Abstract

In this work, we study a distributed source coding problem with multiple encoders, a central decoder and a joint distortion criterion. The encoders observe correlated sources which they quantize and communicate to a central decoder through a rate constrained, noiseless channel. The objective is to minimize the distortion as measured by a joint distortion criterion that depends on the sources and the reconstruction. We are interested in characterizing a computable inner bound to the optimal rate-distortion region.

We first consider a special case of the general problem where the sources are jointly Gaussian and the decoder is interested in reconstructing a linear function of the sources under mean square error distortion criterion and present an achievable rate-distortion region. We demonstrate a coding scheme involving structured nested lattice codes that achieves this bound by encoding in such a fashion that the decoder is able to reconstruct the function directly. For certain source distributions, this approach yields a larger rate-distortion region compared to when the decoder reconstructs lossy versions of the sources first and then estimates the function from them. We then extend this approach to the case of reconstructing a linear function of an arbitrary number of jointly Gaussian sources.

Next, we consider the general distributed source coding problem involving multiple discrete sources and a central decoder with a joint distortion criterion. This formulation includes as a special case the problem of computing a function of the sources to within some distortion and also the classic Slepian-Wolf problem, Berger-Tung problem, Wyner-Ziv problem, Yeung-Berger problem and the Ahlswede-Korner-Wyner problem. We present a new achievable rate-distortion region for this problem based on “good” structured nested random codes built over abelian groups. We demonstrate rate gains for this problem...
over traditional coding schemes involving unstructured random codes. For certain sources and distortion functions, the new rate region is strictly bigger than that based on the Berger-Tung rate region, which has been the best known achievable rate region for the problem till now. Further, there is no known unstructured random coding scheme that achieves these rate gains.

In the case of both discrete and continuous sources, our coding schemes and the rate gains they offer rely critically on the structure of the codebooks considered. For the case when the sources were jointly Gaussian and the decoder employs a mean-square error distortion criterion, nested lattice codes with certain properties were found to offer rate gains over unstructured codes. This raises the question as to whether nested lattice codes with similar properties can be employed for the more general case of arbitrary continuous alphabets and arbitrary additive distortion measures. Also, the rate region that we have derived lacks an information-theoretic characterization. Such a characterization would give insights into the nature of coding gains offered by nested lattice codes and facilitate comparison with other rate regions for the problem. In the case of discrete sources, we showed that good nested codes can be built over abelian groups. A natural extension of this problem would be to investigate whether good codes can be built over non-abelian groups. We hope to address some of these problems in the future.

1 Introduction

In this work, we consider a general distributed source coding problem involving multiple sources, a central decoder and a joint distortion criterion. We first study a special case of the problem when the sources are jointly Gaussian and the decoder is interested in reconstructing a linear function of the sources under mean square distortion. For the case of discrete sources, we present an inner bound to the optimal rate-distortion region for the case of a general distortion criterion employed at the decoder. Our approach for both these problems involves the use of structured random codes which offer rate gains otherwise unattainable using unstructured random codes.

In the following section, we explain the distributed source coding problem that we study.

2 Distributed Source Coding

Since its inception in 1973 by Slepian and Wolf, the problem of distributed source coding has been a source of inspiration for information/communication/data-compression theory community because of its formidable nature (in its full generality) and its wide scope of practical applications. In this problem, a collection of $K$ correlated information sources, with $i$th source having an alphabet $\mathcal{X}_i$, is observed separately by $K$ encoders. Each encoder maps its observations into a finite-valued set. The indices from these sets are transmitted over $K$ noiseless but rate-constrained channels to a joint decoder. The decoder is interested in obtaining $L$ reconstructions with $L$ fidelity criteria (one for each). The $i$th reconstruction has an alphabet $\mathcal{Y}_i$, and the $i$th fidelity criterion is a
mapping from the product of alphabets of a subset of the sources and $\hat{Y}_i$ to the set of nonnegative real numbers.

The goal is to find a computable performance limit for this communication problem. The performance limit, also referred to as the optimal rate-distortion region, is expressed as the set of all $(K + L)$-tuples of rates of the $K$ indices transmitted by the encoders and distortions of the $L$ reconstructions of the decoder that can be achieved in the usual Shannon sense. A schematic of the problem is shown in Figure 1.

