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Exact and asymptotic conditions on traveling wave solutions of the Navier–Stokes equations

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We derive necessary conditions that traveling wave and other solutions of the Navier–Stokes equations must satisfy in the pipe, Couette, and channel flow geometries. Some conditions are exact and must hold for any traveling wave solution or periodic solution irrespective of the Reynolds number (Re). Other conditions are asymptotic in the limit $\mathrm{Re}\to\infty$. For the pipe flow geometry, we give computations up to $\mathrm{Re}=100\,000$ showing the connection of our asymptotic conditions to critical layers that accompany vortex structures at high Re. © *2009 American Institute of Physics*. [doi:10.1063/1.3244660]

The Navier–Stokes equations (NSEs), which model the evolution of the velocity field $\mathbf{u}$ of an incompressible fluid, are $\partial\mathbf{u}/\partial t+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\nabla p+(1/\mathrm{Re})\Delta\mathbf{u}$, where the velocity field $\mathbf{u}$ must satisfy the incompressibility constraint $\nabla\cdot\mathbf{u}=0$. In the NSE, p denotes pressure and Re denotes the Reynolds number. Traveling wave solutions of the form $\mathbf{u}(\mathbf{x},t)=\tilde{\mathbf{u}}(\mathbf{x}-\mathbf{c}t)$ form our main topic. As the motion of turbulent fluids is characterized by disordered and intermittent fluctuations about a mean,[1] the significance of traveling wave solutions may seem limited. However, there is some evidence connecting traveling wave solutions to transitional flows,[2–4] although the connections that have been made are not conclusive.

Further, certain lower branch traveling waves exhibit *critical layers* at high Re (Refs. 5 and 6) that are far beyond the reach of ordinary direct numerical simulation. The ability to compute critical layers in fully resolved numerical solutions of the NSE could be significant, as critical layers occur in many important situations.[7,8] The gigantic trailing vortices that escape from the boundary layers of airplanes during take-off may develop critical layers,[9] so could vortices shed by wind turbines, with possible implications for the optimal arrangement of turbines in a wind farm.

Many linearly unstable (and nonlaminar) traveling wave solutions and equilibria (which are special cases of traveling waves with $\mathbf{c}=0$) of Couette,[10,11] channel,[3,12,13] and pipe[14,15] geometries have now been computed. A notably systematic and extensive effort is due to Gibson *et al.*[16,17] If the streamwise velocity is averaged in the streamwise direction, the resulting field, denoted by $U(y,z)$ in the case of the Couette flow and by $W(r,\theta)$ in the case of pipe flow, is called a streak. Streaks appear crucial to connections of traveling waves with observed phenomena.[14,18] All the conditions that we derive apply to the streaks alone.

We now turn to the derivation of the asymptotic conditions. In the velocity field $\mathbf{u}=(u,v,w)$ of the NSE, u, v, and w are the streamwise (coordinate axis x), wall-normal (y), and spanwise (z) components, respectively, for the rectangular Couette and channel geometries. In the case of pipe flow, u, v, and w are the radial (r), polar (θ), and streamwise (or axial) (z) components, respectively.

In plane Couette flow, the walls at $y=\pm 1$ move in the x direction with speeds equal to ± 1. The boundary conditions in the streamwise and spanwise directions are periodic, with the periods taken to be $2\pi\Lambda_x$ and $2\pi\Lambda_z$, respectively. For pipe flow, we assume the axial or streamwise boundary condition to be periodic with period $2\pi\Lambda$. The walls are no slip in all cases. The derivations are given mainly for the plane Couette flow geometry.

Traveling waves normally arise from saddle-node bifurcations with increasing Re.[10,12,14,15,19] The branch corresponding to lower energy dissipation is called the lower branch. We will now derive certain scalings with respect to increasing Re that are characteristic of the lower branch families.

