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Analysis QR Jan 12, 2013

1. $$\lim_{x \mapsto \infty} \int_x^{x+1} f'(x)\,dx =$$

$$\lim_{x \mapsto \infty} \left[f(x+1) - f(x)\right] = a - a = 0.$$

But, By the mean value thm we have that $\int_x^{x+1} f'(x)\,dx =$

$f'(\xi_x)$ for some $\xi_x \in [x, x+1]$

$$\Rightarrow \lim_{x \mapsto \infty} f'(\xi_x) = 0$$

$$= \lim_{x \mapsto \infty} f'(x) = b \Rightarrow b = 0.//$$

2. Note that

$$\lim_{R\to\infty} \int_0^R \frac{\sin x}{x}\,dx$$

$$= \lim_{R\to\infty} \int_0^R \int_0^\infty e^{-xy} \sin x \, dy\, dx$$

Now since $\displaystyle\int_0^R \int_0^\infty |e^{-xy} \sin x| \, dy\, dx$

$= \displaystyle\int_0^R \left|\frac{\sin x}{x}\right| dx < \infty$ since

$\left|\frac{\sin x}{x}\right|$ is continuous on $\mathbb{R}$ and we have that the ~~continuous~~ continuous image of $[0, R]$ is compact $\Rightarrow$ $\left|\frac{\sin x}{x}\right|$ attains a maximum on $[0, R]$, say $M \Rightarrow \displaystyle\int_0^R \left|\frac{\sin x}{x}\right| dx \le RM$

$\Rightarrow$ We can invoke Fubini on $[0, R] \times (0, \infty)$

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Thus, $\lim_{R\to\infty} \int_0^R \int_0^\infty e^{-xy} \sin x \, dy \, dx =$

$$\lim_{R\to\infty} \int_0^\infty \int_0^R e^{-xy} \sin x \, dx \, dy$$

$$= \lim_{R\to\infty} \int_0^\infty -\frac{1}{1+y^2}\left(y e^{-xy} \sin x + e^{-xy} \cos x\right)\Big|_0^R dy$$

$$= \lim_{R\to\infty} \int_0^\infty \left[\frac{1}{1+y^2} - \frac{e^{-Ry}}{1+y^2}\left(y \sin R + \cos R\right)\right] dy$$

We wish to show $\lim_{R\to\infty} \int_0^\infty \frac{e^{-Ry}}{1+y^2}(y \sin R + \cos R) \, dy = 0$

Let $f_R(y) = \frac{e^{-Ry}}{1+y^2}(y \sin R + \cos R)$. (*)

Then we have $f_R(y) \leq \frac{e^{-y}}{1+y^2}(y+1)$

for $R > 1$

But, $\frac{e^{-y}}{1+y^2}(y) \leq e^{-y}$

$+ \frac{e^{-y}}{1+y^2} \leq e^{-y} \Rightarrow (*) \leq 2e^{-y}$

But, $\|2e^{-y}\|_{L_1(0,\infty)} = -2e^{-y}\Big|_0^\infty = 2$

$\Rightarrow 2e^{-y}$ dominates the f_R, so we can apply the Dominated Convergence Thm and so,

$$\lim_{R \to \infty} \int_0^\infty \frac{e^{-Ry}}{1+y^2}(y\sin R + \cos R)\,dy$$

$$= \int_0^\infty \lim_{R \to \infty} \frac{e^{-Ry}}{1+y^2}(y\sin R + \cos R)\,dy$$

$$= \int_0^\infty 0\,dy = 0$$

It now follows that the integral we original wanted to solve

$$= \int_0^\infty \frac{1}{1+y^2}\,dy = \tan^{-1}(y)\Big|_0^\infty$$

$$= \pi/2 \,.//$$

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Let $E = \{x \in X : |f(x)| > t\}$

3. We have that

$$\int_0^\infty \mu\{x \in X : |f(x)| > t\}\, dt \quad (\Delta)$$

$$= \int_0^\infty \int_X \mathbb{1}_E(x)\, d\mu\, dt \overset{(*)}{=}$$ Now, $\mathbb{1}_E(x)$ is

a ~~positive~~ nonnegative measurable

on $(0,\infty) \times X \Rightarrow$ We can invoke Fubini.

Thus, we have $(*) =$

$$\int_X \int_0^\infty \mathbb{1}_E(x)\, dt\, d\mu = \int_X \int_0^{|f(x)|} dt\, d\mu$$

$$= \int_X |f(x)|\, d\mu$$

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Going by to (Δ), we have

$$\int_X |f(x)|\,d\mu = \int_0^\infty \mu\{x \in X : |f(x)| > t\}\,dt$$

$$= \int_0^1 \mu\{x \in X \mid |f(x)| > t\}\,dt + \int_1^\infty \mu\{x \in X \mid |f(x)| \geq t\}\,dt$$

$$\leq 1\,\mu(X) + \int_1^\infty \frac{K}{t^p}\,dt$$

$$= \mu(X) + \left.\frac{t^{1-p}}{1-p}K\right|_1^\infty$$

$$= \frac{K}{p-1} + \mu(X) < +\infty$$

$$\Rightarrow \|f\|_1 < +\infty \Rightarrow f \in L_1.\ //$$

4. By the Lebesgue Differentiation Theorem, we have that

$$(*) \lim_{|I_x| \to 0} \frac{1}{|I_x|} \int_{I_x} \left| \mathbb{1}_E(y) - \mathbb{1}_E(x) \right| dy = 0$$

for a.e. $x \in [0,1]$ since $\mathbb{1}_E$ is Lebesgue Integrable.

