RAMIFICATION IN THE COHOMOLOGY OF ALGEBRAIC SURFACES

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Abstract. Let $K$ be a field of characteristic zero, which is complete with respect to a discrete valuation, and whose residue field is perfect of an odd positive characteristic. We study the ramification in the cohomology of a smooth proper surface $X$ defined over $K$, under the assumption that $X$ admits an integral model $\mathcal{X}$ whose special fibre has at worst ordinary double points. Our main result is a formula that expresses the ramification in the cohomology of $X$ in terms of a numerical invariant of $\mathcal{X}$. It gives rise to a refined formula when $X$ is a K3 surface with a lattice-polarization. As an application, we determine the action of the inertia group on the primitive cohomology of certain lattice-polarized K3 surfaces parametrized by an open subset of the projective plane.

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1. Introduction

Let $K$ be a field of characteristic zero which is complete with respect to a discrete valuation. Assume that its residue field $k$ is perfect of characteristic $p > 0$. We fix an algebraic closure $\overline{K}$ of $K$, and let $G_K$ be the Galois group $\text{Gal}(\overline{K}/K)$. There is a short exact sequence

$$1 \rightarrow I_K \rightarrow G_K \rightarrow G_k \rightarrow 1$$

where $I_K$, called the inertia subgroup of $G_K$, is the largest subgroup of $G_K$ which acts trivially on the residue field of $K$. The quotient $G_k$ is naturally identified with the absolute Galois group of $k$.

When $X/K$ is an algebraic variety and $\ell$ is a prime, the étale cohomology group $H^1(X/\overline{K}, \mathbb{Q}_\ell)$ is naturally equipped with a $G_K$-action. The analysis of this $G_K$-action is of a great interest, which is illustrated by the celebrated theorem of Serre-Tate and Neron-Ogg-Shafarevich.

**Theorem** (Serre-Tate, Neron-Ogg-Shafarevich). Let $A/K$ be an abelian variety and $\ell$ be any fixed prime different from $p$. The action of $I_K$ on $H^1(A/\overline{K}, \mathbb{Q}_\ell)$ is trivial if and only if $A$ has good reduction.

When $X$ is a smooth proper variety over $K$, we say $X$ has good reduction if there is a smooth proper map $X \rightarrow \text{Spec}(R)$, where $R$ is the valuation ring of $K$, whose generic fibre is isomorphic to $X$. The above theorem is established for elliptic curves by Neron-Ogg-Shafarevich, and for abelian varieties by Serre-Tate.

More generally, if $X$ is a smooth proper variety which has good reduction then all of its $\ell$-adic étale cohomology groups with $\ell \neq p$ are unramified. However, the converse is not true, to which curves of genus at least two are counterexamples. The implication of the form

$$H^i(X/\overline{K}, \mathbb{Q}_\ell) \text{ is unramified for all } i \implies X \text{ has good reduction}$$

being false, one naturally considers its variations.

First, one may strengthen the assumption so that the good reduction of $X$ can be deduced. This approach is taken by T. Oda [7], where it was established that if $X$ is a curve then the triviality the outer action of $I_K$ on suitable quotients of the étale fundamental group of $X$ implies that $X$ has good reduction. Its $p$-adic analogue has been established by Andreatta-Iovita-Kim [1].

Second, one may consider a particular type of varieties, and try to show the converse. For example, Liedtke-Matsumoto [4] considered a K3 surface which admits a potential semistable model, and established that the unramifiedness of $H^2(X/\overline{K}, \mathbb{Q}_\ell)$ implies the good reduction of $X$ over an unramified extension of $K$.

Third, one may weaken the conclusion, by aiming something weaker than good reduction. A prominent result in this direction is due to Rapoport-Zink [8]. They start with an integral model $X/R$ which is semistable, which gives rise to a filtration, called the weight filtration, of $H^i(X/\overline{K}, \mathbb{Q}_\ell)$. The weight-monodromy
conjecture says that $I_K$ acts trivially on the graded quotients of the weight filtration. Conditionally on the weight-monodromy conjecture, the cohomology $H^i(X/K, \mathbb{Q}_\ell)$ is unramified if and only if the weigh filtration on it has at most one non-trivial graded quotient. Although the weight-monodromy conjecture is not fully known, it is established for surfaces by Rapoport-Zink [8] and for set theoretic complete intersections in a smooth projective toric variety by Scholze [9]. Their $p$-adic analogues also exist, notably due to Hyodo-Kato [3] and Mokrane [5].

In the present article, we consider a combination of the second and third variations, in that the relation between the $I_K$-action on $H^2(X/K, \mathbb{Q}_\ell)$ and properties of integral models of a smooth surface $X$ is investigated. Instead of starting from a semistable model, we will assume the existence of an integral model $X$ of $X$ such that the special fibre $X_k$ of $X$ has at worst ordinary double points. We further assume that $p \nmid 2\ell$.

Our aim is rather modest in that we will start from a smooth surface $X/K$ with an integral model $X$ whose special fibre $X_k$ has only ordinary double points as its singularities. The ordinary double point is the singularity at the origin of the affine equation $z_0z_1 + z_2^2 = 0$, which is arguably the mildest singularity that can occur in a surface.

To each singular point $x$ of $X_k$, we will define a numerical invariant $n_x$, a positive integer, which is determined by the formal neighborhood of $x$ in $X$. We will define an invariant

$$g(X) \in \mathbb{Z}_{\geq 0},$$

a nonnegative integer, to be the number of singular points $x$ such that $n_x$ is odd. Our main theorem relates $g(X)$ to the $I_K$-action on $H^2(X/K, \mathbb{Q}_\ell)$.

Theorem. The codimension of $H^2(X/K, \mathbb{Q}_\ell)^{I_K}$ in $H^2(X/K, \mathbb{Q}_\ell)$ is equal to $g(X)$. Furthermore, the quotient $H^2(X/K, \mathbb{Q}_\ell)/H^2(X/K, \mathbb{Q}_\ell)^{I_K}$ is a direct sum of ramified quadratic characters.

While the above theorem regards a smooth surface $X$, it can be used to prove a refined result when $X$ is a K3 surface equipped with an additional structure, called a lattice-polarization. The notion of a lattice-polarization is due to Nikulin [6], where K3 surfaces over the field of complex numbers are mainly considered. This notion was extended in [2] to families of K3 surfaces over a possibly non-simply connected base.

For us, a lattice-polarized K3 surface over $K$, or a K3 surfaces polarized by a lattice, is a triple $(X, \Lambda, \lambda)$ of a K3 surface $X$, a subgroup $\Lambda \subset \text{Pic}(X/K)$ of the Picard group, and an element $\lambda \in \Lambda$, subject to some natural conditions. In particular, we allow $G_K$ to act non-trivially on $\Lambda$.

Given a lattice-polarized K3 surface $(X, \Lambda, \lambda)$, it is natural to consider the primitive cohomology $P_\ell(X)$ of it. By definition, $P_\ell(X)$ is the cokernel of the $\ell$-adic Chern character from $\Lambda \otimes \mathbb{Q}_\ell$ into $H^2(X/K, \mathbb{Q}_\ell(1))$, where $\mathbb{Q}_\ell(1)$ is the Tate twist of $\mathbb{Q}_\ell$. Since we are mainly interested in the action of $I_K$, we may assume
that $k$ is algebraically closed, in which case the Tate twist has no effect. In such a case, we will regard $P_\ell(X)$ as a quotient of $H^2(X/K, \mathbb{Q}_\ell)$.

Under the assumption that $k$ is algebraically closed, there is a unique nontrivial quadratic character $\psi$ of $G_K$, which defines the unique ramified quadratic extension $L/K$. For a lattice-polarized K3 surface $(X, \Lambda, \lambda)$, we define a nonnegative integer

$$a(X, \Lambda)$$

(1.0.1) to be the dimension of the $\psi$-isotypic component of $\Lambda \otimes \mathbb{Q}$. If the choice of $\Lambda$ is clear from the context, we simply write $a(X)$ instead of $a(X, \Lambda)$. As we will show later, we have $g(X) \geq a(X)$. Our main theorem gives rise to the following.

