Lattices for Distributed Source Coding - Reconstruction of a Linear function of Jointly Gaussian Sources

Dinesh Krithivasan and S. Sandeep Pradhan

University of Michigan, Ann Arbor
Presentation Overview

• Problem Formulation

• A straightforward coding scheme using random coding

• A new coding scheme using lattice coding
  – Motivation for the coding scheme
  – An overview of lattices and lattice coding
  – A lattice based coding scheme

• Results and Extensions

• Conclusion
Problem Formulation

- Source \((X_1, X_2)\) is a bivariate Gaussian.
- \(X_1, X_2 \sim \mathcal{N}(0, 1), \mathbb{E}(X_1X_2) = \rho > 0.\)
- Encoders observe \(X_1, X_2\) separately and quantize them at rates \(R_1\) and \(R_2\).
- Decoder interested in a linear function \(Z \triangleq X_1 - cX_2.\)
- Lossy reconstruction to within a mean squared error distortion \(D.\)
- Objective: Achievable rates \((R_1, R_2)\) at distortion \(D.\)
Berger-Tung based Coding Scheme

- Encoders: Quantize $X_1$ to $W_1$ and $X_2$ to $W_2$. Transmit $W_1$ and $W_2$.
- Decoder: Reconstruct $\hat{Z} \triangleq \mathbb{E}(Z | W_1, W_2)$.
- Can use Gaussian test channels for $P(W_1 | X_1)$ and $P(W_2 | X_2)$ to derive achievable rates and distortion.
- Known to be optimal for $c < 0$ with Gaussian test channels.
Motivation for our Coding Scheme

- Korner and Marton considered lossless coding of $Z = X_1 \oplus X_2$
- Coding Scheme if the encoding is centralized?
Motivation for our Coding Scheme

- Korner and Marton considered lossless coding of $Z = X_1 \oplus X_2$

- Coding Scheme if the encoding is centralized?
  - Compute $Z = X_1 \oplus X_2$. Compress it to $f(Z)$ and transmit.
  - $f(\cdot)$ is any good source coding scheme.
Motivation for our Coding Scheme

• Korner and Marton considered lossless coding of $Z = X_1 \oplus X_2$

• Coding Scheme if the encoding is centralized?
  – Compute $Z = X_1 \oplus X_2$. Compress it to $f(Z)$ and transmit.
  – $f(\cdot)$ is any good source coding scheme.

• Suppose $f(\cdot)$ distributes over $\oplus$?
Motivation for our Coding Scheme

- Korner and Marton considered lossless coding of $Z = X_1 \oplus X_2$

- Coding Scheme if the encoding is centralized?
  - Compute $Z = X_1 \oplus X_2$. Compress it to $f(Z)$ and transmit.
  - $f(\cdot)$ is any good source coding scheme.

- Suppose $f(\cdot)$ distributes over $\oplus$?
  - Compress $X_1$ and $X_2$ as $f(X_1)$ and $f(X_2)$.
  - Decoder computes $f(X_1) \oplus f(X_2) = f(X_1 \oplus X_2) = f(Z)$
Motivation for our Coding Scheme

- Korner and Marton considered lossless coding of $Z = X_1 \oplus X_2$

- Coding Scheme if the encoding is centralized?
  - Compute $Z = X_1 \oplus X_2$. Compress it to $f(Z)$ and transmit.
  - $f(\cdot)$ is any good source coding scheme.

- Suppose $f(\cdot)$ distributes over $\oplus$?
  - Compress $X_1$ and $X_2$ as $f(X_1)$ and $f(X_2)$.
  - Decoder computes $f(X_1) \oplus f(X_2) = f(X_1 \oplus X_2) = f(Z)$

- No difference between centralized and distributed coding.

- Choosing $f(\cdot)$ as a linear code will work.
An overview of Lattices

• An $n$ dimensional lattice $\Lambda$ - collection of integer combinations of columns of a $n \times n$ generator matrix $G$.