![Figure 1: A general distributed source coding problem](image)

Toward this goal, progress has been made in a number of directions. In the following we restrict our attention to the case of the collection of stationary memoryless sources. In [1], a solution to the problem was given for the case when the decoder wishes to reconstruct all the sources losslessly. In [2, 3], the case of lossless “one-help-one” problem was resolved. Here the decoder wishes to reconstruct only one of the sources ($K = L + 1 = 2$). In [4], the case of lossy “one-help-one” problem was resolved for the case when the rate of the helper is greater than its entropy (also referred to as the Wyner-Ziv problem). In [5, 6], an inner bound, and an outer bound (also known as the Berger-Tung inner and outer bounds respectively) to the performance limit are given for the case where (a) $K = L = 2$ and (b) the fidelity criterion of each source does not depend on the other source (also referred to as independent fidelity criteria). In [7], an inner bound to the performance limit is

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1 The source which does not enter into any of the fidelity criteria is referred to as a helper. When the rate at which the helper is transmitted is greater than its entropy, the helper is also referred to as side information.
given for the case when the decoder wishes to reconstruct a function of $K$ sources losslessly. It was also shown that this is optimal for the case when the sources are conditionally independent given the function. In [8], the performance limit is given for reconstructing losslessly the modulo-2 sum of two binary correlated sources, and was shown to be tight for the symmetric case. This has been extended to several cases in [9] (see Problem 23 on page 400) and [11]. An improved inner bound was provided for this case in [12]. The key point to note is that the performance limits given in [8, 11, 12] are outside the inner bound given in [7].

With regard to above set of results, we would like to make the following observations. (a) Most of the above approaches, except that of [8] and its extensions in [9, 11, 12], use random vector quantization followed by independent random binning (see Chapter 14 of [10]) of the quantizer indices. (b) The four exceptions, which consider only lossless source coding problems, deviate from this norm, and instead use structured random binning based on linear codes on finite fields. Further, the binning operation of the quantizers of the sources are “correlated”. This incorporation of structure in binning appears to give improvements in the rates especially for those cases that involve reconstruction of a function of the sources. Moreover, it is still not known whether it is possible to approach this performance without the structured codes. (c) For some distributed source coding problems (that belong to the first category), whose performance limits were derived using random coding and random binning, it is well-known that these limits can also be approached using structured codes. For example structured codes were considered for (a) the Slepian-Wolf problem in [13], (b) the Wyner-Ziv problem for the binary case with Hamming distortion and for the quadratic Gaussian case in [18], (c) the Berger-Tung inner bound for the two terminal quadratic Gaussian problem with independent fidelity criteria in [18] and (d) high-resolution distributed source coding problem with independent fidelity criteria in [17].

3 Contributions

Motivated by the rate gain offered by structured codes over unstructured codes for certain problems, we adopt a similar approach to that of [8] for the general problem of distributed source coding. In particular, we demonstrate the existence of good nested structured codes whose components are “good” codes for source and channel coding for certain appropriately defined notions of “goodness”. We consider two problems below - (a) reconstructing a linear function of jointly Gaussian sources, (b) discrete sources with a joint distortion criterion.

3.1 Linear Function of Gaussian Sources

We consider a lossy distributed source coding problem with $K$ jointly Gaussian sources with one reconstruction, i.e., $L = 1$. The fidelity criterion has the additional structure that is given by the following. The decoder wishes to reconstruct a linear function of the sources with squared error as the fidelity criterion. We consider a coding scheme with the following structure: sources are quantized using structured vector quantizers followed
by “correlated” structured binning. That is, the binning operations of the quantizers of the sources are not performed “independently”. The structure used in this process is given by nested lattice codes. We provide an inner bound to the optimal rate-distortion region. We show that the proposed inner bound is better for certain parameter values than an inner bound that can be obtained by using a coding scheme that uses random vector quantizers following by independent random binning. For this purpose we use the machinery developed by [15, 16, 18, 19, 20] for the Wyner-Ziv problem in the quadratic Gaussian case.