**Theorem.** Let $(X, \Lambda, \lambda)$ be a lattice-polarized K3 surface with primitive cohomology $P_\ell(X)$. Then the codimension of $P_\ell(X)^{1_K}$ in $P_\ell(X)$ is equal to $g(X) - a(X, \Lambda)$. Furthermore, the quotient $P_\ell(X)/P_\ell(X)^{1_K}$ is $\psi$-isotypic.

We sketch its proof. The crucial ingredients are the Rapoport-Zink spectral sequence and the weight-monodromy theorem. In order to apply these tools, we first construct a (potential) semistable model $X^{\text{ss}}$ of $X$. More precisely, we construct a particular semistable model $X^{\text{ss}}/R_L$ of $X/L$, where $L/K$ is the unique quadratic extension of $K$ with valuation ring $R_L$. This semistable model further enjoys the property that the Galois action of $\text{Gal}(L/K)$ on $X/L$ extends to $X/R_L$ in a way that it is compatible with its action on $R_L$. The analysis of the $\text{Gal}(L/K)$-action on the Rapoport-Zink spectral spectral sequence for $X^{\text{ss}}/R_L$ will yield the above theorem.

The techniques employed in the above sketch of proof are rather similar to those in [4]. The authors of [4] investigated K3 surfaces $X/K$ such that it has a potential semistable reduction over some Galois extension $K'/K$. Although such a smooth model $X$ of $X/K'$ may not admit an action of $\text{Gal}(K'/K)$, they showed that there is always a birational modification $X''$ of $X$, such that $\text{Gal}(K'/K)$ acts on $X''$, and that $X''_k$ has at worst rational double points.

Note that the ordinary double point is the simplest kind among the rational double points, so in terms of the singularities involved our scope is more limited. We have a relative advantage to be able to construct an explicit semistable model over $L$ with Galois action. In fact, this advantage is crucial for the application, which is discussed below.

As an application, we will consider a two dimensional family of lattice-polarized K3 surfaces. These K3 surfaces carry polarizations by a lattice of rank 14, and we will apply our theorem to them. The parameter space for this family is an open subset $U$ of $\mathbb{P}^2$. For $y \in U(K)$, we will have a lattice polarized K3 surface $X_y$.

Under the assumption that the reduction of $y$ lies at a particular divisor $\Delta$ at infinity, we will construct an integral model $X_y$ whose special fibre has at worst ordinary double points. Furthermore, we will compute both of the invariants $g(X_y)$ and $a(X_y)$ in terms of the intersection multiplicity of $y$ and $\Delta$ as $R$-schemes. As
a consequence of our theorem we will be able to determine the action of $I_K$ on $P_I(X_y)$.

We outline the contents of the paper. In §2, we review lattice-polarized K3 surfaces and their primitive cohomology. In §3, we state the main result without proof, which is postponed to §4. In §5, we give an application of the main result.

What is not attempted in the present article includes two natural questions. One is whether our method can be generalized to rational double points. The other is to find an approach which does not rely on the assumption $p > 2$, whence it works for all residue characteristics.

2. THE PRIMITIVE COHOMOLOGY OF LATTICE-POLARIZED K3-surfaces

In this section we define a lattice-polarized K3 surface and its primitive cohomology.

2.1. Lattice-polarized K3 surfaces. Let $K$ be a field of characteristic zero. We fix an algebraic closure $\overline{K}$ of $K$, and let $G_K$ be the Galois group of $\overline{K}/K$. A lattice-polarized K3 surface over $K$ is a triple $(X, \Lambda, \lambda)$, where $X$ is a K3 surface over $K$, $\Lambda \subset \text{Pic}(X/K)$ is a subgroup of the Picard group, and $\lambda$ is an element of $\Lambda$, such that the following conditions are satisfied.

1. $\Lambda$ is $G_K$-stable.
2. $\lambda$ is $G_K$-invariant.
3. The self-intersection number of $\lambda$ is positive.

The self-intersection number of any element in $\text{Pic}(X/\overline{K})$ is even, so the self-intersection number of $\lambda$ is $2d$ for some positive integer $d$. The integer $2d$ is called the degree of the lattice-polarized K3 surface.

We often write simply $X$ to denote a lattice-polarized K3 surface when $\Lambda$ and $\lambda$ are clear from the context. Given a triple $(X, \Lambda, \lambda)$, the subgroup $\Lambda \subset \text{Pic}(X/\overline{K})$ is a lattice with respect to the intersection pairing. If the rank of $\Lambda$ is $r$, then the signature of this lattice is $(1, r - 1)$ by Hodge Index Theorem.

We defined a lattice polarized K3 surface as a triple $(X, \Lambda, \lambda)$, according to which $\Lambda$ is a subgroup of $\text{Pic}(X/\overline{K})$. An alternative way to define a lattice-polarized K3 surface is to consider an abstract lattice $\Lambda$ and introduce the notion of $\Lambda$-polarizations on a K3 surface. Suppose $\Lambda$ is an abstract lattice of rank $r$ and signature $(1, r - 1)$ with a marked element $\lambda$ of positive length. Then a $\Lambda$-polarization on $X$ is an embedding $j: \Lambda \to \text{Pic}(X/\overline{K})$ such that $j(\Lambda)$ is $G_K$-stable and $j(\lambda)$ is $G_K$-invariant. If we identify $\Lambda = j(\Lambda)$, then we get a lattice-polarized K3 surface $(X, \Lambda, \lambda)$.

2.2. Primitive cohomology. For any prime number $\ell$, let

\begin{equation}
\text{ch}_\ell: \text{Pic}(X/\overline{K}) \longrightarrow H^2(X/\overline{K}, \mathbb{Q}_\ell(1))
\end{equation}

be the $\ell$-adic Chern class map into the second $\ell$-adic etale cohomology group of $X/\overline{K}$. Note that the $\text{ch}_\ell$ is injective when $X$ is a K3 surface. The ($\ell$-adic)
primitive cohomology of a lattice-polarized K3 surface \((X, \Lambda, \lambda)\) is defined to be

\[(2.2.2) \quad P_\ell(X) := \frac{H^2(X/K, \mathbb{Q}_\ell(1))}{\text{ch}_\ell(\Lambda) \otimes \mathbb{Q}_\ell} \]

where \(\Lambda\) and \(\lambda\) are suppressed from the notation.

3. The statement of the main theorem

Let \(p\) be an odd prime. We assume that \(K\) is a field which is complete with a discrete valuation, with residue characteristic \(p\). We fix a uniformizer \(\pi\) of the valuation ring \(R\) of \(K\), generating the maximal ideal \(\mathfrak{p}\) of \(R\). The residue field of \(R\) is denoted by \(k\), assumed to be perfect of characteristic \(p\).

3.1. Integral models. By an integral model of a smooth proper surface \(X\), we mean a proper flat \(R\)-scheme \(\mathcal{X}\) equipped with an isomorphism

\[ \mathcal{X} \times \text{Spec } R \cong X. \]

The choice of a particular isomorphism \(j\) will be unimportant to us, and we will always identify \(\mathcal{X} \times \text{Spec } R \cong X\).

Throughout, we assume that \(X/K\) admits an integral model \(\mathcal{X}\) such that \(\mathcal{X}_k\) is irreducible with at worst ordinary double points. That is to say, the formal neighborhood of a singular point of \(\mathcal{X}_k\) is isomorphic to \(z_0z_1 + z_2^2 = 0\) over \(\bar{k}\).

3.2. A numerical invariant \(g(\mathcal{X})\). We would like to define a numerical invariant of an integral model \(\mathcal{X}\) of a smooth proper surface \(X/K\). In fact, it is determined by \(X/\hat{K}\) where \(\hat{K}\) is the maximal unramified extension of \(K\). We will assume, in this subsection, that \(k\) is separably closed, and that \(K = \hat{K}\).

