Online Appendix for
A Pooling Analysis of Two Simultaneous Online Auctions

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A. Equilibrium Analysis

This section provides theoretical support for the strategy outlined in §2.2. For convenience, to ease the presentation of the case in which auctioneer a's reservation price is not met, we imagine there are m_a+1 "reservation" dedicated bidders installed in auction a at the market's outset, where each such bidder's valuation equals auctioneer a's reservation price. Thus in this section of the appendix we assume without loss of generality that $d_a \geq m_a + 1$, $a = 1, 2$. See §2.1 for informational assumptions. Note that the equilibrium analysis presented below generalizes the model from the main text, by allowing for non-identically distributed bidder valuations.

Note that the second sentence of σ^*(e) implies that shared bidders wait for all dedicated bidders to enter and finish their bidding before bidding in a situation in which both auctions have the same standing bid. This implication is used directly within the proofs of Claims 3 and Lemma 2 (subcase (b2) in the latter).

Solution concept. To show σ^* is a Bayesian Nash Equilibrium (Proposition 1), we need to show that if all players $j \neq i$ play σ^* during the game, then player i cannot gain by deviating from σ^*. The argument has two major steps: First we bound from below i's payoff when all players use σ^* (with the help of Lemma 1), then prove that this is indeed an upper bound for any strategy i might choose to play (aided by Lemma 2). Along the way, virtually every proof is by contradiction.

Notation. Several new terms are required: Let

$$\begin{aligned}
W_a^t &\triangleq \text{set of winning bidders in auction } a \text{ at stage } t \text{ when } \sigma^* \text{ is played} \\
&\quad\ \text{by all bidders,} \\
g_a^t &\triangleq \text{standing bid in auction } a \text{ at stage } t, \\
p_a^* &\triangleq \text{ending price in auction } a \text{ when } \sigma^* \text{ is played by all bidders,} \\
v_j &\triangleq \text{the valuation of bidder } j, \text{ and} \\
AE\,(H,p) &\triangleq \{j \in H \mid v_j \geq p\}.
\end{aligned}$$

As the price in an isolated, m-item auction under σ^* is simply the largest value less than or equal to at least $m+1$ bidders' valuations, for our pooled setup we define sets below that characterize each auction's price in this same fundamentally straightforward manner, only with attention necessarily paid to the shared bidders' behavior. To this end, for $a = 1, 2$ define

$$p_a \triangleq \max\{0, \delta, 2\delta, \ldots\} \text{ such that } |AE\,(d_a, p_a)\,| \geq m_a + 1, \tag{A1}$$

$$p_{as} \triangleq \max\{0, \delta, 2\delta, \ldots\} \text{ such that } |AE\,(d_a \cup S, p_{as})\,| \geq m_a + 1, \text{ and} \tag{A2}$$

$$p_{12s} \triangleq \max\{0, \delta, 2\delta, \ldots\} \text{ such that } |AE\,(d_1 \cup d_2 \cup S, p_{12s})\,| \geq m_1 + m_2 + 1. \tag{A3}$$

Equilibrium Proof. Most of the setup so far has been toward properly defining the values p_1, p_2, p_{1s}, p_{2s}, and p_{12s}; using these values, Lemma 1 establishes an upper bound on each auctions' transaction price when σ^* is played, and Lemma 2 shows that unilateral deviations by i cannot reduce i's transaction price below these bounds. While each proof along the way requires slightly different variables, we try to economize new notation by reusing labels with a spirit of consistency: For example, t denotes whatever stage is of interest for the argument at hand, and we denote bidders who play (or are shown to play) σ^* by the letters h, j, l and k, and reserve the letter i for the bidder who deviates from σ^*. Hopefully these commonly-used labels will improve readability more than they confound it. First, we present three claims that will be useful when proving Lemma 1.

Claim 1. *If $p_{1s} \geq p_2$ and $p_{2s} \geq p_1$, then $p_1 \leq p_{12s}$ and $p_2 \leq p_{12s}$.*

Proof. We prove $p_2 \leq p_{12s}$; the proof of $p_1 \leq p_{12s}$ follows by symmetry. Suppose that $p_2 > p_{12s}$; we produce a contradiction. With $p_2 > p_{12s}$, the definition of p_{12s} in equation (A3) implies

$$m_1+m_2 \geq |AE(d_1 \cup d_2 \cup S, p_2)| = |AE(d_2, p_2)|+|AE(d_1 \cup S, p_2)| \geq m_2+1+|AE(d_1 \cup S, p_2)|,$$

where the equality follows from $d_1,\ d_2,\ S$ disjoint and the last inequality follows from (A1). Thus $m_1 - 1 \geq |AE(d_1 \cup S, p_2)|$, which together with the Claim's assumption $p_{1s} \geq p_2$ yields a contradiction to (A2). □

