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QR January 7, 2012

(1) We have $f(z) \cdot f\left(\frac{z}{z+1}\right) = z$

on $\{ 1/2, 1/3, \ldots 1/n, \ldots \} \cup (0)$

Since f is continuous at 0 and

$\lim_{z \to 0} f(z) \cdot f\left(\frac{z}{z+1}\right) = f^2(0) = 0$

Now $\left|\frac{z}{z+1}\right| < 1$ on $\frac{1}{2}\mathbb{D}$ so

$g(z) = f(z) \cdot f\left(\frac{z}{z+1}\right) = z$ makes

sense if we restrict the domain to

$\frac{1}{2}\mathbb{D}$. Now, $g'(z) = 1$

$= f'(z) f\left(\frac{z}{z+1}\right) + \frac{(z+1) - z}{(z+1)^2} f'\left(\frac{z}{z+1}\right)$

$\Rightarrow g'(0) = 1 = f'(0) f(0) + \frac{1}{(0+1)^2} f'(0)$

$\Rightarrow 1 = 0 \Rightarrow \rightarrow\leftarrow$ //

$\Rightarrow$ No such f can exist. //

(2) Let R be the radius of convergence. Then we have that

$$R = \lim_{n\to\infty} \cancel{\frac{|c_{n+1}\, n!|}{|c_n\, n|}} = \lim_{n\to\infty} \frac{|c_{n+1}(n+1)!|}{|c_n\, n!|}$$

$= \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| n.$ Since f is analytic on $\mathbb{D} \Rightarrow \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| \geq 1 \Rightarrow R = +\infty.$

Now, $c_n = \dfrac{1}{2\pi i} \displaystyle\int_{C_r} \frac{f(z)}{z^{n+1}}\, dz$ where

C_r is a circle centered at the origin of radius r, $r \in (0,1)$. Then we have that

$$|c_n| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(re^{it})|\, r}{r^{n+1}}\, dt$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{r^n}\, dt = \frac{1}{r^n} \to 1 \text{ as } r \to 1$$

$$\Rightarrow |c_n| \leq 1 \Rightarrow |F(z)| = \left|\sum_{n=0}^{\infty} \frac{c_n z^n}{n!}\right|$$

$$\leq \sum_{n=0}^{\infty} \frac{|c_n||z|^n}{n!} \leq e^{|z|}. //$$

(3) f is non-vanishing on $\mathbb{D} \Rightarrow \frac{1}{f}$ is analytic on $\mathbb{D} \Rightarrow$ by the maximum modulus principle, $\frac{1}{f}$ attains its max on $\partial\mathbb{D}$ (Since $|f| = 1$ on $\partial\mathbb{D}$ + f is continuous on $\overline{\mathbb{D}} \Rightarrow \frac{1}{f}$ is)

So $\frac{1}{f}$ has max 1. But, maximizing $\frac{1}{f}$ is equivalent to minimizing $f \Rightarrow$ $|f| \equiv 1$ on $\overline{\mathbb{D}}$. This means $f(\mathbb{D}) \subset S^1$. But, no subset (other than $\emptyset$) of S^1 is open in $\mathbb{C}$ $\Rightarrow f$ is not open $\Rightarrow$ by the open mapping theorem f is constant $\Rightarrow f = e^{i\theta}$. //

(b) It is easy to see must only have a finite number of zeros, say $z_1, \ldots, z_n$ on $\mathbb{D}$. ~~Define~~

$$\cancel{g(z) = \alpha \frac{z-2}{1-2z} \cdot \frac{z-z_1}{1-\bar{z}_1 z} \cdots \frac{z-z_n}{1-\bar{z}_n z}}$$

Note that

$$f(z) = g(z) \frac{z-2}{1-2z} \cdot \frac{z-z_1}{1-\bar{z}_1 z} \cdots \frac{z-z_n}{1-\bar{z}_n z}$$

~~This is clear since f has a simple pole at $z=1/2$, $z=2$ is~~ Where $g(z)$ is analytic on $\mathbb{D}$ with no zeros in $\mathbb{D}$.

