RIGID INNER FORMS OVER LOCAL AND GLOBAL FIELDS (FOR JUNIOR NUMBER THEORY DAYS AT JHU)

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1. Introduction

Goal: State Langlands correspondence for quasi-split reductive groups, then use the fact that every group is an inner form of a quasi-split group to state Langlands correspondence for general groups. This entails being careful about how we deal with the data of an inner twist (namely, passing to rigid inner twists).

Notation:
- $F =$ local or global field;
- $\overline{F}$ fixed algebraic closure, $F^s \subseteq \overline{F}$ separable closure;
- $\Gamma = \mathrm{Gal}(F^s/F)$, $W_F =$ Weil group of F.

2. Local Langlands correspondence

- $G =$ connected reductive group over F;
- $\widehat{G} =$ dual group of G obtained by dualizing based root datum, connected reductive group over $\mathbb{C}$, has Γ-action from action on root datum of G;
- $W_F' = W_F \times \mathrm{SL}_2(\mathbb{C})$, ${}^LG = \widehat{G} \rtimes W_F$.
- A *tempered L-parameter* $\varphi \colon W_F' \to {}^LG$ is a homomorphism satisfying some nice properties (namely, it is a morphism of W_F-extensions, sends W_F to semisimple elements, and has image whose projection to $\widehat{G}$ is bounded).

Conjecture A; There exists a map with finite fibers

$$LL \colon \Pi_{\mathrm{temp}}(G(F)) \to \Phi_{\mathrm{temp}}(G),$$

where the left-hand side is isomorphism classes of tempered representations of $G(F)$ and the right-hand side is ${}^LG$-conjugacy classes of tempered L-parameters of G. This map satisfies some additional nice properties that we won't discuss here.

- For a fixed parameter φ, we want to understand the fiber $\Pi_\varphi = LL^{-1}([\varphi])$, which is called an *$L$-packet*.
- Assume G is *quasi-split*; this means it has a Borel subgroup over F; a non-example is $\mathrm{SL}_1(D)$, the group of units in a central simple F-algebra D with reduced norm equal to 1.

Conjecture B: There exists an injection (bijection if F is non-archimedean)

$$\iota\colon \Pi_{\varphi} \to \mathrm{Irr}(\mathcal{S}_{\varphi}),$$

where $S_{\varphi} = Z_{\widehat{G}}(\varphi)$ and $\mathcal{S}_{\varphi} = \pi_0(S_{\varphi}/Z(\widehat{G})^{\Gamma})$, the latter of which is a finite group. This map satisfies some additional nice properties that we will not go into here.

3. General case

Problem: Conjecture B is false for general G.

Idea: For a general connected reductive G over F, there is a unique quasi-split inner form G' of G.

- Recall from Galois cohomology that we have bijections

$$\{\text{Forms of G}/\cong\} \leftrightarrow H^1(\Gamma, \mathrm{Aut}_G(F^s)),$$

$$\{\text{Inner forms of G}/\cong\} \leftrightarrow H^1(\Gamma, \mathrm{Inn}_G(F^s)) = H^1(F, G_{\mathrm{ad}}).$$

- For example, if $G = \mathrm{GL}_n$, then $\mathrm{U}(n)$ is an outer form and $\mathrm{GL}_m(D)$, where D is a central F-division algebra with $\dim(D) = d^2$, $md = n$, is an inner form.
- **Fact**: $\{\text{Inner forms of } G\} = \{G' \mid {}^L(G') = {}^LG\}$.

Idea: Use data of (G', ψ), G' quasi-split, $G_{F^s} \xrightarrow{\sim,\psi} G'_{F^s}$ to state LLC for G.

Problem: There are automorphisms of the inner twists (G', ψ) which permute $\Pi_{\varphi}(G')$.

Example; $G = \mathrm{SL}_2$, $F = \mathbb{R}$, $(G', \psi) = (\mathrm{SL}_2, \mathrm{id})$; then $\mathrm{Ad}(g)$, $g = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathrm{PGL}_2(\mathbb{R})$ gives an automorphism of $(\mathrm{SL}_2, \mathrm{id})$ which is an outer automorphism of $\mathrm{SL}_2(\mathbb{R})$, and thus nontrivially permutes the representations of the latter group; more specifically, it exchanges holomorphic and anti-holomorphic discrete series representations.