• Nearest neighbor quantizer
  
  $$Q_{\Lambda}(x) \triangleq \{\lambda \in \Lambda: \|x - \lambda\| \leq \|x - \lambda'\| \ \forall \lambda' \in \Lambda\}.$$  

• Quantization error : $x \mod \Lambda \triangleq x - Q_{\Lambda}(x)$.

• Voronoi region $V_0(\Lambda) \triangleq \{x \in \mathbb{R}^n: Q_{\Lambda}(x) = 0^n\}$.

• The second moment $\sigma^2(\Lambda)$ of lattice $\Lambda$ - expected value per dimension of a random vector uniformly distributed in $V_0(\Lambda)$.

• Normalized second moment $G(\Lambda) = \frac{\sigma^2(\Lambda)}{V^{2/n}(\Lambda)}$. $V = \int_{V_0} dx$. 
Introduction to Lattice Codes

- Lattice code - a subset of the lattice points are used as codewords.
- Have been used for both source and channel coding problems.
- Various notions of goodness:
  - A good source $D$-code if $\log(2\pi e G(\Lambda)) \leq \epsilon$ and $\sigma^2(\Lambda) = D$.
  - A good channel $\sigma_{\bar{z}}^2$-code if it achieves the Poltyrev exponent on the unconstrained AWGN channel of variance $\sigma_{\bar{z}}^2$. 
Introduction to Lattice Codes

- Lattice code - a subset of the lattice points are used as codewords.
- Have been used for both source and channel coding problems.
- Various notions of goodness:
  - A good source $D$-code if $\log(2\pi eG(\Lambda)) \leq \epsilon$ and $\sigma^2(\Lambda) = D$.
  - A good channel $\sigma^2_\delta$-code if it achieves the Poltyrev exponent on the unconstrained AWGN channel of variance $\sigma^2_\delta$.
- Why lattice codes?
Introduction to Lattice Codes

- Lattice code - a subset of the lattice points are used as codewords.

- Have been used for both source and channel coding problems.

- Various notions of goodness:
  - A good source $D$-code if $\log(2\pi e G(\Lambda)) \leq \epsilon$ and $\sigma^2(\Lambda) = D$.
  - A good channel $\sigma_z^2$-code if it achieves the Poltyrev exponent on the unconstrained AWGN channel of variance $\sigma_z^2$.

- Why lattice codes?
  - Lattices achieve the Gaussian rate distortion bound
  - Lattice encoding *distributes* over the function $Z = X_1 - cX_2$
Nested Lattice Codes

- $(\Lambda_1, \Lambda_2)$ is a nested lattice if $\Lambda_1 \subset \Lambda_2$.

- Nested lattice codes widely used in multiterminal communications.

- Need nested lattice codes such that both $\Lambda_1$ and $\Lambda_2$ are good source and channel codes.

- Such nested lattices termed “good nested lattice codes”.

- Known that such nested lattices do exist in sufficiently high dimension.
The Coding Scheme

- A good nested lattice code \((\Lambda_{11}, \Lambda_{12}, \Lambda_2)\) with \(\Lambda_2 \subset \Lambda_{11}, \Lambda_{12}\).

- Dithers \(U_i \sim \text{Unif } \mathcal{V}_0(\Lambda_{1i}), i = 1, 2\).
- \(\sigma^2(\Lambda_{11}) = q_1, \sigma^2(\Lambda_{12}) = \frac{D\sigma^2_Z}{\sigma^2_Z - D} - q_1, \sigma^2(\Lambda_2) = \frac{\sigma^4_Z}{\sigma^2_Z - D}\)
- Note: Second encoder scales the input before encoding.
The Coding Scheme contd.

- Rate of a nested lattice \((\Lambda_1, \Lambda_2)\) is \(R = \frac{1}{2} \log \frac{\sigma_2^2(\Lambda_2)}{\sigma_2^2(\Lambda_1)}\).

- Thus, Encoder rates are

\[
R_1 = \frac{1}{2} \log \frac{\sigma_Z^4}{q_1(\sigma_Z^2 - D)}, \quad R_2 = \frac{1}{2} \log \frac{\sigma_Z^4}{D\sigma_Z^2 - q_1(\sigma_Z^2 - D)}.
\]

- Sum rate \(R_1 + R_2 = \log \frac{2\sigma_Z^2}{D}\) achievable at distortion \(D\).
The Coding Scheme contd.