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Then, $g(z) = \cancel{\frac{f(z)(1-2z)}{z-2}} \frac{z-z_1}{1-\bar{z}_1 z} \cdots$

$$\frac{f(z)(1-2z)}{z-2} \frac{1-\bar{z}_1 z}{z-z_1} \cdots \frac{1-\bar{z}_n z}{z-z_n},$$

$\Rightarrow$ $\frac{1}{g}$ is analytic on $\mathbb{D}$ $\Rightarrow$

g attains max and min on $\partial\mathbb{D}$

$\Rightarrow$ g constant. But, on $\partial\mathbb{D} \ni z$

$$|f(z)| = |g(z)| \left|\frac{z-2}{1-2z}\right| \left|\frac{z-z_1}{1-\bar{z}_1 z}\right| \cdots \left|\frac{z-z_n}{1-\bar{z}_n z}\right|$$

$$= |g(z)| \left|\frac{z-2}{\bar{z}-2}\right| \left|\frac{1-2\bar{z}}{1-2z}\right| \left|\frac{1-z_1\bar{z}}{1-\bar{z}_1 z}\right| \cdots \left|\frac{1-z_n\bar{z}}{1-\bar{z}_n z}\right|$$

$$= |g(z)| = 1 \Rightarrow g = e^{i\theta}$$

$$\Rightarrow f(z) = e^{i\theta} \frac{z-2}{1-2z} \cdot \frac{z-z_1}{1-\bar{z}_1 z} \cdots \frac{z-z_n}{1-\bar{z}_n z}. //$$

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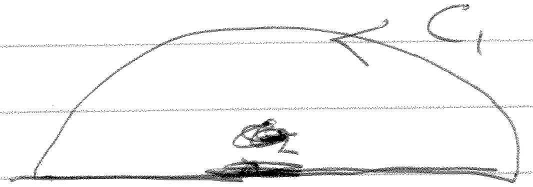
(4) Evaluate $\displaystyle\lim_{N\to+\infty}\int_{-N}^{N}\frac{x\sin x}{x^2+1}\,dx$

Consider $f(z)=\dfrac{ze^{iz}}{z^2+1}$ and

the Contour C



Then, we have

$$\cancel{\int_C f(z)\,dz = 2\pi i\,\frac{(i)\sin(i)}{2i}}$$

$$\cancel{= \int_{C_1} f(z)\,dz \to 0 + \int_{-\infty}^{\infty}\frac{x\sin x}{x^2+1}\,dx}$$

$$\cancel{\Rightarrow i\pi\sin(i)}$$

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$$\int_C f(z)\,dz = \frac{2\pi i (i) e^{-1}}{2i} = \frac{i\pi}{e}$$

$$= \underbrace{\int_{C_1} f(z)\,dz}_{\to 0} + \int_{-\infty}^{\infty} \frac{x(\cos x + i\sin x)}{x^2+1}\,dx$$

$$\Rightarrow \frac{\pi}{e} = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}\,dx. \quad //$$

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Saturday Jan 7, 2012

(1) Extend $f : [a, b] \mapsto \mathbb{R}$ to $f : [a, \infty) \mapsto \mathbb{R}$

by defining $f(x) = f(b)$, $x \geq b$.

Then, Let $f_n(x) = n\left[f(x + 1/n) - f(x)\right]$

$\lim_{n \to \infty} f_n(x) = f'(x) \ \forall x$

Each of the f_n is a measurable function.