Note that $(*) \geq$

$$\lim_{|I_x| \to 0} \frac{1}{|I_x|} \left| \int_{I_x} \mathbb{1}_E(y)\, dy - \int_{I_x} \mathbb{1}_E(x)\, dy \right|.$$

If $x \notin E$, then the limit reduces to

$$\lim_{|I_x| \to 0} \frac{1}{|I_x|} \int_{I_x} \mathbb{1}_E(y)\, dy =$$

$$\lim_{|I_x| \to 0} \frac{m(E \cap I_x)}{|I_x|} = 0 \text{ for a.e. } x \notin E$$

But, By assumption, $m(E \cap I_x) \geq \alpha m(I_x)$ $\forall I_x \Rightarrow \lim_{|I_x| \to 0} \frac{m(E \cap I_x)}{|I_x|} \geq \alpha \neq 0$

$\Rightarrow x \notin E$ for at most countable pts $\Rightarrow m(E) = 1.$ //

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(a)

5. Let $X = [0,1]$ and let μ be normal measure. Define $f_n = n^2 \mathbb{1}_{[0,1/n]}$

then, $\lim_{n\to\infty} f_n = 0$ which is integrable as are the f_n. So,

$$\int f_n d\mu = \|f_n\|_1 = n \Rightarrow \lim_{n\to\infty} \|f_n\|_1 = \infty$$

This proves the claim!

(b) We have $\lim_n \|f_n - f\| \geq \left|\lim_n \int (f_n - f) d\mu\right|$

$= \left|\lim_n \int f_n d\mu - \int f d\mu\right| = 0$ by assumption

$\Rightarrow \lim_n \|f_n - f\| = 0$, i.e. $f_n \to f$ in L_1.

//

(1) $f: \mathbb{H}_R \mapsto \Delta$, $f(5)=0$ can be written as $f = g \circ h$

Where $h: \mathbb{H}_R \mapsto \Delta$ is a conformal map sending $5 \mapsto 0$ and

$g: \Delta \mapsto \Delta$ is an arbitrary analytic function with $g(0)=0$.

Then, $f'(5) = g'(0)\,h'(5)$

By Schwartz, $|g'(0)| \leq 1$

Taking $g(z)=z$ we obtain the max.

So, $|f'(5)| = |h'(5)|$

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$$\mathbb{H} \mapsto \mathbb{B}$$

$$e^{\pi i/2} z$$

$$\frac{iz - \overline{B}}{iz - \overline{B}}$$

$$\frac{iz - 5i}{iz + 5i}$$

(1) Continued

The map $z \mapsto iz$ rotates $\mathbb{H}_R \mapsto \mathbb{H}_U$ (i.e. the upper Half plane) Also its clearly conformal

Now, $\dfrac{z-\beta}{z-\bar{\beta}} : \mathbb{H}_U \longmapsto \mathbb{D}$ conformal for

$\beta \in \mathbb{H}_U$, then $\dfrac{iz-\beta}{iz-\bar{\beta}} : \mathbb{H}_R \longmapsto \mathbb{D}$

Choose $\beta = i5$, then $\beta \in \mathbb{H}_U$ and

$\dfrac{iz-i5}{iz+i5} = \dfrac{z-5}{z+5} = a(z)$ is a map taking $\mathbb{H}_R \mapsto \mathbb{D}$

with $a(5) = 0$. $|a'(5)| =$

$$\left| \frac{(z+5)-(z-5)}{(z+5)^2}\Big|_5 \right| = \frac{10}{10^2} = \frac{1}{10}$$

So by Reimann Mapping Thm we have that

$|f'(5)| = 1/10$. //

(2) Take u^2, then $(u^2)_{xx} + (u^2)_{yy} = 0$

on $\mathbb{R}$ But, $(u^2)_{xx} + (u^2)_{yy} =$

$\cancel{2uu_x + u_{xx}}$

$(2uu_x)_x + (2uu_y)_y = 2uu_{xx} + 2u_x^2$

$+ 2uu_{yy} + 2u_y^2 = 0$ on $\mathbb{R}$ $=$

$2u_x^2 + 2u_y^2 \Rightarrow u_x = u_y = 0$ on $\mathbb{R}$

Since $f = \cancel{u+iv}\ u + iv$ is entire $\Rightarrow$

$f'(z) = u_x + iv_x$. But, we have
is entire

$v_x = -u_y = 0$ on $\mathbb{R} \Rightarrow f'(z) = 0$ on $\mathbb{R}$.