In the case of plane Couette flow or channel flow, if a traveling wave solution is given by $\tilde{\mathbf{u}}(\mathbf{x}-\mathbf{c}t)$, the velocity field $\tilde{\mathbf{u}}(\mathbf{x})$ can be decomposed as

$$\mathbf{u}_0(y,z)+\sum_{n=1}^{\infty}[\mathbf{u}_n(y,z)\exp(in\alpha x)+\text{c.c.}], \tag{1}$$

where $\alpha=1/\Lambda_x$. We take $\mathbf{u}_0=(U,v_0,w_0)$ and $\mathbf{u}_n=(u_n,v_n,w_n)$ for $n\geq 1$. For pipe flow, the decomposition analogous to Eq. (1) is given by $\mathbf{u}_0(r,\theta)+\Sigma_{n=1}^{\infty}[\mathbf{u}_n(r,\theta)\exp(in\alpha z)+\text{c.c.}]$, with $\alpha=1/\Lambda$. We take $\mathbf{u}_n=(u_n,v_n,w_n)$ for $n\geq 1$ as for Couette flow, but $\mathbf{u}_0=(u_0,v_0,W)$ for pipe flow.

The scalings of the lower branch family that are already known apply to the streaks (U or W), rolls [(v_0,w_0) or (u_0,v_0)], or magnitude of modes such as $\mathbf{u}_1$. It is an empirical fact (but see Refs. 3, 20, and 21) that the rolls and the $\mathbf{u}_1$ mode diminish in magnitude approximately at the rate Re^{-1}. Higher modes with $n>1$ appear to diminish even faster. The derivations assume these scalings. In addition, the dissipation rate of the lower branch families decrease with increasing Re, assuming that the dissipation rate of the laminar solution is normalized to be 1.[5,6]

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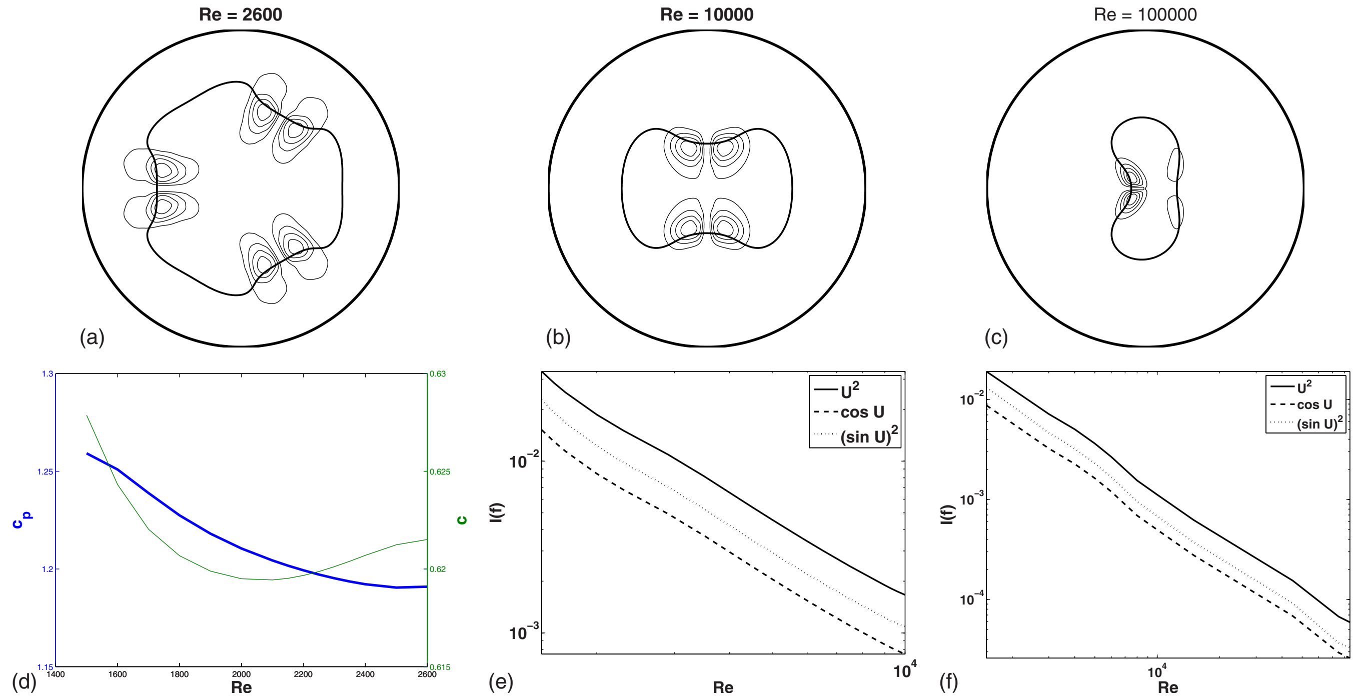

