31 August 2013

1. We have $X \setminus \{x\} \simeq S^1 \times S^1 \simeq T$,

$X \simeq$ point. Consider the exact sequence

$$0 \to \tilde{H}_3(X \setminus \{x\}) \to \overset{0}{\tilde{H}_3(X)} \to \tilde{H}_3(X, X \setminus \{x\}) \to$$

$$\overset{\mathbb{Z}}{\tilde{H}_2(X \setminus \{x\})} \to \overset{0}{\tilde{H}_2(X)} \to \tilde{H}_2(X, X \setminus \{x\}) \to \cdots$$

So, $\tilde{H}_{n+1}(X, X \setminus \{x\}) \simeq \tilde{H}_n(T)$

$\Rightarrow \tilde{H}_3(X, X \setminus \{x\}) \simeq \mathbb{Z}$, $\tilde{H}_2(X, X \setminus \{x\}) \simeq \mathbb{Z} \oplus \mathbb{Z}$

$\Rightarrow X$ is not a manifold. If It was, then

top homology group $\tilde{H}_i(X, X \setminus \{x\}) \simeq \mathbb{Z}$,

and all rest 0.

2. Let K be closed in $X \times Y$. Suppose $p(K)$ not closed. Let $x \notin p(K)$ be a limit point to $p(K)$. Then $\{x\} \times Y$ is closed and disjoint from K.

Let $\mathcal{O} = \{ U_{(x,y)} \mid y \in Y,\ U_{(x,y)}$ open set containing (x,y), $U_{(x,y)}$ disjoint from $K \}$

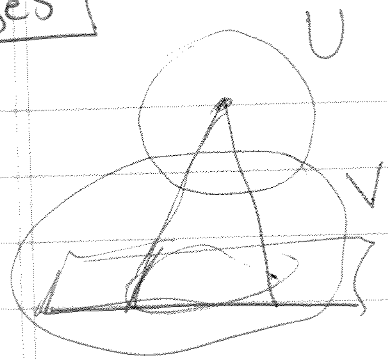
Since Y is compact, there is a finite subset of $\mathcal{O}$, say $U_1, \dots, U_N$ such $\{x\} \times Y \subset \bigcup_{i=1}^{N} U_i = V \times Y$ open

But $V \times Y$ open in $X \times Y \Leftrightarrow V$ open in $X \Rightarrow$

V open set around $\{x\}$ disjoint from $p(K) \Rightarrow$

$p(K)$ contains limit points $\Rightarrow$ closed $\Rightarrow$ p is a closed mapping.

$U \simeq *,\ V \simeq Y,\ U \cap V \simeq X$

3. 

Using the Van-Kampen Thm we have the following

$$\begin{tikzcd} \pi_1(X) \arrow[r] \arrow[d, "f_*"'] & \pi_1(*) \arrow[d] \\ \pi_1(Y) \arrow[r] & \pi_1(Cf) \end{tikzcd}$$

f_* homotopic to the inclusion map $i : \pi_1(X) \to \pi_1(Y)$

$$\Rightarrow \pi_1(Cf) \simeq \pi_1(Y) / f_*\pi_1(X) = 0 \iff$$

$$f_*\pi_1(X) = \pi_1(Y) \Rightarrow f_* \text{ onto.} //$$

4. (a) ~~Let $g : [-1,1] \to \mathbb{R}$ be a mapping that sends z coordinate in S^2 to image in $\mathbb{R}^3$ projected onto z axis.~~

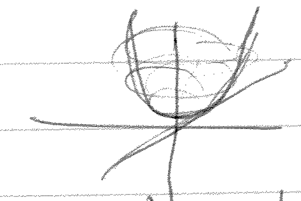
~~Let~~ Consider $\pi \circ f : S^2 \to \mathbb{R}$ where $\pi(x,y,z) = z$.

Then $\pi \circ f$ is the composition of continuous fns.

We have that $\pi \circ f(S^2)$ is a compact connected subset of $\mathbb{R} \Rightarrow \pi \circ f(S^2) = [a,b]$. Then the planes $z=a$, $z=b$ are tangent planes to $f(S^2)$. They are clearly $\parallel$. Suppose $a=b$, $\Rightarrow f(S^2)$ lives in the plane $z=a$. $\Rightarrow H_2(f(S^2)) = 0 \neq \mathbb{Z} \Rightarrow \Rightarrow\Leftarrow$. Therefore $a \neq b$ and we are done.

4. (b) Consider $g: \mathbb{R}^2 \mapsto \mathbb{R}^3$ given by

$g(x,y) = (x, y, x^2+y^2)$.