For every singular point \(x\) of \(\mathcal{X}_k\), the formal neighborhood of \(x\) in \(\mathcal{X}\) has the coordinate ring isomorphic to

\[(3.2.1) \quad \frac{R[[z_0, z_1, z_2]]}{z_2^2 + z_0z_1 + r}\]

with some \(r \in \mathfrak{p}\). Because \(X\) is smooth, \(r\) is unequal to zero. Let \(n_x\) be the \(\pi\)-adic order of \(r\), so \(r \in \mathfrak{p}^{n_x}\) and \(r \not\in \mathfrak{p}^{n_x+1}\).

**Proposition 3.2.2.** The number \(n_x\) is well-defined.

**Proof.** Consider the sheaf \(\Omega^1_{\mathcal{X}/R}\) of relative differentials of \(\mathcal{X} \rightarrow \text{Spec } R\). Let \(I_x\) be the stalk of the zeroth fitting ideal of \(\Omega^1_{\mathcal{X}/R}\). If \(\mathcal{O}_x\) is the local ring of germs of functions near \(x\), then \(n_x\) is equal to the length of \(\mathcal{O}_x/I_x\) as an \(\mathcal{O}_x\)-module. This provides an intrinsic definition of \(n_x\). \(\square\)

**Definition 3.2.3.** Let \(X\) be a smooth surface with an integral model \(\mathcal{X}\) such that \(X_k\) has at worst ordinary double points. We define \(g(\mathcal{X})\) to be the number of singular points \(x\) in \(X_k\) such that \(n_x\) is odd.

Without assuming \(K = \hat{K}\), we simply define \(g(\mathcal{X})\) to be \(g(\mathcal{X}/\hat{R})\), where \(\hat{R}\) is the valuation ring of \(\hat{K}\).
3.3. A numerical invariant $a(X)$. Here we assume that $X$ is a K3 surface equipped with a polarization by a lattice $\Lambda$. We would like to introduce a numerical invariant of a lattice polarized K3 surfaces. This invariant will be determined by $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ as a $\text{Gal}(\overline{K}/\hat{K})$-module, so we assume $K = \hat{K}$.

Let $\psi: G_K \to \{\pm 1\}$ be a nontrivial quadratic character. Since $K = \hat{K}$, such a character is unique, and must be ramified.

**Definition 3.3.1.** Define $a(X, \Lambda)$ to be the $\mathbb{Q}$-dimension of the $\psi$-isotypic component of $\Lambda_{\mathbb{Q}}$. When the polarization given to $X$ is clear from the context, we simply write $a(X)$ for $a(X, \Lambda)$.

3.4. The statement of the main theorem and its consequences. Here we state our main theorem and its consequences, whose proofs will be given in §4. We keep the assumptions of the previous subsection. In particular, $\overline{K} = \hat{K}$ and $X$ is an integral model of a smooth surface $X/K$, such that $X_k$ has at worst ordinary double points. Our main theorem is the following.

**Theorem 3.4.1.** Let $X/K$ be a smooth algebraic surface. The codimension of $H^2(X/\overline{K}, \mathbb{Q}_\ell)^{I_K}$ in $H^2(X/\overline{K}, \mathbb{Q}_\ell)$ is equal to $g(X)$. Furthermore, the quotient $H^2(X/\overline{K}, \mathbb{Q}_\ell)/H^2(X/\overline{K}, \mathbb{Q}_\ell)^{I_K}$ is $\psi$-isotypic.

As a corollary, we obtain:-

**Corollary 3.4.2.** Suppose that $\mathcal{X}$ and $\mathcal{X}'$ are two integral models of a smooth surface $X/K$ such that their special fibres have at worst ordinary double points. Then we have $g(\mathcal{X}) = g(\mathcal{X}')$.

If $X$ is a K3 surface carrying a lattice polarization by $\Lambda$, then there is a refinement of the above formula involving $a(X) = a(X, \Lambda)$. We begin with a proposition.

**Proposition 3.4.3.** We have $g(\mathcal{X}) \geq a(X)$.

Our main theorem interprets the non-negative integer $g(\mathcal{X}) - a(X)$ in terms of the ramification in the primitive cohomology $P_{\ell}(X)$.

**Theorem 3.4.4.** Let $\ell \nmid p$ be any prime. Then the codimension of $P_{\ell}(X)^{I_K}$ in $P_{\ell}(X)$ is equal to $g(\mathcal{X}) - a(X, \Lambda)$. Furthermore, the quotient $P_{\ell}(X)/P_{\ell}(X)^{I_K}$ is $\psi$-isotypic.

As a special case of our theorem, we obtain a numerical characterization for the unramifiedness of $P_{\ell}(X)$.

**Corollary 3.4.5.** Let $X/K$ be a K3 surface. The followings are equivalent.

- We have $g(\mathcal{X}) = a(X)$.
- $P_{\ell}(X)$ is unramified for some $\ell \nmid p$.
- $P_{\ell}(X)$ is unramified for all $\ell \nmid p$. 
4. THE PROOF OF THE MAIN THEOREM

We will prove Theorem 3.4.1 in this section, and deduce its consequences. We will first construct a (potential) semistable model, from which the theorem will follow by invoking the Rapoport-Zink spectral sequence and the weight-monodromy theorem.

4.1. Constructing a good semistable model. Recall that \( \pi \) is a fixed uniformizer of \( R \). Consider the equation

\[
z_2^2 + z_0z_1 + \pi^n = 0 \tag{4.1.1}
\]

for some positive integer \( n \), and let \( Z_n/R \) be the formal neighborhood of the singular point in the reduction of (4.1.1). Our aim is to describe a potential semistable model of \( Z_n \). In fact, our semistable model will be defined over \( L \), which is the field cut out by the unique quadratic character \( \psi \) of \( G_K \). Our semistable model will have an additional feature that the Galois group \( \text{Gal}(L/K) \) acts on it in a way that the action extends the natural Galois action on the generic fibre, and that the action is compatible with its action on \( R_L \), the valuation ring of \( L \).

We denote respectively the generic and the special fibre of \( Z_n \) by \( Z_n \) and \( Z_n \). Let \( x \in Z_n(k) \) be the singular point with coordinate \((0, 0, 0)\). Let \( \text{Bl}_x Z_n \) be the blowup of \( Z_n \) at \( x \).

**Proposition 4.1.2.** Suppose that \( n > 2 \). Then the special fibre of \( \text{Bl}_x Z_n \) is reduced and has two irreducible components which intersect transversally. One component is the minimal resolution of \( Z_n \). The other component has an ordinary double point whose formal neighborhood in \( \text{Bl}_x Z_n \) is isomorphic to \( Z_n - 2 \).

**Proof.** Let

\[
A_n = R[z_0, z_1, z_2]/(z_2^2 + z_0z_1 + \pi^n) \tag{4.1.3}
\]

and let \( I_n \) be the ideal generated by \( z_0, z_1, z_2, \) and \( \pi \). The Rees algebra \( A_n' \) of \( I_n \) has equation

\[
A_n' = A_n[w_0, w_1, w_2, w_3]/I_n' \tag{4.1.4}
\]

where \( I_n' \) is generated by the determinants of \( 2 \times 2 \) minors of the matrix

\[
\begin{bmatrix}
z_0 & z_1 & z_2 & \pi \\
w_0 & w_1 & w_2 & w_3
\end{bmatrix}
\]

together with

\[
z_0w_1 + z_2w_2 + \pi^{n-1}w_3, \quad w_0w_1 + w_2^2 + \pi^{n-2}w_3^2. \tag{4.1.5}
\]

Its special fibre has two components, that are respectively defined by \( z_0 = z_1 = z_2 = 0 \) and \( w_3 = 0 \). The chart with \( w_3 \neq 0 \) has the coordinate ring

\[
R[w_0, w_1, w_2]/(w_0w_1 + w_2^2 + \pi^{n-2}) \tag{4.1.6}
\]
which has a unique singular point in the special fibre whose neighborhood is $\mathcal{Z}_{n-2}$. The chart with $w_2 \neq 0$ has the coordinate ring
\[(4.1.8) \quad \frac{R[w_0, w_1, w_3, z]}{(w_3z_2 + \pi, w_0w_1 + w_3^2\pi^{n-2} - 1)}\]
which is semistable. The chart with $w_1 \neq 0$ has the coordinate ring
\[(4.1.9) \quad \frac{R[w_0, w_3, z_2]}{(w_3z_1 + \pi)},\]
and the chart with $w_0 \neq 0$ has the coordinate ring
\[(4.1.10) \quad \frac{R[w_1, w_3, z_0]}{(w_3z_0 + \pi)},\]
both of which are semistable. □

**Proposition 4.1.11.** Suppose that $n = 2$. Then the special fibre of $\text{Bl}_x\mathcal{Z}_n$ is reduced and has two irreducible components which intersect transversally. One component is the minimal resolution of $\mathcal{Z}_n$. The other component is isomorphic to a smooth quadric surface.

