6 Jan 2014

1. $X = M \amalg_f N \approx$ Klien Bottle

$\Rightarrow H_n(X) = 0,\ n \geq 2,\ H_1(X) = \mathbb{Z} \times \mathbb{Z}_2,$

$H_0(X) = \mathbb{Z}$. Alternatively use MV.

We have $U = M, V = N, U \cap V = S^1$

$0 \to \tilde{H}_2(S^1) \to \tilde{H}_2(M) \oplus \tilde{H}_2(N) \to$

$\tilde{H}_2(X) \to \tilde{H}_1(S^1) \to \tilde{H}_1(M) \oplus \tilde{H}_1(N) \to$

$\tilde{H}_1(X) \to 0$ Which reduces to

$0 \xrightarrow{a} \tilde{H}_2(X) \xrightarrow{b} \mathbb{Z} \xrightarrow{c} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{d}$

$1 \mapsto 2(1,1)$

$\tilde{H}_1(X) \xrightarrow{e} 0$

$\mathrm{im}(a) = 0 = \mathrm{Ker}(b),\ \mathrm{im}(b) = \mathrm{Ker}(c) = 0$

$\Rightarrow \tilde{H}_2(X) = 0$

$\mathrm{im}(d) = \mathrm{Ker}(e) = \tilde{H}_1(X)$. We have $\mathbb{Z} \oplus \mathbb{Z} / \mathrm{Ker}(d) \approx \tilde{H}_1(X)$

$\Rightarrow \mathbb{Z} \oplus \mathbb{Z} / 2\langle 1,1 \rangle \approx \tilde{H}_1(X) \Rightarrow \langle (1,0),(1,1) \rangle / \langle 2\langle 1,1 \rangle \rangle \approx \tilde{H}_1(X)$

$\approx \langle a,b \rangle / 2b \approx \mathbb{Z} \times \mathbb{Z}_2 \Rightarrow$ $H_n(X) = 0,\ n \geq 2$

$H_1(X) \approx \mathbb{Z} \times \mathbb{Z}_2,\ H_0(X) \approx \mathbb{Z}$ //

2. $d\pi_x : T_x(SL_n\mathbb{R}) \mapsto \mathbb{R}^n = \pi : T_x(SL_n\mathbb{R}) \mapsto \mathbb{R}^n$.

$\det : M_n(\mathbb{R}) \mapsto \mathbb{R}$. $SL_n\mathbb{R} = \det^{-1}(1)$

is a submanifold of $M_n(\mathbb{R})$ since I is a regular value of det. Let $A \in SL_n(\mathbb{R})$.

Then $T_A(SL_n(\mathbb{R})) = \ker d(\det)_A$

$$d(\det)_A(B) = \lim_{t \mapsto 0} \frac{\det(A+tB) - \det(A)}{t}$$

Let $A = (a_1, \dots, a_n)$. Suppose $B = (a_i, 0, 0, \dots, 0)$, $i \neq 1$

$$\text{Then } d(\det)_A(B) = \lim_{t \mapsto 0} \frac{\det(a_1 + t a_i, a_2, \dots, a_n) - \det(A)}{t}$$

$$= \lim_{t \mapsto 0} \frac{\det(A) + t\det(a_i, \dots, a_i, \dots, a_n) - \det(A)}{t}$$

$$= \det(a_i, \dots, a_i, \dots, a_n) = 0$$

Now, let $B = (a_1, -a_2, 0, \dots, 0)$

Then $d(\det)_A(B) = \lim_{t \to 0} \dfrac{\det(a_1 + t a_1, a_2 - t a_2, a_3, \dots, a_n) - \det(A)}{t}$

$= \lim_{t \to 0} \dfrac{\det(A) + t\det(A) - t\det(A) - \det(A)}{t} = 0$

$\Rightarrow \operatorname{Span}\{a_1, a_2, \dots, a_n\} \subseteq \pi(T_A(SL_n(\mathbb{R})))$

$\Rightarrow \pi(T_A(SL_n(\mathbb{R}))) = \mathbb{R}^n \Rightarrow \pi$ is a submersion for $n > 1$. //

3. It is easy to see that the space deformation retracts to a torus.

$\Rightarrow \pi_1(Y) = \pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1)$

$= \mathbb{Z} \times \mathbb{Z}$. //

4. (a)

Let $x, y \in \mathbb{R}^\omega$

Let $f: [0,1] \to \mathbb{R}^\omega$ be given by

$f(t) = x + (y-x)t = (x_1 + (y_1 - x_1)t, x_2 + (y_2 - x_2)t, \ldots$

Since each component is continuous + $f(0) = x$, $f(1) = y$

$\Rightarrow$ f continuous by Product Topology.

$\Rightarrow$ f is a path from x to y. $\Rightarrow \mathbb{R}^\omega$ is path connected

$\Rightarrow$ connected

(b) (i) Y is clearly Hausdorff since

if $x \neq y$, $x, y \in Y \Rightarrow \exists\, i$ s.t. $x_i \neq y_i$.

Then $\exists\ U_{x_i}, U_{y_i}$ open, $U_{x_i} \cap U_{y_i} = \emptyset$

since R Hausdorff. So, $\prod_{j=1}^{i-1} R_j \times U_{x_i} \times \prod_{j=i+1}^{\infty} R_j$

and $\prod_{j=1}^{i-1} R_j \times U_{y_i} \times \prod_{j=i+1}^{\infty} R_j$ work.