We present our coding theorem for the two user case below. Consider a pair of jointly Gaussian sources \((X_1, X_2)\) such that the source sequence \((X_1^n, X_2^n)\) is independent over time. Assume without loss of generality that the sources have unit variance and are correlated according to \(E(X_1X_2) = \rho > 0\). Let the decoder be interested in reconstructing a linear function \(Z = X_1 - cX_2\) of the sources to within a distortion \(D\) as given by the mean square error distortion criterion \(d(X_1, X_2, Z) = E((X_1 - cX_2 - Z)^2)\).

**Theorem 1.** The set of all tuples of rates and distortions \((R_1, R_2, D)\) that satisfy

\[
2^{-2R_1} + 2^{-2R_2} \leq \left(\frac{\sigma_Z^2}{D}\right)^{-1}
\]

are achievable. Here \(\sigma_Z^2 = 1 + c^2 - 2\rho c\) is the variance of the function \(Z\) to be reconstructed.

The two encoders use the nested lattice codes \((\Lambda_{11}, \Lambda_2)\) and \((\Lambda_{12}, \Lambda_2)\) for encoding their sources. We require that the fine codes \(\Lambda_{11}, \Lambda_{12}\) be good source codes and \(\Lambda_2\) be a good channel code as defined in [18]. The details of the proof and the extension of the coding theorem to arbitrary number of jointly Gaussian sources can be found in [23]. As part of the proof of this coding theorem, we also prove the existence of good nested lattice codes with an arbitrary finite level of nesting where all the component codes are simultaneously good source and channel codes. The details can be found in [21] and the proof is an extension of the proof in [20] and uses similar techniques.

### 3.2 Discrete Sources with a Joint Distortion Criterion

We then study the general problem of distributed source coding with a joint distortion criterion for the case when the sources are discrete and the distortion is additive. We demonstrate the rate gains obtainable over the Berger-Tung based coding scheme for this problem using a coding scheme which is very similar in spirit to the coding scheme of Korner and Marton [8] and the lattice based coding scheme described earlier. Our approach relies on the use of nested group codes for encoding. The binning operation of the encoders are done in a “correlated” manner as dictated by these structured codes. This use of “structured quantization followed by correlated binning” is in contrast to the more prevalent “quantization using random codes followed by independent binning” in distributed source coding. This approach unifies all the known results in distributed source coding such as the Slepian-Wolf problem [1], Korner-Marton problem [8], Wyner-Ahlswede-Korner problem [2, 3], Wyner-Ziv
problem [4], Yeung-Berger problem [14] and Berger-Tung problem [6], under a single framework while recovering their respective rate regions. Moreover, this approach performs strictly better than the standard Berger-Tung based approach for certain source distributions.

The encoders use nested group codes built over abelian groups in our coding scheme. We exploit a classic result from abstract algebra [24] that says any finite abelian group is isomorphic to the direct sum of possibly repeating primary cyclic groups. This enables us to focus solely on primary cyclic groups in the proofs of our coding theorems. Sources over general abelian groups are then treated as vector sources with their components taking values in primary cyclic groups. These vector sources are encoded sequentially one digit at a time. At every decoding instance, the decoder has the previously decoded digits available to it as side information with which to decode the current digit.

### 3.2.1 Group Codes - Definition and Results

A group code over the primary cyclic group \( \mathbb{Z}_{p^r} \) is defined as a subgroup \( C \) of \( \mathbb{Z}_{p^r}^n \). As part of the proof of our coding theorem, we derive bounds on the sizes of “good” group source and channel codes for certain natural notions of goodness. These results are paraphrased below.

**Lemma 2.** Assume without loss of generality that \( Z \) takes values over the primary cyclic group \( \mathbb{Z}_{p^r} \). A good group channel code \( C \) exists for the triple \((Z, S, P_{ZS})\) for sufficiently large \( n \) if its cardinality satisfies

\[
\frac{1}{n} \log |C| \leq \log p^r - \max_{0 \leq i < r} \left( \frac{r}{r - i} \right) (H(Z|S) - H([Z]_i|S))
\]

where \([Z]_i\) is a random variable taking values over the set of distinct cosets of \( p^i \mathbb{Z}_{p^r} \) in \( \mathbb{Z}_{p^r} \).