FIG. 1. (Color online) Three traveling wave solutions of pipe flow are shown. Units are chosen to make the pipe radius and the center line speed of the laminar solution equal to 1. The boundary condition fixes the mass flux (and not the pressure gradient) to be that of the laminar solution. Solutions (a) and (b) correspond to the c family and the upper dashed-dotted curve of Fig. 7 of Ref. 15, respectively. For more on (c), see Refs. 5 and 22. In the upper plots, the four contour levels of $|u_1|$ are equispaced in (0, max), with maximum being 0.0094, 0.0011, and 0.0002, respectively. The thick line is the critical curve. The plots below correspond to the ones directly above. Plot (d) (the thick line is for c_p) can be validated against Fig. 7 of Ref. 15 after replacing Re and c by c_p Re and c/c_p, respectively. In (e) and (f), the legends show the choice of $f(U)$ and $I(f)$ denotes Eq. (7) ($\gamma \approx 1.5$).

At this point we assume $\mathbf{c}=(c,0,0)$ so that the traveling wave moves in the streamwise direction only with wavespeed c. The wall normal or y component of the $n=1$ mode of the NSE gives

$$i\alpha(U-c)v_1 = -\partial_y p_1 + \mathrm{Re}^{-1}(-\alpha^2 v_1 + \partial_y^2 v_1 + \partial_z^2 v_1) + \cdots \tag{2}$$

for plane Couette or channel flow. The first neglected terms in Eq. (2) are $-v_0\partial_y v_1 - w_0\partial_z v_1 - v_1\partial_y v_0 - w_1\partial_z v_0$. The analogous equation for the radial component of pipe flow is $i\alpha(W-c)u_1 = -\partial_r p_1 + \mathrm{Re}^{-1}\ \Delta_r u_1$, where Δ_r corresponds to the usual form of the Laplacian in the radial component of the NSE. Terms such as $-v_1\partial_\theta v_0/r$ are neglected.

Using Eq. (2), Wang *et al.*[6] estimated that most of the variation in v_1 is concentrated in a region around the critical curve $U=c$, with the width of that region scaling as $\mathrm{Re}^{-1/3}$. In the case of pipe flow, an identical argument gives $W=c$ as the equation of the critical curve. The top set of plots in Fig. 1 illustrates the critical layer in the case of pipe flow.

To derive further scalings, we consider the streamwise component of the $n=0$ mode of NSE, which is

$$v_0\partial_y U + w_0\partial_z U = \mathrm{Re}^{-1}(c_p + \Delta U) + M \tag{3}$$

for plane Couette or channel flow. Here c_p/Re gives the pressure gradient in the streamwise direction, with $c_p=0$ for plane Couette flow and $c_p>0$ for channel flow. The pipe flow analog is $u_0\partial_r W + (v_0/r)\partial_\theta W = \mathrm{Re}^{-1}(c_p+\Delta W)+M$. In Eq. (3) and its pipe flow analog, $-M$ equals the $n=0$ mode of the streamwise component of $[(\tilde{\mathbf{u}}-\mathbf{u}_0)\cdot\nabla](\tilde{\mathbf{u}}-\mathbf{u}_0)$. From here on we restrict the derivation to plane Couette flow or to channel flow.

Since $\mathbf{u}_0=(U,v_0,w_0)$ has zero divergence, we can find a function $\psi(y,z)$ such that $v_0=\partial_z\psi$ and $w_0=-\partial_y\psi$. We then get

$$L\psi = \mathrm{Re}^{-1}(c_p + \Delta U) + M, \tag{4}$$

where

$$L = (\partial_y U)\partial_z - (\partial_z U)\partial_y. \tag{5}$$

The skew symmetry of the linear operator L is the key to deducing further scalings.

Let $\phi(y,z)$ and $\psi(y,z)$ have z periods of $2\pi\Lambda_z$ and be sufficiently smooth. The following calculation uses integration by parts:

$$\begin{aligned}\int_0^{2\pi\Lambda_z}\int_{-1}^{1}\phi L\psi\,dy\,dz &= \int_{-1}^{1}\phi\psi U_y\big|_0^{2\pi\Lambda_z}dy - \int_0^{2\pi\Lambda_z}\int_{-1}^{1}(\psi U_y\phi_z + \psi\phi U_{yz})dy\,dz \\ &\quad - \int_0^{2\pi\Lambda_z}\phi\psi U_z\big|_{-1}^{1}dz \\ &\quad + \int_0^{2\pi\Lambda_z}\int_{-1}^{1}(\psi U_z\phi_y + \psi\phi U_{yz})dy\,dz,\end{aligned}$$

where the subscripts are for partial derivatives. On the right hand side, two double integral terms cancel and the single integral terms are both zero because U_y is periodic in z and

U_z is zero at the walls (from no slip). We are left with $-\int\int \psi L\phi dydz$ on the right, verifying skew symmetry of the operator L.