**Proof.** We first note that the computation of Rees algebra in the proof of Proposition 4.1.2 is valid for $n = 2$. In particular, $\text{Bl}_x\mathcal{Z}_n$ is represented by (4.1.14) with $n = 2$, and its affine charts are given by (4.1.7), (4.1.8), (4.1.9), and (4.1.10). In particular, the component of the special fibre defined by $z_0 = z_1 = z_2 = 0$ has the (homogeneous) coordinate ring
\[(4.1.12) \quad \frac{k[w_0, w_1, w_2, w_3]}{(w_0w_1 + w_2^2 + w_3^2)}\]
which is a smooth quadric surface. □

Proposition 4.1.2 shows that a semistable model for $\mathcal{Z}_n$ be obtained from that of $\mathcal{Z}_{n-2}$. Proposition 4.1.11 shows that for even $n$, a semistable model of $\mathcal{Z}_n$ can be found by iterated blowups.

When $n$ is odd, it suffices to obtain a semistable model for $\mathcal{Z}_1$, for which we pass to the quadratic extension $L$ of $K$. Recall that $R_L$ is the valuation ring of $L$.

**Proposition 4.1.13.** Suppose that $n = 1$. The special fibre of $\text{Bl}_x(\mathcal{Z}_1/R_L)$ is reduced and has two irreducible components which intersect transversally. The special fibre of $\text{Bl}_x(\mathcal{Z}_1/R_L)$ is decomposed into $\mathcal{Z}_{1,0} \cup \mathcal{Z}_{1,1}$ where $\mathcal{Z}_{1,0}$ is the minimal resolution of $\mathcal{Z}_1$ and $\mathcal{Z}_{1,1}$ is a smooth quadric surface defined by (4.1.12). The Galois group $\text{Gal}(L/K)$ acts on $\text{Bl}_x\mathcal{Z}_1/R_L$ in the manner that it acts trivially on $\mathcal{Z}_{1,0}$ and acts on $\mathcal{Z}_{1,1}$ by sending $w_3$ to $-w_3$.

**Proof.** Without loss of generality, we choose $\pi_L$ to be a uniformizer of $L$ such that $\pi_L^2 = \pi$. Then, $\mathcal{Z}_1/R_L$ has the form
\[(4.1.14) \quad z_2^2 + z_0z_1 + \pi_L^2 = 0\]
to which Proposition 4.1.11 applies. Moreover, the center of blowup is defined by the ideal \((z_0, z_1, z_2, \pi_L)\), which is stable under the action of \(\text{Gal}(L/K)\). Hence the Galois group \(\text{Gal}(L/K)\) acts on \(\text{Bl}_L (Z_1/R_L)\) in a way that it extends the Galois action on \(X/L\), and that it is compatible with its action on \(R_L\). By Proposition 4.1.11, \(Z_{1,1}\) is defined by
\[
(w_2^2 + w_0 w_1 + w_3^2 = 0) \quad (4.1.15)
\]
where the variable \(w_i\) for \(i = 0, 1, 2\) corresponds to \(z_i\) and \(w_3\) corresponds to \(\pi_L\).

The unique nontrivial element of \(\text{Gal}(L/K)\) fixes \(w_i\) for \(i = 0, 1, 2\), and sends \(w_3\) to \(-w_3\).

Now we are ready to construct semistable models \(Z_{n, \text{ss}}\) of \(Z_n\). We consider even and odd \(n\)'s separately. Let \(m\) be any positive integer and let \(n = 2m\). We obtain \(Z_{n, \text{ss}}\) by iteratively blowing up the ordinary double point in the singular fibre \(m\)-times. When \(n = 2m - 1\), then we iteratively blow up the ordinary double point in the special fibre \((m - 1)\)-times. After that, we base change it to \(R_L\), and blow up the ordinary double point in the singular fibre. The following propositions describe the special fibres of \(Z_{n, \text{ss}}\).

**Proposition 4.1.16.** Let \(m\) be any positive integer and let \(n = 2m\). The semistable \(R\)-model \(Z_{n, \text{ss}}\) of \(Z_n\) has has \(m + 1\) components in its special fibre. One of them is the minimal resolution of \(Z_n\), which we denote by \(Z_{n,0}\). One can write
\[
Z_n = Z_{n,0} \cup Z_{n,1} \cup \cdots \cup Z_{n,m} \quad (4.1.17)
\]
where \(Z_{n,i}\) is the component introduced at the \(i\)-th blowup for each \(i \geq 1\). In particular, \(Z_{n,i}\) is isomorphic to a smooth quadric surface for each \(i \geq 1\), and the intersection of two adjacent components is isomorphic to a projective line.

**Proof.** It follows from applying Proposition 4.1.2 \((m - 1)\)-times and applying Proposition 4.1.11. \(\square\)

**Proposition 4.1.18.** Let \(m\) be any positive integer and let \(n = 2m - 1\). The semistable \(R_L\)-model \(Z_{n, \text{ss}}\) has has \(m + 1\) components in its special fibre. One of them is the minimal resolution of \(Z_n\), which we denote by \(Z_{n,0}\). One can write
\[
Z_n = Z_{n,0} \cup Z_{n,1} \cup \cdots \cup Z_{n,m} \quad (4.1.19)
\]
where \(Z_{n,i}\) is the component introduced at the \(i\)-th blowup for each \(i \geq 1\). In particular, \(Z_{n,i}\) is isomorphic to a smooth quadric surface for each \(i \geq 1\), and the intersection of two adjacent components is isomorphic to a projective line. For \(i - j \geq 2\), \(Z_{n,i} \cap Z_{n,j}\) is empty. For all \(i\), \(Z_{n,i}\) is stable under the action of \(\text{Gal}(L/K)\). If \(i < m\), the action is trivial. If \(i = m\), the action is nontrivial, which is equivalent to the action of \(\text{Gal}(L/K)\) on \(Z_{1,1}\) in Proposition 4.1.13.

**Proof.** Applying Proposition 4.1.2 \((m - 1)\)-times, we get \(Z_1\), to which we apply Proposition 4.1.11 after taking the base change to \(R_L\). \(\square\)
Now we return to the case of a smooth surface $X$ over $K$. For a given integral model $X$ with at worst ordinary double points in $X_k$, we define $X^{ss}$ to be the $R$-scheme obtained by applying the above construction to each singular point of $X_k$.

