SOLUTIONS TO THE MAY 7, 2011 UNIVERSITY OF MICHIGAN QUALIFYING EXAM IN TOPOLOGY

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1. Problem 1

Define a topological space X by taking the disjoint union of two copies of the torus $S^1 \times S^1$, choosing one point in each copy, and identifying those points (the topology on X is the quotient topology). Is this space homotopy equivalent to a 2-manifold? Justify your answer.

The space described is just the wedge product of two tori. It is clear that if it were a manifold, then it would have to be a compact 2-manifold. However, $\pi_1(X) = F_2 * F_2$ and by classification of compact 2-manifolds we have that X cannot be a connected sum of $\mathbb{R}P^2$. However, it is also clearly not the connected sum of tori, and therefore X cannot be a manifold.

2. Problem 6

A two-point compactification of a Hausdorff space X is a compact Hausdorff space Y such that X is a dense subspace of Y, $Y - X$ consists of exactly two points. Prove that no two-point compactification of the Euclidean plane $\mathbb{R}^2$ exists. Let $Y = \mathbb{R}^2 \cup \{\infty_1, \infty_2\}$ and let U_{∞_1} and U_{∞_2} be disjoint open neighborhoods around ∞_1 and ∞_2 respectively. Since Y is compact, we have that the space $Y \backslash \{U_{\infty_1} \cup U_{\infty_2}\}$ is compact as well. Therefore $Y \backslash \{U_{\infty_1} \cup U_{\infty_2}\} \subset \{(x, y) \in \mathbb{R}^2 \mid ||(x, y)|| \leq M\}$ for some $M \in \mathbb{R}$. Then $V = \{(x, y) \in \mathbb{R}^2 \mid ||(x, y)|| > M\}$ is a connected set in $\mathbb{R}^2$, and $U_{\infty_1} \backslash \{\infty_1\}$ and $U_{\infty_2} \backslash \{\infty_2\}$ are disjoint open sets covering V. This is a contradiction.

3. Problem 9

Suppose $f : X \to Y$ is a smooth immersion between smooth manifolds of the same dimension. Given that X is closed and Y is connected, prove that Y is also closed.

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