- Rate of a nested lattice \((\Lambda_1, \Lambda_2)\) is \(R = \frac{1}{2} \log \frac{\sigma^2(\Lambda_2)}{\sigma^2(\Lambda_1)}\).

- Thus, Encoder rates are

\[
R_1 = \frac{1}{2} \log \frac{\sigma_Z^4}{q_1(\sigma_Z^2 - D)}, \quad R_2 = \frac{1}{2} \log \frac{\sigma_Z^4}{D\sigma_Z^2 - q_1(\sigma_Z^2 - D)}.
\]

- Sum rate \(R_1 + R_2 = \log \frac{2\sigma_Z^2}{D}\) achievable at distortion \(D\).

**Theorem 1** The set of all rate-distortion tuples \((R_1, R_2, D)\) that satisfy

\[
2^{-2R_1} + 2^{-2R_2} \leq \left(\frac{\sigma_Z^2}{D}\right)^{-1}
\]

are achievable.
Outline of the Proof

• Key Idea: Distributive property of lattice mod operation

\[(x \mod \Lambda + y) \mod \Lambda = (x + y) \mod \Lambda \quad \forall x, y\]

• Using this, one of the mod-$\Lambda_2$ operation can be removed from the signal path.

• Simplified but equivalent coding scheme is

\[\begin{array}{c}
e_{q_1} \\
X_1^n \\
\downarrow \\
+ \\
\downarrow \\
\text{mod } \Lambda_2 \\
\downarrow \\
\hat{Z} \\
\downarrow \\
1 - \frac{D}{\sigma_z^2} \\
e_{q_2} \\
cX_2^n \\
\downarrow \\
\end{array}\]

• $e_{q_1}$ and $e_{q_2}$ are subtractive dither quantization noises independent of the sources.
Proof Outline contd.

- Effective dither is $e_q = e_{q_1} - e_{q_2}$ with variance $\frac{D\sigma^2_Z}{\sigma^2_Z - D}$.
- Decoder operation can be described as
  \[
  \hat{Z} = \left( \frac{\sigma^2_Z - D}{\sigma^2_Z} \right) \left( (Z + e_q) \mod \Lambda_2 \right).
  \]
- But $\Lambda_2$ is a good $\frac{\sigma^4_Z}{\sigma^2_Z - D}$ channel code.
- Implies that $((Z + e_q) \mod \Lambda_2) = (Z + e_q)$ with high probability.
- Decoder reconstruction $\hat{Z} = \left( \frac{\sigma^2_Z - D}{\sigma^2_Z} \right) (Z + e_q)$.
- Can be checked that $\mathbb{E}(Z - \hat{Z})^2 = D$. 
Comments about the Lattice Coding Scheme

• Larger rate region than the random coding scheme for some source statistics and some range of $D$.

• Coding scheme here is lattice vector quantization followed by “correlated” lattice-structured binning.

• Reconstructing a function of more than two sources
  
  $Z = \sum_{i=1}^{K} c_i X_i$
  
  – Some sources coded together using the lattice coding scheme. Others coded separately using Berger-Tung coding.
  – Possible to present a unified rate region combining all such strategies.
Comparison of the Rate Regions

- Compare sum rates for $\rho = 0.8$ and $c = 0.8$.

- Lattice based scheme - lower sum rate for small distortions.
- Time sharing between the two schemes - Better rate region than either scheme alone.
Range of Values for Lower Sum Rate

- Shows where \((\rho, c)\) should lie for lattice sum rate to be lower than Berger-Tung sum rate for some distortion \(D\).

- Contour marked \(R\) - Lattice sum rate lower by \(R\) units.

- Improved performance only for \(c > 0\).
Conclusion

• Considered lossy reconstruction of a function of the sources in a distributed setting.

• Presented a coding scheme of vector quantization followed by “correlated” binning using lattices.

• Improves upon the rate region of the natural random coding scheme.

• Currently working on extending the scheme to discrete sources and arbitrary functions.