4.2. Rapoport-Zink spectral sequence. We review the Rapoport-Zink spectral sequence $E_{\ast\ast}$ associated to $X^{ss}$, with an emphasis on the terms that contribute to $H^2(X/K, \mathbb{Q}_\ell)$. Let $n$ be the number of irreducible components of $X^{ss}$, and let $V_i$ for $i = 1, 2, \cdots, n$ be its irreducible components. Note that if $i_1, i_2, i_3$ are distinct, then $V_{i_1} \cap V_{i_2} \cap V_{i_3} = \emptyset$. Also note that for all distinct pairs $(i, j)$, the cohomology of $V_i \cap V_j$ is concentrated in even degrees, because $V_i \cap V_j$ is either empty or isomorphic to the projective line. These two facts about $X^{ss}$ greatly simplify the spectral sequence. Indeed, the part of the spectral sequence which computes $H^2(X/K, \mathbb{Q}_\ell)$ is given by a 3-term sequence

$$\bigoplus_{1 \leq i < j \leq n} H^0(V_i \cap V_j) \xrightarrow{d} \bigoplus_{1 \leq i \leq n} H^2(V_i) \xrightarrow{d'} \bigoplus_{1 \leq i < j \leq n} H^2(V_i \cap V_j)$$

where cohomology of an empty space is regarded as zero. The three terms are respectively $E^{1,2}_{RZ}$, $E^{0,2}_{RZ}$, and $E^{1,2}_{RZ}$. The first differential $d$ is given by the map $H^0(V_i \cap V_j) \rightarrow H^2(V_i)$ which is the Gysin map if $i = \mu$ or $j = \mu$, and zero otherwise. Note that in this case the image of the Gysin map is generated by the cycle class of $V_i \cap V_j$. The second differential $d'$ is the sum of maps $H^2(V_\mu) \rightarrow H^2(V_i \cap V_j)$, which is the pullback if $i = \mu$, the negation of the pullback if $j = \mu$, and zero otherwise.

**Proposition 4.2.2.** We have a natural $\text{Gal}(\overline{K}/K)$-equivariant isomorphism

$$\text{Ker}(d')/\text{Im}(d) \cong H^2(X/\overline{K}, \mathbb{Q}_\ell).$$

Furthermore, the action of $\text{Gal}(\overline{K}/L)$ on $H^2(X/\overline{K}, \mathbb{Q}_\ell)$ is trivial.

**Proof.** The $\text{Gal}(\overline{K}/L)$-equivariant isomorphism is due to Rapoport-Zink. The triviality of the $\text{Gal}(\overline{K}/L)$-action follows from the weight-monodromy conjecture, which is known for surfaces by Rapoport-Zink [8].

Since the semistable model has an action of $\text{Gal}(L/K)$, it induces an action on the sequence (4.2.1). The functoriality of the Rapoport-Zink spectral sequence implies that this action induces the action of $\text{Gal}(L/K)$ on $H^2(X/\overline{K}, \mathbb{Q}_\ell)$ via the above isomorphism. \[\square\]

Now we would like to analyze the action of $\text{Gal}(L/K)$ on $H^2(X/\overline{K}, \mathbb{Q}_\ell)$. In order to do that, we compute the Galois action on $E^{i,2}_{RZ}$ for $i = -1, 0, 1$. Note that $\text{Gal}(\overline{K}/L)$ acts trivially on $E^{i,2}_{RZ}$ for all $i = -1, 0, 1$, and it remains to determine the action of $\text{Gal}(L/K)$.

**Proposition 4.2.4.** The action of $\text{Gal}(L/K)$ is trivial on $E^{-1,2}_{RZ}$ and $E^{1,2}_{RZ}$.

**Proof.** It follows from the observation that the action of $\text{Gal}(L/K)$ is trivial on $V_i \cap V_j$ for every $i, j$ with nonempty intersection. Indeed, if we let $V_i$ be defined by the
equation \( w_2^2 + w_0w_1 + w_3^2 = 0 \), on which \( \text{Gal}(L/K) \) acts by sending \( w_3 \) to \( -w_3 \).

Our construction of the semistable model shows that the subvariety \( V_i \cap V_j \subset V_i \) is defined by \( w_2^2 + w_0w_1 + w_3^2 = 0 \) and \( w_3 = 0 \), and it follows that the action of \( \text{Gal}(L/K) \) on \( V_i \cap V_j \) is trivial.

It remains to determine the action of \( \text{Gal}(L/K) \) on \( E_{RZ}^{0,2} \). Recall that \( \psi \) is the unique quadratic character of \( G_K \) which induces the isomorphism \( \text{Gal}(L/K) \xrightarrow{\sim} \{\pm 1\} \).

Proposition 4.2.5. The dimension of the \( \psi \)-isotypic component of \( E_{RZ}^{0,2} \) is equal to \( g(X) \).

Proof. Let \( x \) be an ordinary double point of \( X_k \) with odd \( n_x \). Let \( m \) be the positive integer with \( n_x = 2m - 1 \). By our construction of \( X^{ss}_k \), there is a natural map \( q : X^{ss}_k \to X_k \), \( q^{-1}(x) \) is the union of \( m \) smooth quadric surfaces. Among the \( m \) smooth quadric surfaces, there is a unique surface, say \( V_x \), on which \( \text{Gal}(L/K) \) nontrivially, and the action is described in Proposition 4.1.13. As a \( \text{Gal}(L/K) \)-representation, \( H^2(V_x) \) is isomorphic to \( 1 \oplus \psi \), where \( 1 \) is the trivial character. Therefore the dimension of the \( \psi \)-isotypic component of \( E_{RZ}^{0,2} \) is at least \( g(X) \).

On the other hand, if an irreducible component \( V_i \) of \( X^{ss}_x \) is not equal to \( V_x \) for some \( x \) with odd \( n_x \), then the action of \( \text{Gal}(L/K) \) on \( V_i \) is trivial. Hence the dimension of the \( \psi \)-isotypic component of \( E_{RZ}^{0,2} \) is exactly \( g(X) \).

We can finally connect \( g(X) \) to \( H^2(X/K, \mathbb{Q}_\ell) \).

Proposition 4.2.6. The dimension of the \( \psi \)-isotypic component of \( H^2(X/K, \mathbb{Q}_\ell) \) is equal to \( g(X) \).

Proof. It follows from combining Proposition 4.2.5 with Proposition 4.2.2 and Proposition 4.2.4.

4.3. Proofs. We are ready to give the postponed proofs. By Proposition 4.2.6, \( g(X) \) is the dimension of \( \psi \)-isotypic component of \( H^2(X/K, \mathbb{Q}_\ell) \). Note that it is also equal to the codimension of \( H^2(X/K, \mathbb{Q}_\ell)^{1\chi} \) in \( H^2(X/K, \mathbb{Q}_\ell) \) because the action of \( G_K \) on \( H^2(X/K, \mathbb{Q}_\ell) \) factors through \( \text{Gal}(L/K) \) by Proposition 4.2.2. This yields Theorem 3.4.1.

Proposition 3.4.3, which is the inequality \( g(X) \geq a(X) \), follows from the injectivity of the \( G_K \)-equivariant Chern class map (2.2.1). Theorem 3.4.4 also follows directly from it, when combined with Theorem 3.4.1.

Corollary 3.4.2 and Corollary 3.4.5 follow from both of \( g(X) \) of \( a(X) \) being independent of \( \ell \).

5. Application

In this section, we apply Theorem 3.4.4 to determine the action of the inertia group on the primitive cohomology of some lattice-polarized K3 surfaces over \( K \).
The K3 surfaces to be considered here form a family over an open subset of $\mathbb{P}^2$. We will first describe the family, and then proceed to determine the ramification in their primitive cohomology.

5.1. A family of lattice-polarized K3 surfaces. Let $\mathbb{P}^2$ be a projective plane with homogeneous coordinates $z_0, z_1, z_2$. Let $D \subset \mathbb{P}^2$ be the subvariety defined by

$$z_0z_1z_2(z_0 + z_1 + z_2) = 0$$

which is the union of four lines. If we regard $D$ as a quadrilateral, then there are three diagonals. The union of the three diagonals is defined by

$$(z_0 + z_1)(z_1 + z_2)(z_2 + z_0) = 0$$

which we denote by $\Delta$. Let

$$U = Y - \Delta$$

be the open subset of $Y$ obtained by removing $\Delta$.

We will construct a family

$$f : X \to U$$

of lattice polarized K3 surfaces. Let $K$ be a field, and $y \in U(K)$. We will describe the fibre $X_y$ and describe the natural lattice polarization carried by it.

