, for $|\varepsilon|$ small, for g_2 we have

$|f|^{p+\varepsilon}$ is dominated by $|f|^4 \Rightarrow$

$\lim_{\varepsilon \to 0} g_2(p+\varepsilon) = g_2(p) \Rightarrow$ $g_1 + g_2$ continuous

$\Rightarrow p \mapsto g_1(p) + g_2(p) = g(p) = \|f\|_p^p$ is

continuous $\Rightarrow p \mapsto \|f\|_p$ is continuous. //

(3) We will use a change of variables to solve this problem. $\int_{(0,1)} f(x^2)^q \, dx =$

$$\frac{1}{2} \int_{(0,1)} \frac{1}{\sqrt{y}} f(y)^q \, dy = \frac{1}{2} \left\| \frac{1}{\sqrt{y}} f(y)^q \right\|_1$$

$$\leq \frac{1}{2} \left\| \frac{1}{\sqrt{y}} \right\|_\alpha \left\| f(y)^q \right\|_\beta < +\infty \text{ if}$$

$$q\beta = 4, \ \alpha < 2 \Rightarrow \beta \in (2, 4]$$

$\Rightarrow q \in [2,4) \Rightarrow$ These work.

Consider $f(y) = y^{-1/4}$, then

$\|f(x^2)\|_q = +\infty$ for $q \geq 2 \Rightarrow q \in [2,4)$. //

(5) Suppose $\forall x \in [0,1]$, $\exists$ an interval I_x centered at x such that $m(E \cap I) < \frac{m(I_x)}{8}$. $[0,1]$ is compact and $\{I_x\}_{x \in [0,1]}$ form an open cover $\Rightarrow$ $\exists$ a finite subcover which cover $[0,1]$. From this finite subcover, we can find $I_1, \ldots, I_N$ such that I_m intersects only I_{m-1} & I_{m+1} and there union is $[0,1]$. Thus we have that

$$m(E) \leq \sum_{n=1}^{N} m(E \cap I_n) < \sum_{n=1}^{N} \frac{m(I_n)}{8}$$

$$\leq \frac{2}{8} m([0,1]) = 1/4 \Rightarrow \times$$

This proves the claim. //

Analysis QR Part II

(1) $g(z)$ entire $\Longleftrightarrow \frac{\partial}{\partial \bar{z}}(g) \equiv 0$.

Let $g(z) = f(h(z))$ where $h(z) = 2z + \bar{z}$.

Then we have that

$$\frac{\partial}{\partial \bar{z}}(f \circ h) = \left(\frac{\partial f}{\partial z} \circ h\right)\frac{\partial h}{\partial \bar{z}} + \left(\frac{\partial f}{\partial \bar{z}} \circ h\right)\frac{\partial \bar{h}}{\partial \bar{z}}.$$

But f entire $\Rightarrow \frac{\partial f}{\partial \bar{z}} \equiv 0$

$\Rightarrow \left(\frac{\partial f}{\partial z} \circ h\right)\frac{\partial h}{\partial \bar{z}} = 0$. But,

$$\frac{\partial h}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x}(3x - iy) + i\frac{\partial}{\partial x}(3x - iy)\right)$$

$$= \frac{1}{2}\left[3 + 1\right] = 2$$

$\Rightarrow \frac{\partial f}{\partial z} \circ h \equiv 0$. But, $h(z) = 3x - iy$

$\Rightarrow h$ is bijective $\Rightarrow \frac{\partial f}{\partial z} \circ h \equiv 0 \Longleftrightarrow$

$\frac{\partial f}{\partial z}$ is $\Longleftrightarrow \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Longleftrightarrow f$ constant. //

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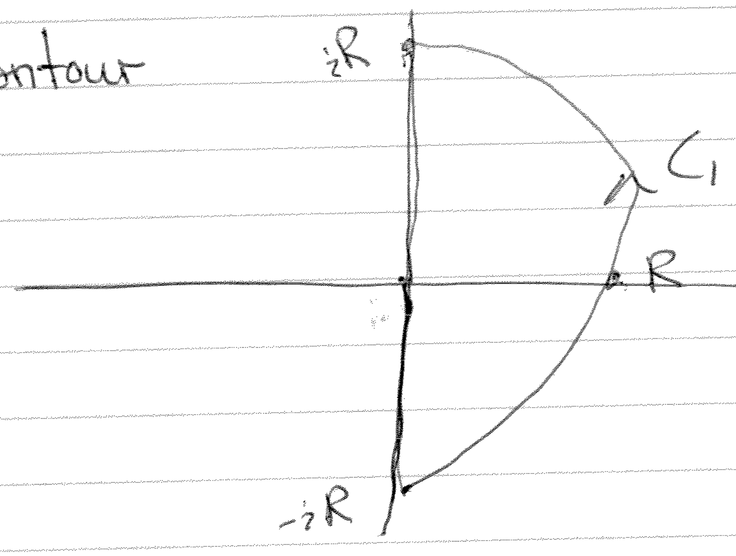
(2) Consider the contour



Then, for R big enough we have that all the zeros are contained in the contours since

$p(iR) = R^8 - 10R^3 i - 50Ri + 1$

$\neq 0 \Rightarrow$ No zeros on contour. $p(z)$ has no poles so we have that $\int_C \frac{p'(z)}{p(z)} dz$

$= 2\pi i (N)$, where N is # of zeros counting multiplicity inside of C. as $R \mapsto +\infty$

$p(z)$ behaves like $z^8 \Rightarrow$ between $-\pi/2$ & $\pi/2$,

$p(C_1)$ goes around 0 almost 4 times.