From direct substitution into Eq. (5), it is evident that $L[f(U)]=0$ for *any* smooth f. Thus the functions $f(U)$ are in the kernel of the antisymmetric operator L. Since the linear system (4) can be solved for ψ (or equivalently for the rolls), the Fredholm alternative implies that

$$\int_0^{2\pi\Lambda_z}\int_{-1}^{1} f(U)(c_p+\Delta U+\mathrm{Re}\ M)dydz=0.$$

For lower branch traveling wave families with $\tilde{\mathbf{u}}-\mathbf{u}_0$ of magnitude $\mathrm{Re}^{-\gamma}$ with $\gamma\approx 1$, the magnitude of M is approximately Re^{-2} in the limit $\mathrm{Re}\to\infty$. We have

$$\int f(U)[\Delta U(y,z)+c_p]dydz=O(\mathrm{Re}^{-\gamma}) \quad (6)$$

for any smooth f, where the integral is over the cross section of the channel. Here $\gamma>1$ is possible if there are cancellations in the integral of M over the cross section. The analogous condition for pipe flow is given by

$$\int f(W)[\Delta W(r,\theta)+c_p]rdrd\theta=O(\mathrm{Re}^{-\gamma}), \quad (7)$$

with γ as above.

Both conditions (6) and (7) apply to the streaks alone. At $\mathrm{Re}=\infty$, the asymptotic conditions become exact. The balance between streaks and rolls is fundamental to the existence of lower branch solutions at large Re.[3,20] If the theory is to be made exact, the streaks must satisfy the conditions we have derived.

In addition to the pipe families of Fig. 1, we computed a lower branch equilibrium family and a traveling wave family of plane Couette flow up to Re=45 000 and Re=7000. The (a) and (b) families of Fig. 1 could not be continued to Re much higher than shown in the top plots. For a given resolution, we cannot expect to find solutions if the rolls, which diminish in magnitude with Re, are too small to be detected. Even after using sufficient resolution, the GMRES-hookstep iterations (see Refs. 5 and 23) became very slow. Even though the residual error could be made quite small, the norm of the Newton steps became quite large and increased with iteration. Although it is uncertain if all lower branch families exist in the $\mathrm{Re}\to\infty$ limit, Fig. 1 amply demonstrates that they exist for large enough Re for the predicted scalings to hold.

The critical curves are away from the pipe walls and have an inward indentation where the counter-rotating vortices face each other. Since this behavior is evident even for Re=2600, we suspect that critical curves or surfaces may give a way to visualize the structure of puffs in transitional pipe flow. For puffs, see Ref. 24. For a view of transition to turbulence based on shears, see Ref. 25.

The exact conditions, whose derivation we now give, apply to traveling wave solutions and periodic solutions and to relative periodic solutions that do not have a spanwise motion. Relative periodic solutions can capture some aspects of turbulent fluctuations.[23] The conditions we derive are consequences of the Reynolds averaged equations. Nevertheless they appear not to have been noticed in literature on traveling waves and other exact solutions of the NSE.

Let $\mathbf{u}(\mathbf{x},t)=\tilde{\mathbf{u}}(\mathbf{x}-\mathbf{c}t)$ be a traveling wave solution of plane Couette flow. We assume $\mathbf{c}=(c,0,0)$ so that the traveling wave moves in the streamwise direction only. If $\tilde{\mathbf{u}}=(u,v,w)$, the streamwise or x component of the NSE gives

$$-c\partial_x u+(u\partial_x u+v\partial_y u+w\partial_z u)=-\partial_x p+\frac{c_p}{\mathrm{Re}}+\frac{1}{\mathrm{Re}}\Delta u. \quad (8)$$

Let $U(y,z)$ denote $(2\pi\Lambda_x)^{-1}\int_0^{2\pi\Lambda_x}u(x,y,z)dx$, which is the mean streamwise component of u. From Eq. (8), we get

$$\overline{(u\partial_x u+v\partial_y u+w\partial_z u)}=\frac{c_p+\Delta U}{\mathrm{Re}}, \quad (9)$$

where the overline denotes streamwise averaging. At the walls $y=\pm 1$, $\partial_x u=0$, and $v=w=0$ because of no slip. For the same reason, $\partial_{zz}U=0$ at the walls. Therefore,

$$c_p+\partial_{yy}U=0 \quad (10)$$

must hold at the walls.