Note that, setting \( r = 1 \) in the above expression recovers the known fact that linear codes built over Galois fields can achieve the entropy bound when used for lossless compression of memoryless sources. Each term in the maximization in equation (2) corresponds to a non-trivial subgroup of \( \mathbb{Z}_{p^r} \). Thus, we incur a rate penalty for building codes over groups with large number of subgroups.

**Lemma 3.** Assume without loss of generality that \( U \) takes values over the primary cyclic group \( \mathbb{Z}_{p^r} \). A good group source code \( C \) exists for the triple \((X, U, P_{XU})\) for sufficiently large \( n \) if its cardinality satisfies

\[
\frac{1}{n} \log |C| \geq \log p^r - \min\{H(U|X), r|H(U|X) - \log p^r - 1 + |^+\}
\]

where \(|x|^+ \triangleq \max\{x, 0\} \).

Note that, setting \( r = 1 \) in the above expression recovers the known fact that linear codes over Galois fields do not achieve the Shannon rate-distortion bound unless the reconstruction random variable \( U \) is uniformly distributed over \( U \). Thus, we incur a rate penalty for imposing the group structure over the codebook.
3.2.2 Coding Theorem

Though group codes have an inferior performance when compared to unstructured random codes, exploiting
the structure present in such codes can give overall rate gains in distributed source coding problems. Consider
the general distributed source coding problem with two discrete sources \((X, Y)\) and a central decoder interested
in minimizing the expected joint distortion \(Ed(X, Y, \hat{Z})\). Suppose the encoders quantize \(X\) to \(U\) and \(Y\) to \(V\)
and transmit these quantized values to the central decoder. Let \(G: \mathcal{U} \times \mathcal{V} \rightarrow \hat{Z}\) be the optimal reconstruction
function that minimizes the expected distortion. Suppose \(G(U, V)\) can be represented as equivalent to addition
in a finite abelian group \(G\) which is decomposable into primary cyclic groups, i.e., \(G \cong \mathbb{Z}_{p_1^{e_1}} \oplus \ldots \mathbb{Z}_{p_k^{e_k}}\). Then,
the encoders use a sequential encoding approach where each digit is encoded using a nested group code built
over the appropriate primary cyclic group. This coding scheme is illustrated below for the case when \(G = \mathbb{Z}_{p^r}\).

Let \(U, V\) be random variables taking values over the primary cyclic group \(\mathbb{Z}_{p^r}\). Fix the joint distribution
\(P_{XUV} = P_{XY}P_{UX|Y}P_{V|Y}\). Let the optimal reconstruction function \(G(U, V)\) that minimizes the expected dis-
tortion \(Ed(X, Y, G(U, V))\) be equivalent to addition in \(\mathbb{Z}_{p^r}\). Let \(Z \triangleq U \oplus_{p^r} V\). The encoders use the nested
group codes \((C_{11}, C_2)\) and \((C_{12}, C_2)\) for encoding the sources. Here, \(C_{11}\) is a good group source code for the triple
\((X, U, P_{XUV})\), \(C_{12}\) is a good group source code for the triple \((Y, V, P_{YV})\) and \(C_2\) is a good channel code for the triple
\((Z, \{0\}, P_Z)\). The \(X\)-encoder looks for a typical sequence \(u^n \in C_{11}\) that is jointly typical with the source
sequence \(x^n\). Provided \(C_{11}\) satisfies the cardinality bound of Lemma 3, such a sequence \(u^n\) is found with high
probability. If it finds such a sequence, it transmits the coset index of the coset in \(\mathbb{Z}_{p^r}\) in which \(u^n\) lies.
The \(Y\)-encoder’s operation is similar. Since \(z^n = u^n \oplus_{p^r} v^n\), the decoder can compute the coset of \(C_2\) in \(\mathbb{Z}_{p^r}\)
in which the sequence \(z^n\) lies from the received indices. Since \(C_2\) is a good channel code for the distribution
\(P_Z\), the decoder can recover \(z^n\) with high probability. The rates of these nested codes are easily computed as
\(R_1 = \frac{1}{n} \log \frac{|C_{11}|}{|C_2|}\) and similarly for \(R_2\). From the cardinality bounds of Lemmas 2 and 3, the following rate region
can be derived.