Claim 2. *Suppose all bidders play σ^* in the game. Then the auction prices cannot both exceed p_{12s}.*

Proof. Suppose not. Assume without loss of generality that auction 2 is the first to reach standing bid $p_{12s} + \delta$. Let t be the stage at which the standing bid in auction 1 jumps from p_{12s} to $p_{12s} + \delta$ (w.o.l.o.g. we assume the auctioneers update standing bids at the beginning of each stage), and let j be the bidder who submits the unsuccessful bid causing this jump. Note that by σ^*, $j \cup W_1^t \cup W_2^t$ is a set of $m_1 + m_2 + 1$ unique bidders, and $j \cup W_1^t \cup W_2^t \subseteq AE(d_1 \cup d_2 \cup S, p_{12s} + \delta)$, since σ^* stipulates that bids not exceed true valuations. Hence, $m_1 + m_2 + 1 \leq |AE(d_1 \cup d_2 \cup S, p_{12s} + \delta)|$, which contradicts (A3). □

Claim 3. *Suppose all bidders play σ^* in the game. If $S \cap W_a^t \neq \emptyset$, then $g_a^t \leq p_{-a}^*$, that is, during the game no shared bidder is ever winning at a standing bid that exceeds the cheapest final price among the two auctions.*

Proof. Suppose $S \cap W_1^t \neq \emptyset$ and $g_1^t > p_2^*$. We derive a contradiction. (The argument when a shared bidder is winning in auction 2 is symmetric.) Let $p \triangleq g_1^t$, i.e., p is the auction 1 standing bid at stage t. Let j be the last bidder in $S \cap W_1^t$ to bid p successfully in auction 1, and let $t' \leq t$ denote the stage at which this bid was placed. Since $p > p_2^* \geq g_2^{t'}$ yields $p - \delta \geq g_2^{t'}$, σ^*(d) implies j's bid must have been cast when the auction 1 and 2 standing bids were tied (at $p - \delta$). Consider $k \in AE(d_1, p)$. Because j and k follow σ^*, j's bid would have been submitted only after all d_1 bidders eligible to bid p had done so (by σ^*(e)). Since j's bid was successful, this implies $AE(d_1, p) \subset W_1^{t'+1}$. Since no more shared bidders bid p successfully in auction 1 during the interval $[t' + 1, t - 1]$, and the above shows that all bidders in d_1 who are capable of bidding p are already winning an item at stage $t' + 1$, we have that the stage t standing bid of p in auction 1 must be set by an unsuccessful bid of p by a shared bidder l during the interval $[t' + 1, t - 1]$. By σ^*(e), bidder l bids p unsuccessfully in auction 1 only if the standing bid in each auction equals $p - \delta$ and the most recent m_a

bids in auction a have been successful, for $a = 1, 2$. (I.e., bidding p would be unsuccessful in either auction.) After bidding p unsuccessfully in auction 1, bidder l is still left without an item. Suppose that at her next turn, the standing bid in auction 2 is still only $p - \delta$. By σ^*(d) l will bid p in auction 2, which will be unsuccessful. Thus, the final price of auction 2, p_2^*, is at least p – contradicting our assumption that $p > p_2^*$, and hence proving the claim. $\square$

We are now ready to establish the first of two lemmas used to prove the equilibrium proposition.

Lemma 1 (Upper bound on p_a^*). *Suppose all bidders play σ^*, and let p_a^* be the final price in auction a. Then*

$$(p_1^*, p_2^*) \leq \begin{cases} (p_1, p_{2s}) & \text{if } p_1 > p_{2s}; \\ (p_{1s}, p_2) & \text{if } p_{1s} < p_2; \\ (p_{12s}, p_{12s}) & \text{otherwise.} \end{cases} \tag{A4}$$

Proof. Note that the three cases in equation (A4) are the analogues to U_1, U_2, and E from the paragraph following the statement of Proposition 2 from the main text. We break up the proof of Lemma 1 along these three cases, in each case combining bidder behavior dictated by σ^* with the formal definitions of p_1, p_2, p_{1s}, p_{2s}, and p_{12s} to yield a contradiction if Lemma 1 does not hold.