So, By the Identity principle, since

$f'(z) = 0$ on a sequence with limit point

Same thing

or since its 0 on $\mathbb{R}$, either way we

$f' = 0$ on $\mathbb{C} \Rightarrow f$ is Constant. //

(3) Since $1-\cos z$ and $100z^2$ are both holomorpic on $\mathbb{C}$ and since $100z^2$ is nonvanishing on the unit circle, S^1

We can invoke ~~the identity principle~~ Rouché.

Since $\max_{S^1} |100z^2| = 100 >$

$\max_{\text{on } S^1} |1-\cos z| < 20$

$\Rightarrow$ $100z^2$ and $100z^2 + (1-\cos z)$ have the same number on roots on $\mathbb{D}$, namely 2. //

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Continued, Now for finding the 2 solutions.

(3)

$$\cos z = 1 + 100z^2 \Rightarrow$$

$$1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots = 1 + 100z^2$$

$$\Rightarrow -\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots = 100z^2$$

$$\Rightarrow \frac{(200+1)z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \ldots = 0$$

$\Rightarrow$ 0 is a solution with multiplicity 1, and so we're Done. //

(4)

(a) Showing $I(t, y)$ is well-defined can be accomplished by showing the integral is finite since

$f(x+iy)e^{-ixt}$

$= [u(x+iy) + iv(x+iy)]e^{-ixt}$ where

$u + v$ are harmonic and thus are continuous and differiable. I E

$\frac{d}{dx} u(x+iy) = u_x(x+iy)$ and $\frac{d}{dx} v(x+iy) = v_x(x+iy)$

$\Rightarrow$ integral makes sense if it is finite

$$\left| \int_{\mathbb{R}} f(x+iy) e^{ixt} dx \right| \leq \int_{\mathbb{R}} |f(x+iy)| dx$$

$$\leq C \int_{\mathbb{R}} \frac{1}{1+x^2} dx = C \tan^{-1}(x) \Big|_{-\infty}^{\infty}$$

$$= \pi C.$$

(a) I just showed its bounded.

(a) We show that $I(t,y)$ is continuous.

$$\lim_{h\to 0} \left| I(t+h,y) - I(t,y) \right|$$

$$\leq \lim_{h\to 0} \int_{\mathbb{R}} \frac{c}{1+x^2} \underbrace{\left| e^{-ixh} - 1 \right|}_{f_h} dx$$

$$\leq \lim_{h\to 0} 2C \int_{\mathbb{R}} \frac{1}{1+x^2} dx = 2\pi C$$

Here we used $2 \geq \left| e^{-ixh} - 1 \right|$

So $\frac{2C}{1+x^2}$ dominates the f_h

+ By Dominated Convergence Thm

$$\lim_{h\to 0} \int_{\mathbb{R}} \frac{c}{1+x^2} \left| e^{-ixh} - 1 \right| dx = \int_{\mathbb{R}} 0 \, dx = 0$$

$\Rightarrow I(t,y)$ is continuous in first coordinate. //

(5) Suppose 0 is a removable singularity then $\lim\limits_{z \to 0} \left| e^{f(z)} \right| \leq \lim\limits_{z \to 0} e^{|f(z)|} = e^{|a|}$

$\Rightarrow z = 0$ is Not Pole. Now, suppose $f(z)$ has a pole of order n

$$\Rightarrow e^{f(z)} = e^{P_n(z^{-1}) + g(z)}$$

Where $g(z)$ is entire, $P_n(z)$ is a polynomial of degree n, we group the constant term with $p_n(z^{-1}) \Rightarrow \lim\limits_{z \to 0} g(z) = 0$

First, note that $e^{p_n(z)}$ is not a polynomial since $p_n(z)$ nonconstant ~~entire~~ $\Rightarrow$ ~~$p_n(z)$ takes on one of $2\pi i$ or~~ $e^{p_n(z)}$ is nonconstant + entire $\Rightarrow$ Never is zero $\Rightarrow$ Not a nonconstant polynomial $\Rightarrow$ $e^{p_n(z)}$ has an essential singularity at ∞

$\Rightarrow e^{p_n(1/z)}$ has an essential sing at 0

Since $\lim\limits_{z \to 0} e^{f(z)} = \lim\limits_{z \to 0} e^{p_n(z^{-1})} \cdot e^{g(z)}$

$= \lim\limits_{z \to 0} e^{p_n(z^{-1})} \Rightarrow z = 0$ is an essential singularity.

Now, suppose that $z=0$ is an essential singularity of $f(z) \Rightarrow$ in any nbd of $z=0$, say N_0, we have $f(N_0) = \mathbb{C}$ minus possibly 1 pt,

So, we note that e^z is entire so

$\exp(f(N_0)) = \exp(\mathbb{C} \setminus \text{maybe one pt})$

$= \mathbb{C} \setminus (0)$ ~~maybe one other point~~

(We know it is $\mathbb{C} \setminus (0)$ since takes all but possibly one pt ∞ many times.

So By Wierstrass Essential

$\Rightarrow$ Can't be Role//