Solution (Vogan): Replace (G', ψ) with (G', ψ, z), the latter of which is called a *pure inner twist*, where $z \in Z^1(F, G)$ maps to $\psi \in Z^1(F, G_{\mathrm{ad}})$ (we identify ψ with the corresponding cocycle).

There is an appropriate notion of isomorphisms of pure inner twists of a group G; this is set up so that automorphisms of a a fixed pure inner twist fix the L-packets.

Problem: Since $H^1(F, G) \to H^1(F, G_{\mathrm{ad}})$ isn't surjective, one can't always enrich an inner twist to a pure inner twist. Using the previous example, SU_2 is a nontrivial inner form of SL_2 which is not captured by $H^1(\mathbb{R}, \mathrm{SL}_2) = \{*\}$.

- Note that $H^1(F,G)$ = fpqc cohomology of G as a sheaf on Sch/F, which is the same as isomorphism classes of fpqc G-torsors on Sch/F.
- **Idea**: Replace $H^1(F,G)$ with "bigger" $H^1_{\mathrm{bas}}(\mathcal{E},G)$ = G-torsors on $\mathcal{E} \to \mathrm{Sch}/F$ (subject to some conditions) such that we have a surjection

$$H^1_{\mathrm{bas}}(\mathcal{E},G) \twoheadrightarrow H^1(F,G_{\mathrm{ad}}).$$

- More specifically, given $\mathcal{E} \to \mathrm{Sch}/F$, our set $Z^1_{\mathrm{bas}}(\mathcal{E},G)$ is all G-torsors $\mathscr{T}$ on $\mathcal{E}$ such that $\mathscr{T} \times^G G_{\mathrm{ad}}$ descends to a G_{ad}-torsor T over F.
- There is an obvious map $Z^1_{\mathrm{bas}}(\mathcal{E},G) \to Z^1(F,G_{\mathrm{ad}})$; we choose $\mathcal{E}$ carefully so that this is surjective.
- By enriching inner twists to triples $(G',\psi,\mathscr{T})$, called *rigid inner twists*, we are now able to enrich *any* inner twist. Like pure inner twists, rigid inner twists have automorphisms which preserve L-packets.

4. What is $\mathcal{E}$?

- Fix an abelian group scheme A over F.
- Recall that we have a correspondence

$$\{A - \text{torsors } /F\} \leftrightarrow \{A_{\overline{F}} - \text{torsors } /\overline{F}, \phi \text{ descent datum}\}.$$

- Fix a a Cech 2-cocycle valued in A with respect to the cover $\mathrm{Spec}(\overline{F}) \to \mathrm{Spec}(F)$.
- An *a-twisted torsor over F* is a pair of an $A_{\overline{F}}$-torsor and an *a-twisted descent datum* ϕ (this means that its differential is not trivial, like a descent datum's is, but rather is translation by the fixed element a).
- $\mathcal{E}_a \to \mathrm{Sch}/F$ is the category whose fiber over U is the set of all a-twisted torsors over U.
- As a remark, note that the map $a \mapsto \mathcal{E}_a$ induces a bijection between the second Cech cohomology of $\overline{F}/F$ valued in A and (isomorphism classes of) A-gerbes (split over $\overline{F}$).
- $\{G\text{-torsors on } \mathcal{E}_a\} \leftrightarrow \{A\text{-equivariant } a\text{-twisted } G\text{-torsors}/F\}$.

What are A and a for us?

- $A = u = \varprojlim_{E,n} \mathrm{Res}_{E/F}(\mu_n)/\mu_n$, where the limit is over all finite Galois extensions E/F and natural numbers n.
- Heuristically, this u comes from using local Poitou-Tate duality to deduce that $\mathrm{Hom}_F(A,Z)$ (and thus A) should capture the norm groups $N_{E/F}(X^*(Z))$ for all E/F finite Galois and for all finite multiplicative Z, and capture n-torsion for all $n \in \mathbb{N}$.
- $a = -1 \in \widehat{\mathbb{Z}} = H^2_{\mathrm{fppf}}(F,u)$, using local class field theory. Note that when F has characteristic zero, one can use Galois cohomology and replace $\mathcal{E}$ with a Galois gerbe and define rigid inner forms using group extensions (Kaletha's original approach).

