SOLUTIONS TO THE SEPTEMBER 1, 2012 UNIVERSITY OF MICHIGAN QUALIFYING EXAM IN TOPOLOGY

DONDI ELLIS

1. Problem 1

A space X is constructed from two disjoint copies of $\mathbb{R}P^3$ and a copy of the unit interval I by gluing one end of I to a point of one copy of $\mathbb{R}P^3$, and gluing the other end of I to the other copy of $\mathbb{R}P^3$.

(1) Describe the universal cover $\tilde{X}$ of X.

The universal cover of X is given by $\coprod_{n\in\mathbb{Z}} S^3_n \cup I_n/\sim$ where $\sim$ is the equivalence relation given by taking some $x_n \in S^3_n$ and $x_{n+1} \in S^3_{n+1}$ and identifying $0 \in I_n$ with x_n and $1 \in I_n$ with x_{n+1}. Visually this creates a sorta beadlike looking pattern.

(2) Compute the homology groups of $\tilde{X}$.

Since $\tilde{X}$ is the universal covering of X, it follows that $\tilde{X}$ is simply connected. Therefore we have that $H_0(\tilde{X}) = \mathbb{Z}$ and $H_n(\tilde{X}) = 0$ for $n > 0$.

2. Problem 3

Let X denote the space $S^2 \cup A$, where $A = \{(x, 0, 0) \in \mathbb{R}^3 : 1 \leq x \leq 2\}$. Show that if $p : X \to Y$ is a covering map, then p must be a homeomorphism, i.e. X cannot cover anything except itself.

Consider an open neighborhood U around $p((1, 0, 0))$. Then $p^{-1}(U)$ will contain a neighborhood around $(1, 0, 0)$ It is clear that no other open neighborhood of any point in $S^2 \cup A$ is homeomorphic to this neighborhood, and thus p must be a 1-sheeted covering, and as such it cannot cover anything but itself.

3. Problem 5

Let X denote the quotient space $\mathbb{R}/\mathbb{Q}$ of the real line obtained by identifying all the rationals to a single point. (This is not the group theoretic quotient.)

(1) is X Hausdorff?

No, X is not Hausdorff. For $x \in \mathbb{R}$, let $[x]$ be its equivalence class in X. I will show that the only open neighborhood around $[0]$ is X.

Date: Sunday, February 24, 2013.

Proof. Let U be an open set in X around $[0]$. Then if $q : \mathbb{R} \to X$ is the quotient map in question, we have that $q^{-1}(U)$ is open in $\mathbb{R}$. In particular, $q^{-1}(U)$ contains $\mathbb{Q}$ which is a dense subset of $\mathbb{R}$. The only open subset of $\mathbb{R}$ containing $\mathbb{Q}$ is $\mathbb{R}$. Therefore, $U = X$. This proves that X is not Hausdorff. $\square$

(2) Is X compact?

Yes it is. We pretty much already proved this. The only open neighborhood around $[0]$ is X. Since any open cover contains an open set around $[0]$, we have proven the claim.

4. Problem 6

Identify the space of all 2×2 real matrices with $\mathbb{R}^4$ so that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ corresponds to (a, b, c, d). Show that the subspace $\sum$ of all matrices with determinant 1 is a smooth 3-dimensional manifold. Let Π denote th hyperplane in $\mathbb{R}^4$ with the equation $x_1 + x_2 + x_3 - x_4 = 0$. Does Π intersect $\sum$ transversely at I.

Consider the map $\det : \mathbb{R}^4 \to \mathbb{R}$ given by $\det(x_1, x_2, x_3, x_4) = x_1x_4 - x_2x_3$. The Jacobian of this map evaluated at an arbitrary point (a_1, a_2, a_3, a_4) is given by $(a_4, -a_3, -a_2, a_1)$. This equals $(0, 0, 0, 0)$ iff $a_1 = a_2 = a_3 = a_4 = 0$. Therefore, 1 is a regular value of det from which we conclude that $\det^{-1}(1)$ is a smooth 3-dimensional submanifold of $\mathbb{R}^4$.

5. Problem 7

The suspension of a space Y is the quotient space of $Y \times I$ obtained by identifying $Y \times \{0\}$ to a point and separately identifying $Y \times \{1\}$ to a point. Let X denote the suspension of $\mathbb{R}P^2$.

(1) Compute $\pi_1(X)$.

Let U be the open set $\mathbb{R}P^2 \times [0, \frac{3}{4})/\sim$ and let V be the open set $\mathbb{R}P^2 \times (\frac{1}{4}, 1]/\sim$. It is clear that U and V both deformation retract to a point. Furthermore, it is clear that their intersection deformation retracts to $\mathbb{R}P^2$. It follows from the Van-Kampen theorem that $\pi_1(X) = 0$.