The surface $X_y$ to be constructed, is determined by the configuration of $D$ and $y$ in $\mathbb{P}^2$ modulo the projective equivalences. Taking the respective duals of $D$ and $y$, we obtain four points $b_1, \cdots, b_4$ and a line $y^*$ in the dual projective plane $\hat{\mathbb{P}}_2$. The blowup of $\hat{\mathbb{P}}_2$ at $b_i$’s shall be denoted by $V_y$. Inside $V_y$, we have the configuration of five disjoint projective lines, consisting of the four exceptional curves and the dual $y^*$ of $y$. The exceptional curve corresponding to $b_i$ shall be denoted by $e_i$. One can identify $V_y$ with the total space of the pencil of conics passing through all $b_i$’s. Among these conics, there are two distinguished conics, denoted by $c_1$ and $c_2$, which meet $y^*$ non-transversally. Figure 1 describes the configuration of these curves. In Figure 1, an open circle represents a simple crossing between two curves, and a closed circle represents a crossing with multiplicity two.

**Proposition 5.1.1.** There is a 4-to-1 cyclic cover $V'_y \to V_y$ branched along $c_1 \cup c_2$. 

![Figure 1. Curves in $V_y$](image-url)
Proof. Let $P$ be the pencil of conics through $\{b_1, b_2, b_3, b_4\}$. There is a natural fibration $V_y \to P$. There is a natural map $y^* \to P$ which sends a point $b$ to the conic which passes through $b$. The ramification locus of $y^* \to P$, say $\delta$, consists of two contact points between $y^*$ and $c_1 \cup c_2$. In order to construct a double cover $\tilde{y}^* \to y^*$ branched along $\delta$ it is enough to find a $K$-rational divisor $x$ such that $\delta$ is rationally equivalent to $2x$. Since $y^*$ is isomorphic to the projective line over $K$, any $K$-rational point can be taken to be $x$. It yields the double cover $\tilde{y}^* \to y^*$, which is independent of the choice of $x$. Let $\tilde{y}^* \to P$ the composition of the two double covers, which is a cyclic 4-1 cover. The pullback $V_y \times_P \tilde{y}^*$ is the cyclic 4-1 cover of $V_y$ branched along $c_1 \cup c_2$. \qed

Let $e_i'$ be the inverse image of $e_i$ in $V'_y$.

**Proposition 5.1.2.** There is a double cover $X_y \rightarrow V'_y$ branched along $e_1' \cup e_2' \cup e_3' \cup e_4'$.

**Proof.** Let $\kappa$ be the canonical class of $V'_y$. The adjunction formula tells us that $2\kappa$ is rationally equivalent to $e_1' \cup e_2' \cup e_3' \cup e_4'$. \qed

**Proposition 5.1.3.** $X_y$ is a K3 surface.

**Proof.** Recall from Proposition 5.1.1 that we constructed a cyclic 4-1 cover $\tilde{y}^*$ of $y^*$. The natural map $X_y \rightarrow \tilde{y}^*$ is a fibration by genus one curves. There are twelve singular fibres. Singular fibers of an elliptic fibration is classified by Kodaira, and every singular fibre in $X_y \rightarrow \tilde{y}^*$ is of type $I_2$. It shows that $X_y$ is a K3 surface. \qed

Let us consider a natural lattice-polarization carried by $X_y$. The pull-back of the hyperplane along the 8-fold covering map $\phi: X_y \rightarrow V_y$ gives you a class $\lambda$ with self-intersection 8. Some other curves in $X_y$ arise in the following way. For each distinct pair $(i, j)$ with $1 \leq i < j \leq 4$, let $l_{ij} \subset V_y$ be the strict transformation of the line connecting $b_i$ and $b_j$. The inverse image $\phi^{-1}(l_{ij})$ is the union of four disjoint projective lines. Let $\Lambda \subset \text{Pic}(X/K)$ be the lattice generated by $\lambda$ and all of the irreducible components of $\phi^{-1}(l_{ij})$. Since there are six pairs of $(i, j)$, $\Lambda$ is generated by $1 + 6 \times 4 = 25$ curves. From now on, we regard $X_y$ as a K3 surface with a lattice-polarization by $\Lambda$.

5.2. **Degenerations.** Let $K$ be a local field with valuation ring $R$, maximal ideal $p = \pi R$, and residue field $k = R/p$. We want to study the degeneration of $X_y$, when $y$ is an element of $U(K)$. We begin with specifying integral models of various spaces. Let $D = \mathbb{P}^2 - Y$ be the complement of $Y$, and we extend $D$ to be a reduced closed flat $R$-subscheme of $\mathbb{P}^2/R$, which we again denote by $D$. This extension is unique. Algebraically, if $f$ is a homogeneous equation of $D/F$, then the equation for $D/R$ is given by $\pi^r f = 0$, where $r$ is the integer characterized by property that $\pi^r f$ has coefficients in $R$ but $\pi^{r-1} f$ does not. The complement $\mathbb{P}^2/R - D/R$ will be denoted by $\mathcal{Y}/R$, which is an integral model of $Y/K$. One defines integral models for $\Delta$ and $U$ similarly.
Throughout, whenever we consider an element $y \in U(K)$, we will assume that we have
\[(5.2.1) \quad y \in U(K) \cap \mathcal{Y}(R),\]
which is equivalent to assuming that $y$ is an element of $U(K)$ such that the reduction mod $p$ of $y$ belongs to $\mathcal{Y}$. One may regard $y$ as a $K$-point of $U$ which possibly degenerates to a point of $\Delta \cap \mathcal{Y}$.

5.3. **Numerical invariants of $y$.** Recall that $\Delta$ has three irreducible components. For $i = 0, 1, 2$, and $\Delta_i$ be the components of $\Delta$, labeled in the way that $\Delta_0$ is defined by $h_0 = z_1 + z_2 = 0$, $\Delta_1$ is defined by $h_1 = z_0 + z_2 = 0$, and $\Delta_2$ is defined by $h_2 = z_0 + z_1 = 0$. Note that the defining polynomials $h_i$’s are well-defined up to multiplication by a unit in $R$.

For a given $y \in U(K) \cap \mathcal{Y}(R)$ and for each $i = 0, 1, 2$, define the integer $n_i$ by
\[(5.3.1) \quad n_i := \text{ord}_\pi (h_i(y)),\]
the $\pi$-adic order of $h_i$ at $y$. It is nothing but the intersection multiplicity between $\Delta_i$ and $y$ at the reduction mod $p$ of $y$.

For any integer $n$, we define
\[(5.3.2) \quad n^* = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ id even.} \end{cases}\]

Now the triple $(n_0^*, n_1^*, n_2^*) \in \{0, 1\}^3$ is an invariant of $X_y$. Our goal is to determine the $G_K$-action on $P_\ell(X)$ in terms of this triple.

If we consider an extension $K'/K$ and regard the $K$-point $y$ as a $K'$-point, then each $n_i$ is multiplied by the ramification index of $K'/K$. In particular, $n_i$ only depends on $y$ viewed as a $\hat{K}$-point. From now, we assume that $k$ is separably closed. Recall that $L/K$ is the unique quadratic extension of $K$, and $\psi$ is the quadratic character associated to the extension $L/K$.

**Theorem 5.3.3.** Let $\ell \nmid p$ be a prime. For $y \in U(K) \cap \mathcal{Y}(R)$, the action of $G_K$ on the primitive cohomology $P_\ell(X)$ factors through $\text{Gal}(L/K)$. Furthermore,
\[2(n_0^* + n_1^* + n_2^*)\]
is the dimension of the $\psi$-isotypic component of $P_\ell(X_y)$.

The rest of the section will be devoted to the proof of the above theorem using Theorem 3.4.4. That is to say, we will construct an integral model $\mathcal{X}_y$ of $X_y$, whose special fibre has at worst ordinary double points, and show that
\[2(n_0^* + n_1^* + n_2^*) = g(\mathcal{X}_y) - a(\mathcal{X}_y)\]
by computing both $g(\mathcal{X}_y)$ and $a(\mathcal{X}_y)$.