Case $p_{1s} < p_2$. Consider g_1^t, the standing bid in auction 1 at some stage t. Let j be the bidder whose bid at some stage $t' < t$ set the standing bid in auction 1 to g_1^t. Since $(j \cup W_1^{t'}) \subseteq (d_1 \cup S)$, $m_1 + 1 = |j \cup W_1^{t'}| \leq |AE\left(d_1 \cup S, g_1^t\right)|$, and hence $g_1^t > p_{1s}$ contradicts (A2). In particular, $p_1^* \leq p_{1s}$, and $g_1^t \leq p_{1s}$ together with $p_{1s} < p_2$ yields $g_1^t \leq p_2 - \delta$ (we make use of this latter result below).

Now, consider $g_2^{t''}$, the standing bid of auction 2 at some stage t''. Suppose $g_2^{t''} \geq p_2 + \delta$; we show a contradiction. Suppose k set the standing bid in auction 2 to $g_2^{t''}$ at some stage $t''' < t''$. Note that by σ^* and $g_1^t \leq p_2 - \delta$ for all t, no S bidder bids more than p_2 in auction 2. Hence, $(k \cup W_2^{t'''}) \subseteq d_2$, so $m_2 + 1 = |k \cup W_2^{t'''}| \leq |AE\left(d_2, g_2^{t''}\right)|$, which contradicts (A1) since $g_2^{t''} \geq p_2 + \delta$. Hence, $g_2^{t''} \leq p_2$ for all stages t'', so $p_2^* \leq p_2$.

Case $p_{2s} < p_1$. By symmetry to case $p_{1s} < p_2$, we have $p_1^* \leq p_1$ and $p_2^* \leq p_{2s}$.

Case $p_{1s} \geq p_2$ and $p_{2s} \geq p_1$. We show $p_1^* \leq p_{12s}$; by symmetry, $p_2^* \leq p_{12s}$ then follows. Suppose $p_1^* \geq p_{12s} + \delta$; we produce a contradiction. Let j be the bidder whose unsuccessful bid set the standing bid of auction 1 to $p_{12s}+\delta$ at some stage t. If $j \in S$, then by σ^* either $g_2^t \geq p_{12s}+\delta$ or j would also be compelled to bid $p_{12s}+\delta$ unsuccessfully in auction 2 (if another bidder had not already done so prior to j's next bid). But, this cannot happen since by Claim 2 $p_1^* > p_{12s}$ implies $p_2^* \leq p_{12s}$. Hence, $j \in d_1$. Since j's bid of $p_{12s}+\delta$ is unsuccessful, we must have $g_1^{t+1} = p_{12s}+\delta$. Furthermore, because Claim 2 implies $p_2^* \leq p_{12s}$, Claim 3 yields $S \cap W_1^{t+1} = \emptyset$. Thus $(j \cup W_1^{t+1}) \subseteq d_1$, so by σ^*, $m_1+1 = |j \cup W_1^{t+1}| \leq |AE\left(d_1, g_1^{t+1}\right)|$, which together with $g_1^{t+1} > p_{12s} \geq p_1$ (the latter inequality by Claim 1) contradicts (A1). Hence, the proof of Lemma 1 is complete. □

We next introduce notation that will be useful in proving Lemma 2. Let $\overline{W}_a^t$ be the analogue to W_a^t when i plays $\bar{\sigma} \neq \sigma^*$ and $j \neq i$ plays σ^* in the game, and let $\bar{p}_a$ denote the ending price in auction a. It is possible that i wins multiple items; to each item won by i create a bidder h_i associated to i and let $\overline{W}_{ai}$ be the set of i's auction a transacting bidders, where clearly $\overline{W}_{ai} \subseteq \overline{W}_a$.

Lemma 2. *Suppose that $j \neq i$ plays σ^* and i plays $\bar{\sigma} \neq \sigma^*$, $|\overline{W}_{1i}| \geq 1$, and auction a ends with price $\bar{p}_a$. Then $\bar{p}_1 \geq p_1^*$.*

Proof. First, suppose $\bar{p}_2 < p_2$. Then since all bidders in d_2 follow σ^* ($|\overline{W}_{1i}| \geq 1$ implies $i \notin d_2$), $AE(d_2, p_2) \subseteq \overline{W}_2$, so $|AE(d_2, p_2)| \leq |\overline{W}_2| = m_2$, contradicting (A1). Hence $\bar{p}_2 \geq p_2$; we make use of this below, where case by case we show that Lemma 1 can be used to prove $\bar{p}_1 \geq p_1^*$.