As the velocity field $\tilde{\mathbf{u}}$ has zero divergence, we may rewrite Eq. (9) as

$$\overline{\nabla\cdot(u^2,uv,uw)}=\partial_y\overline{uv}+\partial_z\overline{uw}=\frac{c_p+\Delta U}{\mathrm{Re}}. \quad (11)$$

Both Eqs. (9) and (11) are Reynolds averaged equations (Ref. 1, Chapter 5). If Eq. (11) is integrated over the cross section, Green's theorem applies to the expression in the middle of Eq. (11). The integral of the middle term must be zero because $v=0$ at the walls and uw is periodic in z. Thus we have

$$\int_0^{2\pi\Lambda_z}\int_{-1}^{+1}(\Delta U+c_p)dydz=0. \quad (12)$$

The derivation of the necessary conditions (10) and (12) applies to channel flow with no change. However, $c_p\neq 0$ for channel flow.

Conditions (10) and (12) must be satisfied by all traveling wave solutions of plane Couette flow or channel flow, whose wave speed vector $\mathbf{c}$ only has a streamwise component. Indeed, those conditions must be satisfied by all periodic solutions with $\mathbf{u}(\mathbf{x},t)=\mathbf{u}(\mathbf{x},t+T)$, T being the period, or relative periodic solutions with $\mathbf{u}(\mathbf{x},t)=\mathbf{u}(\mathbf{x}+\mathbf{s},t+T)$ if the spatial period $\mathbf{s}$ only has a streamwise component. To form U in these instances, one must average both over a single period and in the streamwise direction as a simple modification of our derivation will show. Similarly, c_p must be averaged over the entire period. These conditions have been verified, where applicable, for the relative periodic solutions reported in Ref. 23.

For the case of pipe flow, let $\mathbf{c}=(0,0,c)$ so that the traveling wave travels in the streamwise direction only. Let $W(r,\theta)=(2\pi\Lambda)^{-1}\int_0^{2\pi\Lambda}w(r,\theta,z)dz$ be the mean streamwise velocity. The analog of Eq. (10) requires

$$c_p + \partial_r^2 W + \frac{\partial_r W}{r} = 0 \tag{13}$$

at all points on the circumference. If we assume the pipe radius to be 1, the analog of Eq. (12) is

$$\int_0^{2\pi} \int_0^1 (\Delta W + c_p) r dr d\theta = 0. \tag{14}$$

The derivation of the necessary conditions (13) and (14) for pipe flow is similar to that of their Couette analogs. The conditions must hold for relative periodic solutions with $\mathbf{u}(\mathbf{x},t)=\mathbf{u}(\mathbf{x}+\mathbf{s},t+T)$ as well, if the spatial period $\mathbf{s}$ has streamwise motion only. If the spatial period of the relative periodic motion has spanwise motion, the conditions will not apply.

A reviewer has kindly pointed out to us that a necessary condition such as Eq. (12) can be derived for traveling wave solutions that have wavespeeds in both the spanwise and streamwise directions. In such a case, one has to consider the averaged component of the velocity field in the direction of the wavespeed vector instead of U.

Any velocity field of the form $(u(y,z),0,0)$ is a solution of the Euler equations. However, our asymptotic conditions show that such a velocity field can arise as a limit of lower branch solutions of NSE only if it satisfies the conditions of Eq. (6). The exact condition (12) represents a balance between the viscous, pressure, and inertial terms. Unlike the energy balance between those terms, which uses the entire velocity field, Eq. (12) is a condition on the streaks alone.

Some approaches use a velocity field to model the statistical steady state of high Re turbulence. More speculatively, the sort of conditions we have derived could be useful in that context. In this letter, we have mainly supplied conditions on the streaks that occur as a part of traveling waves but also pointed out the connection between the critical curve and the placement of the rolls.

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