More precisely, we will show that
\[(5.3.4) \quad a(X) = 2(n_0^* + n_1^* + n_2^*)\]
and that
\[ g(\mathcal{X}) = 4(n_0^* + n_1^* + n_2^*) \]
for a particular integral model \( \mathcal{X}_y \) of \( X_y \).

The proof of Theorem 5.3.3 is divided into three parts. In the first part, we prove (5.3.4). In the second part, we construct \( X_y \). In the third part, we prove (5.3.5).

5.4. Proof of (5.3.4). Recall from our construction \( X_y \) as an elliptic surface with twelve singular fibres of type \( I_2 \). Being equipped with an elliptic fibration, \( \text{Pic}(X_y/K) \) can be rationally decomposed into
\[ \text{Pic}(X_y/K) \otimes \mathbb{Q} = \langle \lambda \rangle \oplus B \oplus B' \]
where \( B \) is generated by the irreducible components of fibres, and \( B' \) is the orthogonal complement of \( \langle \lambda \rangle \oplus B \). Our definition of \( \Lambda \) implies that \( \Lambda \otimes \mathbb{Q} = \langle \lambda \rangle \oplus B \).

A fibre \( F \) of type \( I_2 \) has two irreducible components, say \( F = F_1 \cup F_2 \), and there are two transversal intersection points between \( F_1 \) and \( F_2 \). The difference \( F_1 - F_2 \) forms twelve linearly independent elements in \( B \). Together with the class of a general fibre, one obtains a basis for \( B \), whose dimension is 13.

We can further decompose \( B \) into a direct sum of four subspaces. The first subspace is generated by \( F \), the class of a general fibre. To define the remaining three subspaces, it is enough to divide the twelve singular fibres into three kinds. By our construction, \( X_y \) is equipped with a generically 8-to-1 map \( X_y \rightarrow \hat{\mathbb{P}}^2 \), temporarily denoted by \( \phi \). Taking the dual of \( \Delta_i \subset \mathbb{P}^2 \), we get a point \( \Delta_i^* \in \hat{\mathbb{P}}^2 \).

Now we can decompose the set \( S \) of singular fibres into the disjoint union \( S_0 \cup S_1 \cup S_2 \), where a singular fibre \( F \in S \) is \( S_i \) if and only if \( \Delta_i^* \in \phi(F) \). Each \( S_i \) contains four elements. This gives rise to the decomposition
\[ B = \langle F \rangle \oplus B_0 \oplus B_1 \oplus B_2 \]
where \( B_i \) is generated by divisors of the form \( F_1 - F_2 \) with \( F \in S_i \) and \( F = F_1 \cup F_2 \).

Note that the above decomposition of \( B \) is compatible with \( G_K \)-action. Therefore, to prove (5.3.4), it is enough to show that the dimension \( r_i \) of the \( \psi \)-isotypic component of \( B_i \) is equal to \( 2n_i^* \). We will prove two lemmas in order to compute \( r_i \).

**Lemma 5.4.1.** The 4-fold covering \( V_y' \rightarrow V_y \) is locally by an equation of the form \( f = w^4 \), where \( f = 0 \) defines the branch locus \( c \). The coefficients of \( f \) are of degree two in terms of the coordinates of \( y \).

**Proof.** Let us identify \( c \) with its image in \( \hat{\mathbb{P}}^2 \). The equation of \( c \) can be found as follows. Let \( x \in \hat{\mathbb{P}}^2 \) be a point. It determines a quadratic equation \( Q_x = 0 \), which cuts out the conic passing through \( b_1, b_2, b_3, b_4, \) and \( x \). The coefficients of \( Q_x \) are polynomials of degree two in the coordinates of \( x \). The intersection of \( y^* \) and \( Q_x \) is not transversal if and only if a quartic polynomial \( f \) vanishes. Elementary algebraic argument shows that the coefficients of \( f \) are quadratic in the coordinates of \( y \). The equation \( f = 0 \) defines a quartic curve in \( \hat{\mathbb{P}}^2 \). This curve
has to be \( c = c_1 \cup c_2 \), because \( c_1 \) and \( c_2 \) are the conics characterized by the two conditions that they are elements in the pencil of conics determined by \( b_i \)'s and that the intersection with \( y^* \) is not transversal.

Let \( L \) be a line which connects two of \( b_j \)'s. Then there is a unique element \( i \in \{0, 1, 2\} \) such that \( \Delta_k^i \in L \). Let \( f_L \) be the restriction of \( f \) to \( L \). Since \( L \) is disjoint from \( c, f_L \) is an element of \( K^\times \). Since \( c \) is the branch locus of a 4-to-1 cover, \( f \) is well-defined up to fourth powers in \( K^\times \). It follows that \( f_L \) is well defined up to multiplication by an element of \( (K^\times)^4 \). In particular, its \( \pi \)-adic order is well-defined modulo 4.

**Lemma 5.4.2.** The \( \pi \)-adic order of \( f_L \) is equal to \( 2n_1 \) modulo 4.

**Proof.** Let \( y = (z_0 : z_1 : z_2) \) be the coordinate of \( y \). Without loss of generality we may assume \( i = 0 \), in that case \( \Delta_k^0 = (0 : 1 : 1) \) in coordinates and \( \Delta_0 \) is defined by \( z_1 + z_2 = 0 \). On the other hand, if we view \( f_L \) as a polynomial in \( z_i \)'s, then the support of the subscheme defined by \( f_L = 0 \) is exactly \( \Delta_0 \). Since the degree of \( f_L \) is two by Lemma 5.4.1, we conclude \( f_L \) is equal to \((z_1 + z_2)^2\), up to multiplication by a fourth power in \( K^\times \). The number \( n_0 \) is the \( \pi \)-adic order of \( z_1 + z_2 \) by definition, and the lemma follows.

The above lemma shows that the action of \( G_K \) on the basis \( S_i \) of \( B_i \) is given by the action of \( G_K \) on the roots of the polynomial \( T^4 + \pi^{2n_1} \). This shows that \( r_i = 2n_1 \), whence the proof of (5.3.4) is complete.

5.5. **Construction of \( \mathcal{X}_y \).** Our aim in this subsection is to construct an \( R \)-model \( \mathcal{X}_y \) of \( X_y \) whose special fibre has at worst ordinary double points.

As an intermediate step towards constructing \( \mathcal{X}_y \), we first introduce a naive integral model, denoted by \( \mathcal{X}_y^{\text{naive}} \). The definition of \( \mathcal{X}_y^{\text{naive}} \) is based on the description of \( X_y/K \) as an 8-fold covering of \( V_y \). Recall that \( V_y \) is defined as the blowup of \( \mathbb{P}^2/K \) at \( \{b_1, b_2, b_3, b_4\} \). By abuse of notation, we define \( b_i \) to be the natural \( R \)-point on \( \mathbb{P}^2/K \) which extends the \( K \)-point \( b_i \). We define \( V_y \) as the blowup of \( \mathbb{P}^2/K \) at \( \{b_1, b_2, b_3, b_4\} \). By abuse of notation, let \( c \subset V_y \) be the \( R \)-curve obtained by taking the closure of \( c \subset V_y \). The 4-fold cover branched along \( c \subset V_y \) is denoted as \( V'_y \). By abuse of notation, we have four \( R \)-curves \( e_i' \subset V'_y \) for \( i = 1, 2, 3, 4 \), which is the closure of \( e_i \subset V_y \). The double cover of \( V'_y \) branched along \( e_1' \cup e_2' \cup e_3' \cup e_4' \) is denoted by \( \mathcal{X}_y^{\text{naive}} \).

By an \( A_3 \)-singularity, we mean an isolated singular point of the form \( z_2^2 + z_0z_1 = 0 \). In particular, it is not an ordinary double point.

Let \( s \) be the number of \( i \)'s such that \( n_i \) is positive. Since not all \( n_i \)'s can be simultaneously positive, we have \( s < 3 \).