The ptwise limit of a sequence of measurable functions is measurable

$\Rightarrow f'(x)$ is measurable //

(2) Note, $p\int_0^\infty t^{p-1}\mu\{x \mid |f(x)| > t\}\,dt$

$$= p\int_0^\infty \int_X t^{p-1} \mathbb{1}_{\{x \mid |f(x)| > t\}}\,dx\,dt \quad (\Delta)$$

Now, $t^{p-1}\mathbb{1}_{\{x \mid |f(x)|>t\}}$ is non-negative and measurable on $(0,\infty)\times X \Longrightarrow$ by Fubini's Thm for non-negative measurable fns we have (Δ)

$$= p\int_X \int_0^\infty t^{p-1}\mathbb{1}_{\{x \mid |f(x)|>t\}}\,dt\,dx$$

$$= p\int_X \int_0^{|f(x)|} t^{p-1}\,dt\,dx$$

$$= \int_X |f(x)|^p\,dx < \infty \text{ since } f \in L_p$$

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Thus, $p\int_0^\infty t^{p-1}\mu\{x \mid |f(x)|>t\}\,dt<\infty$

$$\Rightarrow p\underbrace{\int_s^{2s} t^{p-1}\mu\{x \mid |f(x)|>t\}\,dt}_{(\Delta)} \mapsto 0$$

as $s>0 \mapsto 0$.

But, $(\Delta) \le p(2s)^{p-1}\mu\{x \mid |f(x)|>s\}\,[s]$

$= p2^{p-1}s^p\mu\{x \mid |f(x)|>s\} \mapsto 0$

as $s \mapsto 0 \Longleftrightarrow s^p\mu\{x \mid |f(x)|>s\} \mapsto$

0 as $s \mapsto 0$.

(3) Note, $|g_n(x) - f(x)|$

$$= \left| n \int_{k/n}^{k+1/n} [f(y) - f(x)]\,dy \right|$$

$$= \left| \frac{1}{|I_n|} \int_{I_n} [f(y) - f(x)]\,dy \right|$$

$$\leq \frac{1}{|I_n|} \int_{I_n} |f(y) - f(x)|\,dy$$

The Lebesgue differentiation says since $f \in L_1$ + I_n are intervals about $x \mapsto 0$ as $n \mapsto \infty \Rightarrow \lim_{n \mapsto \infty} = 0$ for a.e. $X \Rightarrow g_n \mapsto f$ a.e.//

(4) $\mu(X) < \infty$, $f_n \to f$ a.e.

Also, $\lim_{n\to\infty} \int_X |f_n| d\mu \geq \int_X |f| d\mu$

$< \infty$ since $\int_X |f_n| d\mu < \sqrt{\mu(X)}$ $\forall n$

$\Rightarrow |f| < \infty$ a.e. Furthermore,

$\left|\int_E f_n \, d\mu\right| \leq \int_E |f_n| d\mu \leq \sqrt{\mu(E)}$

So, given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon^2}{4}$

and the $\forall\ |E| < \delta$ we have

$\left|\int_E f_n \, d\mu\right| < \varepsilon \Rightarrow (f_n)$ uniformly integrable $\Rightarrow$ We May invoke Vitali Converges theorm which Proves the Claim. //

(5) Let $(q_n)_{n=0}^{\infty}$ be an enumeration of the rational numbers. Consider the function

$$f(x) = \sum_{n=0}^{\infty} 2^{-n} (x-q_n)^{-1/2} \mathbb{1}_{[q_n, 1+q_n]}(x)$$

Then, by Minkowski inequality, we have $\|f\|_1 \le \sum_{n=0}^{\infty} 2^{-n} \|(x-q_n)^{-1/2} \mathbb{1}_{[q_n, 1+q_n]}\|_1$

$$= \sum_{n=0}^{\infty} 2^{1-n} = 4 < \infty.$$ However, we have on the other hand that

$$\|f\|_2 \overset{\text{Minkowski}}{\le} \sum_{n=0}^{\infty} 2^{-n} \|(x-q_n)^{-1/2} \mathbb{1}_{[q_n, 1+q_n]}\|_2$$

$= \infty$ since $\|(x-q_n)^{-1/2} \mathbb{1}_{[q_n, 1+q_n]}\|_2 = \infty$ $\forall n.$ //