Since $p(ti)$ are Re part > 0 $\forall t \Rightarrow$

that $p(C)$ gets stuck and can't wrap around the origin anymore.

So, $\frac{1}{2\pi i} \int_C \frac{p'(z)}{p(z)} dz = \frac{1}{2\pi i} \int_{p(C)} \frac{1}{z} dz$

$= N = 4.$ //

(3) Let $f(z) = \cdots + a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + \cdots$

Then $f(z) + 2f(z^2) =$

$$\left(\cdots + a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + \cdots\right)$$

$$+ \left(\cdots + 2a_{-3}z^{-6} + 2a_{-2}z^{-4} + 2a_{-1}z^{-2} + 2a_0 + \cdots\right)$$

$\Rightarrow a_{-1} = a_{-3} = a_{-5} = \cdots 0$ (i.e. $a_{-\text{odd}} = 0$)

$\Rightarrow \cancel{2a_{-2}}\ 2a_{-1} + a_{-2} = a_{-2} = 0$

$\Rightarrow 2a_{-2} + a_{-4} = a_{-4} = 0$ ect $\Rightarrow$

$a_{-\text{even}} = 0 \Rightarrow f(z)$ does not have a

singularity at $0 \Rightarrow$ No such f exist. //

(4) Claim: Every compact subset of $\mathbb{D}$ is contained in $\alpha\mathbb{D}$, $0<\alpha<1$.

Proof, $\{(1-\frac{1}{n})\mathbb{D}\}_{n=1}^{\infty}$ form an open cover of X compact $\subset \mathbb{D}$ (Since it is a cover of $\mathbb{D}$) $\Rightarrow$ $\exists$ a finite subcover $\Rightarrow X \subset (1-\frac{1}{n})\mathbb{D}$ some $n \Rightarrow$ Claim. Let $X \subset \alpha\mathbb{D}$ be compact Then $f_n(\alpha\overline{\mathbb{D}})$ is compact in $\mathbb{C} \Rightarrow$ Bounded $\Rightarrow \{f_n : \alpha\mathbb{D} \mapsto \mathbb{C}\}_{n=1}^{\infty}$ is a set of holomorphic functions such that $f_n(\alpha\mathbb{D})$ misses $5, 6$ for $n \geq M$, some M since $\operatorname{Re} f_n \mapsto 0$ uniformly on $\alpha\overline{\mathbb{D}}$ $\Rightarrow \{f_n : \alpha\mathbb{D} \mapsto \mathbb{C}\}_{n=M}^{\infty}$ form a normal family $\Rightarrow$ converge on X to holomorphic function f, $f(0)=0$, $\operatorname{Re} f = 0$

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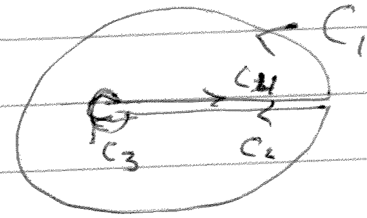
Since f is holomorphic $\Rightarrow \operatorname{Im} f = 0$

~~$\Rightarrow \operatorname{Im} f_n(z) \mapsto 0$~~ $f_n \mapsto 0$

uniformly on compact sets $\Rightarrow$

$\operatorname{Im} f_n \mapsto 0$ uniformly on compact sets. //

(5) Consider the keyhole contour:



Then, $$\int_C \frac{z^t}{(z+1)(z+2)}\,dz =$$

$$\overbrace{\int_{C_1} \frac{z^t}{(z+1)(z+2)}\,dz}^{0} + \int_{C_2} \frac{z^t}{(z+1)(z+2)}\,dz +$$

$$\overbrace{\int_{C_3} \frac{z^t}{(z+1)(z+2)}\,dz}^{0} + \int_{C_4} \frac{z^t}{(z+1)(z+2)}\,dz =$$

$$-\int_0^\infty \frac{e^{t\ln|z| + 2\pi i t}}{(z+1)(z+2)}\,dz + \int_0^\infty \frac{z^t}{(z+1)(z+2)}\,dz =$$

$$2\pi i\,(-1)^t + \frac{2\pi i\,(-2)^t}{-1}$$

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$$\Rightarrow \left(1 - e^{2\pi i t}\right) \int_0^\infty \frac{z^t}{(z+1)(z+2)}\, dz$$

$$= 2\pi i \left[1 - 2^t\right] e^{\pi i t}$$

$$\Rightarrow \frac{e^{-\pi i t} - e^{\pi i t}}{2\pi i} \int_0^\infty \frac{z^t}{(z+1)(z+2)}\, dz = \left[1 - 2^t\right]$$

$$\Rightarrow -\frac{\sin(\pi t)}{\pi} \int_0^\infty \frac{x^t}{(x+1)(x+2)}\, dx = \left[1 - 2^t\right]$$

$$\Rightarrow \frac{\pi\left[2^t - 1\right]}{\sin(\pi t)} = \int_0^\infty \frac{x^t}{(x+1)(x+2)}\, dx$$