5. New conjectures

Theorem (Kaletha in characteristic zero, D. for function fields): Have a functorial isomorphism

$$H^1_{\text{bas}}(\mathcal{E}, G) \xrightarrow{\sim} \pi_0(Z(\widehat{G})^+)^*,$$

where $Z(\widehat{G})^+$ is the preimage of $Z(\widehat{G})^\Gamma$ under the isogeny

$$\widehat{G/Z_{\text{der}}} \to \widehat{G}.$$

This result lets us relate rigid inner forms to $\widehat{G}$; we thus obtain:

Conjecture (Kaletha in characteristic zero, D. for function fields): Let G' be a quasi-split inner twist of G. There is a commutative diagram

$$\begin{array}{ccc} \Pi_\varphi & \xrightarrow{\iota} & \text{Irr}(\pi_0(S_\varphi^+)) \\ \downarrow & & \downarrow \\ H^1_{\text{bas}}(\mathcal{E}, G') & \longrightarrow & \pi_0(Z(\widehat{\widehat{G}})^+)^*, \end{array}$$

where the top map is a bijection, Π_φ is a set of isomorphism classes of representations (x, π), of rigid inner forms of G (that is, $x = (G^*, \psi, \mathscr{T})$ is a rigid inner twist and π is a representation in the φ L-packet of G^*), and S_φ^+ is the preimage of S_φ in $\widehat{G/Z_{\text{der}}}$. This map satisfies some additional nice conditions that we will not discuss here.

6. Global situation

Now G is a connected reductive group over a global field F. We want to understand what the automorphic representations of $G(\mathbb{A})$ are, and, given such a representation π, what its multiplicity is in the discrete spectrum of G. We can help answer these questions using the global Langlands correspondence.

- Fix a *global Arthur parameter* $\varphi \colon L_F \to {}^L G$, where L_F is the conjectural Langlands group of F. This gives local parameters φ_v for every place v. Fix a quasi-split inner form (G', ψ) of G.
- From φ, we have a global L-packet

  $$\Pi_\varphi = \{\otimes_v \pi_v | \pi_v \in \Pi_{\varphi_v}, \pi_v \text{ is a representation of } G, \iota_v(\pi_v) = 0 \text{ for almost all } v\}.$$

- **Proposition** (Kaletha, D.) Π_φ consists of automorphic representations.
- **Problem**: The local bijection ι_v depends on enriching (G'_{F_v}, ψ) to a rigid inner twist for each v.
- We have to do this coherently over all v if we hope to use product formulas over all places to find multiplicities of automorphic representations.

- **Solution** (Kaletha, D.) Enrich (G', ψ) to a *global rigid inner twist* $(G', \psi, \mathscr{T})$ where
$$\mathscr{T} \in Z^1(\mathcal{E}_{\dot{V}}, G),$$
$\mathcal{E}_{\dot{V}}$ = global gerbe.
- Have maps $\mathcal{E}_v \xrightarrow{c_v} \mathcal{E}_{\dot{V}}$ for all v, where $\mathcal{E}_v$ is local gerbe.
- Can define coherent family of enrichments by enriching (G'_{F_v}, ψ) to $(G'_{F_v}, \psi, c_v^* \mathscr{T})$.
- Here $\mathcal{E}_{\dot{V}}$ corresponding to canonical class $\xi \in H^2_{\text{fppf}}(F, P)$, is quite difficult to construct (easy in local case).
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$$P = \varprojlim_{E,S,n} \frac{[\text{Res}_{E/F}(\mu_n^S)/\mu_n^s]}{[\prod_{v \in S} \text{Res}_{E^{\dot{v}}/F}(\mu_n)]/\mu_n},$$
where the limit is over all finite Galois E/F, all $n \in \mathbb{N}$, and all finite sets of places S. Here $\dot{V}$ is a set of lifts of all places of F in F^s and $E^{\dot{v}}/F$ is the decomposition field of $\dot{v}$ in E/F.
- The canonical class ξ is constructed using local canonical classes and complexes of tori.