**Lemma 5.5.1.** The special fibre of \( \mathcal{X}_y^{\text{naive}} \) is smooth if \( s = 0 \). When \( s > 0 \), the special fibre of \( \mathcal{X}_y^{\text{naive}} \) has \( 2s \) isolated singular points of type \( A_3 \). In particular, when \( s > 0 \), the special fibre of \( \mathcal{X}_y^{\text{naive}} \) has singular points that are not ordinary double points.
Proof. The special fibre of the branch locus \( c \subset \mathcal{V}_y \) contains \( s \) nodes. Otherwise it is smooth. Each node gives rise to an \( A_3 \)-singularity in the special fibre of \( \mathcal{V}'_y \). The number of such singularities is equal to \( s \). The branch locus of the double cover \( \mathcal{X}'_y \to \mathcal{V}'_y \) is disjoint from these singularities, so the singular locus of the special fibre of \( \mathcal{X}'_y \) consists of \( 2s \) isolated singularities of type \( A_3 \). □

We will modify \( \mathcal{X}'_y \) by means of a blow up, which replaces an \( A_3 \) singularity with two ordinary double points. In order to describe the modification, consider the chain of double covers,

\[
\mathcal{X}'_y \to \mathcal{V}'_y \to \mathcal{V}_y \to y^* \to \mathcal{P}
\]

of double covers, where \( \mathcal{V}'_y \to \mathcal{V}_y \) is the unique double cover through which the cyclic 4-fold cover \( \mathcal{V}'_y \to \mathcal{V}_y \) factors.

**Lemma 5.5.2.** The special fibre of \( \mathcal{V}'_y \) contains \( s \) ordinary double points.

*Proof.* If we view \( \mathcal{V}'_y \) as the double cover of \( \mathcal{V}_y \) branched along \( c \), then the lemma follows from the observation that the special fibre of \( c \) contains \( s \) nodes. □

In order to modify \( \mathcal{X}'_y \), we will first remove the singularities in \( \mathcal{V}'_y \) by a blow up along a codimension one subscheme, which properly contains the singular locus. The description of the center is given below.

Recall that \( \mathcal{V}_y \) is the total space for a pencil of conics passing through \( b_i \)'s. Let \( \mathcal{P} \) the \( R \)-model of this pencil. There is a natural map \( y^* \to \mathcal{P} \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{V}'_y & \to & \mathcal{V}_y \\
\downarrow & & \downarrow \\
y^* & \to & \mathcal{P}
\end{array}
\]

is cartesian.

**Lemma 5.5.4.** \( \mathcal{V}'_y \) parametrized the pairs \((b',b)\) where \( b \in y^* \), representing a conic \( c \), and \( b' \in c \).

*Proof.* It is nothing but the functor represented by the fibre product. □

Let \( i, j \) be integers such that \( 1 \leq i < j \leq 4 \). Consider the collection of all pairs \((b,b')\) such that \( b \in l_{ij} \cap y^* \) and \( b' \in l_{ij} \), which defines a closed \( R \)-subscheme \( m_{ij} \subset \mathcal{V}_y' \). It is naturally isomorphic to the line \( l_{ij} \subset \mathcal{V}_y \) via the projection \( \mathcal{V}_y' \to \mathcal{V}_y \). Define \( m \) to be

\[
m = m_{12} \cup m_{13} \cup m_{14},
\]

which is a disjoint union. We let

\[
\mathcal{V}''_y \to \mathcal{V}_y
\]

be the blowup of \( \mathcal{V}_y' \) along \( m \).
Lemma 5.5.5. The generic fibre of the map $\tilde{V}'_y \to \mathcal{V}'_y$ is an isomorphism. Also, the special fibre of $\tilde{V}'_y$ is smooth.

Proof. The first assertion follow from the fact that any blowup of a smooth variety along a curve is an isomorphism. In the special fibre, the map $\tilde{V}'_y \to \mathcal{V}'_y$ is, locally around a singular point, the blowup of the equation $w_0w_1 = w_2^2$ along the curve $w_0 = w_2 = 0$. It resolves the singular point $(w_0, w_1, w_2) = (0, 0, 0)$ by replacing it with a projective line. □

Define both $\mathcal{X}_y$ and $\tilde{\mathcal{V}}'_y$ by requiring that the diagram consist of cartesian squares.

Recall that $s$ is the number of $i$’s such that $n_i$ is positive.

Proposition 5.5.6. The special fibre of $\mathcal{X}_y$ contains $4s$ ordinary double points.

Proof. It follows from analyzing the branch locus of the double cover $\tilde{V}'_y \to \tilde{V}'_y$. Note that the branch locus $\tilde{c}'$ of $\tilde{V}'_y \to \tilde{V}'_y$ is the total transform of the branch locus $c''$ of $\mathcal{V}'_y \to \mathcal{V}'_y$. The special fibre of $c''$ contains $s$ nodes. The proof of Lemma 5.5.5 shows that for each node $x$ in the special fibre of $c''$, there are two nodes in the special fibre of $\tilde{c}'$ lying above $x$. It follows that the special fibre of $\mathcal{X}_y$ contains $4s$ ordinary double points. □

5.6. Proof of (5.3.5). Recall that $s$ is the number of $i$’s such that $n_i$ is positive. We have constructed an integral model $\mathcal{X}_y$ whose special fibre has $4s$ ordinary double points, and in this subsection we aim to show that $g(\mathcal{X}_y) = 4(n_0^* + n_1^* + n_2^*)$.

Note that the definition of $g(\mathcal{X}_y)$ only makes use of the fact that the special fibre of $\mathcal{X}_y$ has at worst ordinary double points as singularities, but not the fact that its generic fibre is a K3 surface. Since the special fibre of $\tilde{\mathcal{V}}'_y$ has at worst ordinary double points, we may consider $g(\tilde{\mathcal{V}}'_y)$.

Lemma 5.6.1. We have $g(\mathcal{X}_y) = 2g(\tilde{\mathcal{V}}'_y)$

Proof. The map $\mathcal{X}_y \to \tilde{\mathcal{V}}'_y$ is a double cover whose branch locus is disjoint from the singular locus of $\tilde{\mathcal{V}}'_y \to \text{Spec (R)}$. It implies that $g(\mathcal{X}_y) = 2g(\tilde{\mathcal{V}}'_y)$. □

By the above lemma, to prove (5.3.5), it is enough to show that $g(\tilde{\mathcal{V}}'_y) = 2(n_0^* + n_1^* + n_2^*)$. We will compute $g(\tilde{\mathcal{V}}'_y)$ by looking at the equation of the map $\tilde{V}'_y \to \tilde{V}'_y$ with branch locus $\tilde{c}' \subset \tilde{V}'_y$. Let $x$ be a singular point in the special fibre of $\tilde{V}'_y$. Its image in $\mathcal{V}'_y$ is contained in $\Delta_i^*$ for some $i = 0, 1, 2$. 
Lemma 5.6.2. We have $n_x = n_i$.

Proof. Let $\phi: \tilde{V}_y' \to \tilde{V}_y''$ be the projection. By definition of $\tilde{V}_y''$ and $n_i$, the formal neighborhood of $\phi(x)$ in $\tilde{V}_y''$ is can be explicitly found. Indeed, the map $\tilde{V}_y'' \to V_y''$ is locally the blow up of the equation

$$w_0w_1 + \pi n_i = w_2^2$$

along the locus defined by $w_0 = w_2 - \pi n_i = 0$. It follows that the formal neighborhood around $x \in \tilde{V}_y'$ is given by $w_0w_1 + \pi n_i = w_2^2$. $\square$

Now (5.3.5) follows from counting singular points in the special fibre of $\tilde{V}_y'$. If $n_i$ is positive, there are two singular points in the special fibre of $\tilde{V}_y'$ such that it maps into $\Delta_i^*$. If $x$ is such a singular point, then $n_x = n_i$ by Lemma 5.6.2. It follows that $g(\tilde{V}_y') = 2(n_0^* + n_1^* + n_2^*)$.

References