From this simple sketch it is clear that tangent planes never coincide. Also, clear g is smooth. Since g continuous $\Rightarrow$ inverse image of a compact set is closed. We only need to show inverse image of bounded set is bounded. That is g cannot take unbounded to bounded. But this is clear by how g is defined $\Rightarrow$ g is proper. //

"$\Rightarrow$"

5. Suppose $A \subset \mathbb{R}^\omega$, $\overline{A}$ compact. Let $\pi_i : \mathbb{R}^\omega \mapsto \mathbb{R}$ be projection onto the ith coordinate. Then we have that $\pi_i(\overline{A})$ compact $\forall i \Rightarrow$ bounded $\forall i \Rightarrow$ there is $s = (s_n)$ in $\mathbb{R}^\omega$ such that, for all $a = (a_n)$ in A, and all $n \geq 1$, the inequality $|a_n| \leq |s_n|$ holds. "$\Leftarrow$" If this holds, then $\overline{A} \subseteq$ Product of closed intervals. By Tychonoff, product of compact sets is compact $\Rightarrow \overline{A}$ is closed $\subset$ compact set $\Rightarrow$ compact. //

Afternoon Session

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1. If there did, $\chi(S^{2n}) = 2 = M\chi(X)$

Where $M = \left| \pi_1(X) / p_* \pi_1(S^{2n}) \right| = |\pi_1(X)| = 2K+1$

$\Rightarrow 2 = (2K+1)\chi(X) \Rightarrow \chi(X) = \frac{2}{2K+1} \in (0,1)$

$\Rightarrow \Rightarrow\!\Leftarrow \Rightarrow$ There does not exist such a thing.

2. (a) Suppose $X = \overbrace{U \amalg V}^{\text{open}}$. Since $f^{-1}(y)$ connected

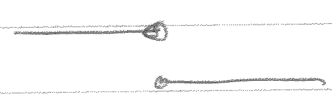
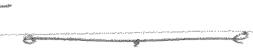
$\Rightarrow f^{-1}(y) \subset U$ or V. Let

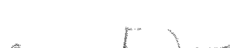
$U' = \{ y \in Y \mid f^{-1}(y) \subset U \}$, $V' = \{ y \in Y \mid f^{-1}(y) \subset V \}$.

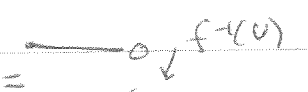
Then $f^{-1}(U') = U$, $f^{-1}(V') = V \Rightarrow U'$ and V' open.

also $U' \amalg V'$ and $U' \amalg V' = Y \Rightarrow Y$ not connected

$\Rightarrow \Rightarrow\!\Leftarrow \Rightarrow X$ is connected. //

(b) Let $X =$ , and let $Y =$ 

Let $f : X \mapsto Y$ be projection. Then if $U =$ 

$f^{-1}(U) =$  $\Rightarrow f^{-1}(U)$ open but U isn't $\Rightarrow$ Not quotient map. But continuous since projection is

3. $S^1 \times S^3$ has 1 4-cell
1 3-cell
1 1-cell
1 0 cell

S^1 and S^3 orientable compact manifolds $\Rightarrow S^1 \times S^3$ is

$\Rightarrow$ Top Homology Group $= \mathbb{Z} \Rightarrow H_4(S^1 \times S^3) = \mathbb{Z}$.

No 2-cells $\Rightarrow$ $H_3(S^1 \times S^3) = \mathbb{Z}$. and $H_2(S^1 \times S^3) = 0$.

$H_1(S^1 \times S^3) = H_1(S^1) \times H_1(S^3) = \mathbb{Z}$. $H_0(S^1 \times S^3) = \mathbb{Z}$

since $S^1 \times S^3$ connected $H_n(S^1 \times S^3) = 0$, $n \geq 5$

4. (a) $dF_{(x,y,u,v)} =$

$$\begin{bmatrix} 2x & -2y & (-2u+2) & 2v \\ 2y & 2x & (-2v) & (-2u+2) \end{bmatrix}$$

$\Rightarrow x^2+y^2=0 \Rightarrow x=y=0$ and

$(-2u+2)^2+4v^2=0 \Rightarrow v=0,\ u=1$

$\Rightarrow$ Worrysome point is $(0,0,1,0)$

$\Rightarrow F(\mathbb{R}^4 \setminus (0,0,1,0))$ are Regular Values.

$F(0,0,1,0) = (1,0)$

Since F onto $\Rightarrow$ Regular Values are

$\mathbb{R}^2 \setminus (1,0)$. //

(b) Note that $T_{(0,1,0,0)}M = \text{Ker } dF_{(0,1,0,0)}$

$$= \text{Ker}\begin{bmatrix} 0 & -2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

$\Rightarrow$ ~~$u = v, x$~~ $u = y, \quad x = v$

$\Rightarrow T_{(0,1,0,0)}M = \left\{ (x,y,u,v) \in \mathbb{R}^4 \,\middle|\, u = y, x = v \right\}$

5. $f: X \mapsto Y$ a regular covering map.

So $f_* \pi_1(X) \cong \mathbb{Z}_2$ is a normal subgroup of $\pi_1(Y)$. Thus, Let $f_* \pi_1(X) = \langle a \rangle$

$\cancel{a\pi_1(Y) = \pi_1(Y)}$ $x\langle a \rangle = \langle a \rangle x \ \forall$ $x \in \pi_1(Y)$. $\Rightarrow \{xa, x\} = \{ax, x\}$

$\Rightarrow xa = ax \ \forall \ x \in \pi_1(Y) \Rightarrow a$ commutes with all $x \in \pi_1(Y)$. Thus $\langle a \rangle$ is in the abelianization of $\pi_1(Y) \Rightarrow$

$f_*: H_1(X) \mapsto H_1(Y)$ is injective. //