Theorem 4. With definitions as above, define the region \(\mathcal{RD}\) as
\[
\mathcal{RD} = \bigcup_{U \times X \times Y \times V : Ed(X, Y, G(U, V)) \leq D} \left\{ R_1 \geq \max_{0 \leq i < r} \left( \frac{r}{p^r - i} \right) (H(Z) - H([Z]_i)) - \min \{ H(U|X), r|H(U|X) - \log p^{r-1} \}^+ \right\}
\]
\[
R_2 \geq \max_{0 \leq i < r} \left( \frac{r}{p^r - i} \right) (H(Z) - H([Z]_i)) - \min \{ H(V|Y), r|H(V|Y) - \log p^{r-1} \}^+
\]
\[
Ed(X, Y, G(U, V)) \leq D
\]
(4)

Then, any \((R_1, R_2, D) \in \mathcal{RD}^*\) is achievable where * denotes convex closure.

A more general coding scheme can be presented by embedding the function to be reconstructed in more
general groups and also by considering a combination of the group codes based scheme and the Berger-Tung
based scheme. The details can be found in [22].
3.2.3 Special Cases

Some of the special cases of Theorem 4 are discussed here. From Lemmas 2 and 3, we get achievable rates for the problems of lossless and lossy source coding using codes built over abelian groups. Specializing the statement of Theorem 4 to the case when the codes are built over a Galois field and setting $Y = 0$, we get the result that nested linear codes achieve the Shannon rate-distortion bound for arbitrary discrete sources and arbitrary additive distortion measures. We also show [22] that nested linear codes can be used to recover the best known rate regions of many distributed source coding problems including the Berger-Tung problem [6], Wyner-Ziv problem [4], Slepian-Wolf problem [1], Korner-Marton problem [8], Yeung-Berger problem [14] and the Ahlswede-Korner-Wyner problem [3]. By considering a lossy version of the Korner-Marton problem, we show that our coding scheme offers rate gains over the Berger-Tung based coding scheme that was the best previously known rate region for this problem. By interpreting this problem as a three-user distributed source coding problem with independent distortion criteria, we can conclude that the Berger-Tung inner bound is not tight for the case of more than two users.

4 Future Work

In this section, we outline some of the extensions of the ideas discussed thus far that we plan to work on in the future.

The result of Theorem 1 is valid only for the case when the sources are jointly Gaussian and the decoder employs the mean-square error distortion criterion. In contrast to this, the result of Theorem 4, which is very similar in spirit, is applicable for arbitrary discrete sources and distortion measures. This leads us to believe that a nested lattice based coding scheme can be used to obtain rate regions for problems involving arbitrary continuous sources and arbitrary additive distortion measures. Before such a coding scheme could be developed, one needs to show the existence of “good” nested lattice codes for stronger notions of goodness than those defined in [18] which were specific to the Gaussian distribution. We plan to investigate these questions with the goal of generalizing the results of Theorem 1 beyond Gaussian sources and mean-square distortion.

In contrast to the norm in information theory, the rate region presented in 1 is not in terms of information-theoretic quantities such as mutual information. It would be of interest to see how this rate region can be recast in the more usual form of an optimization problem where an information theoretic quantity is optimized over the space of some probability distributions subject to certain distortion constraints. Such a formulation would help in characterizing precisely what rate gains nested lattice codes offer and also would facilitate in comparing the coding scheme with other rate regions for this problem.

In the discrete case, the encoders employed nested codes built over abelian groups. It would be of great interest to investigate the conditions under which such codes can be built over non-abelian groups. Apart from
being an interesting problem in its own right, construction of such codes would play a part in enlarging the rate region of Theorem 4 since it would offer more choices of algebraic structures in which to embed the optimal reconstruction function. Our proofs of Lemmas 2 and 3 depend upon the underlying ring structure of \( \mathbb{Z}_{p^r} \) whereas a non-abelian group has no such structure. Also, we construct group codes over abelian groups as the kernels of homomorphisms from \( \mathbb{Z}_{p^r}^n \) to \( \mathbb{Z}_{p^r}^k \). However, such a construction appears to not yield good codes for the non-abelian case. We are currently investigating these and other issues regarding non-abelian group codes.

References