(ii) No. $\overline{\prod_{i=1}^{\infty} C_i} = \prod_{i=1}^{\infty} \overline{C_i}$

$\Rightarrow$ each C_i dense in R $\Rightarrow$ each C_i contains at least 2 points, $\Rightarrow$ a set $\prod_{i=1}^{\infty} \{0,1\} \subset \prod_{i=1}^{\infty} C_i$

$\Rightarrow$ Any dense set has cardinality $\geq 2^{\aleph}$

$\Rightarrow$ No dense set is countable. //

5. (a) Let G be a group. We have that

$G = \langle g_\alpha, \alpha \in A \mid \text{relations} \rangle$

Let $F_A = \langle g_\alpha \mid \alpha \in A \rangle$. Now, consider the inclusion map $i : F_A \longmapsto G$. Then i is a homomorphism and $F_A / \ker i \approx G$. Note that $\ker i$ is a normal subgroup of F_A. Consider the space

$\bigvee_{\alpha \in A} S^1$ which is a wedge of $|A|$ circles. Then,

There is a regular cover corresponding to the normal subgroup $\ker i$ by covering space theory

Call this space $\Gamma(G)$. Then $p_* \pi_1(\Gamma(G)) \approx \ker i$ and $\overset{\approx G}{F_A / \ker i}$ acts on $\Gamma(G)$. $\Rightarrow$

all groups have desired property. //

5. (b) If $\Gamma(G)$ is a tree $\Rightarrow \pi_1(\Gamma(G)) \cong 0$

Since all graphs are free $\Rightarrow G \cong \pi_1(\bigvee_{\alpha \in A} S^1)$

$\Rightarrow G$ free. ONLY free groups.

Afternoon Session

1. $g \circ f : X \mapsto X$ is clearly injective. We show its surjective. Suppose $g \circ f(X) \neq X$.

Let $x \in X \setminus g \circ f(X)$. x has positive distance from $g \circ f(X)$ since compact. Let distance $= d > 0$.

Let $x = x_0$, $x_n = g \circ f(x_{n-1})$.

Then distance(x_0, x_n), $n \neq 0$ $\geq d$ since $g \circ f$ is an isometry, we see distance$(x_n, x_m) \geq d$, $n \neq m$

$\Rightarrow \{x_n\}$ is a sequence without a convergent subsequence. But X compact metric space $\Rightarrow$ sequentially compact. Thus contradiction $\Rightarrow$ $g \circ f(X) = X \Rightarrow g \circ f$ bijection. Now, suppose $f : X \mapsto Y$ not surjective

$g \circ f : X \mapsto X$ surjective $\Rightarrow g : Y \mapsto X$ surjective

$\Rightarrow$ Y not injective $\Rightarrow$ $\Rightarrow\Leftarrow$ $\Rightarrow$ $f : X \mapsto Y$ must be surjective. //

2. $H_1(X \times Y)$ is the abelianization of $\pi_1(X \times Y)$.

So, $H_1(X \times Y) = ab(\pi_1(X \times Y)) = ab(\pi_1(X) \times \pi_1(Y))$

$= ab(\pi_1(X)) \times ab(\pi_1(Y)) = H_1(X) \times H_1(Y)$. //

3. Consider $f: \mathbb{R}^3 \to \mathbb{R}^2$ given by

$f(x,y,z) = (x^2+y^2+z^2,\ x^3 - xz^2 - y^2 z)$ Then $X = f^{-1}(1,0)$

Need to find critical values of f.

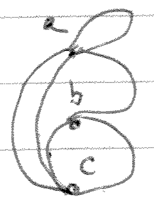
$$df_{(x,y,z)} = \begin{bmatrix} 2x & 2y & 2z \\ (3x^2-z^2) & (-2yz) & (-2xz-y^2) \end{bmatrix}$$

$2y(-2xz - (3x^2-z^2)) = 0, \quad 2y(-2xz - y^2 + 2z^2) = 0$

$2(-2x^2 z - xy^2 - z(3x^2-z^2)) = 0$

Ect...

4. Let $Y = S^1 \vee S^1 = \infty$

Let $X =$ 

Then

$f: X \mapsto Y$ is a covering. Since $|f^{-1}(*)| = 3 \Rightarrow$ if f is regular, then $\mathbb{Z}_3$ is a free action on X. But, $\mathbb{Z}_3$ must take a to b or c, but a is a seperation point and b, and c are not $\Rightarrow$ it can't act on X freely $\Rightarrow$ f is not regular.

5.

$f: M \mapsto N$. Locally $f: \mathbb{R}^n \mapsto \mathbb{R}^m$.

Let $\gamma: [0,1] \mapsto \mathbb{R}^n$ be a path. Then we have $f \circ \gamma : [0,1] \mapsto \mathbb{R}^m$ and $d(f \circ \gamma)_x =$ $df_{\gamma(x)} \circ \gamma_x : [0,1] \mapsto \mathbb{R}^m = 0 \Rightarrow f$ is constant on every path $\Rightarrow f$ is constant. Locally constant on connected set $\Rightarrow$ globally constant $\Rightarrow f$ constant. //