Case $p_{1s} < p_2$. Suppose $\bar{p}_1 < p_{1s}$; we produce a contradiction. We first show that $(S \setminus i) \cap \overline{W}_2 = \emptyset$. If not, for $j \in (S \setminus i) \cap \overline{W}_2$, let t be the stage that j submitted her winning bid in auction 2. Then by j playing σ^* $g_2^t \geq \bar{p}_2 - \delta \geq p_2 - \delta$. But, since $p_2 - \delta \geq p_{1s} > \bar{p}_1 \geq g_1^t$, we have $g_1^t \leq p_2 - 2\delta$, which implies j violates σ^*(d) by bidding in auction 2 at stage t.

Summarizing the above, we have $\bar{p}_1 < p_{1s}$, $\bar{p}_2 \geq p_2$, and $(S \setminus i) \cap \overline{W}_2 = \emptyset$. Then all bidders but i playing σ^* implies $(AE(d_1 \cup S, p_{1s}) \setminus i) \subseteq (\overline{W}_1 \setminus \overline{W}_{1i})$. Equation (A2) implies $|(AE(d_1 \cup S, p_{1s}) \setminus i)| \geq m_1$, which together with $|\overline{W}_{1i}| \geq 1$ implies $|\overline{W}_1| \geq m_1+1$, a contradiction to auction 1 having m_1 items for sale. Hence, $\bar{p}_1 \geq p_{1s}$.

Case $p_1 > p_{2s}$. Suppose $\bar{p}_1 < p_1$. Since $j \neq i$ plays σ^*, $(AE(d_1, p_1) \setminus i) \subseteq (\overline{W}_1 \setminus \overline{W}_{1i})$. Equation (A1) implies $|AE(d_1, p_1) \setminus i| \geq m_1$, and since $|\overline{W}_{1i}| \geq 1$, we get $|\overline{W}_1| \geq m_1+1$, a contradiction.

Case $p_{1s} \geq p_2$ and $p_{2s} \geq p_1$. To yield a contradiction if Lemma 2 fails, suppose $\bar{p}_1 < p_{12s}$. Because $p_2 \geq p_{12s}$ (by Claim 1) and $p_{1s} \geq p_2$ by assumption, we must have $p_{1s} > \bar{p}_1$ – a fact that we make use of below.

Subcase (a): $(S \setminus i) \cap \overline{W}_2 = \emptyset$. Then $j \in AE\,(d_1 \cup S, p_{1s}) \setminus i$ plays σ^* implies $j \in \overline{W}_1 \setminus \overline{W}_{1i}$, and using (A2) and $|\overline{W}_{1i}| \geq 1$ we again get $|\overline{W}_1| \geq m_1 + 1$, a contradiction.

Subcase (b): $\exists\ j \in (S \setminus i) \cap \overline{W}_2$. Bidder j follows strategy σ^*, since $j \neq i$. In the next three sub-subcases, we condition (respectively) on whether or not $\bar{p}_2$ exceeds, equals, or is less than p_{12s}.

Sub-subcase (b1): $\bar{p}_2 > p_{12s}$. We show a contradiction. $\bar{p}_2 > p_{12s} > \bar{p}_1$ implies $\bar{p}_2 - \delta > \bar{p}_1$. Let t be the stage in which j submits her winning bid in auction 2. Since $j \neq i$ follows σ^*, we have $g_2^t \in \{\bar{p}_2 - \delta, \bar{p}_2\}$ and $g_1^t \geq g_2^t$. But, $g_1^t \ \geq\ g_2^t \ \geq\ \bar{p}_2 - \delta \ >\ \bar{p}_1$ contradicts the fact that g_1^t, the standing bid in auction 1 at stage t, cannot exceed the final price in auction 1, $\bar{p}_1$.

Sub-subcase (b2): $\bar{p}_2 = p_{12s}$. Assume w.o.l.o.g. that j is the last bidder in $(S \setminus i) \cap \overline{W}_2$ to bid successfully in auction 2, and suppose this bid occurred at stage t. Bidder $j \neq i$ plays σ^*, hence j's stage t bid must have equalled $\bar{p}_2$ (note that if j bid $\bar{p}_2 + \delta$ at stage t, then σ^*(d) implies $g_1^t \geq \bar{p}_2$, contradicting $\bar{p}_1 \geq g_1^t$ and the assumption $\bar{p}_2 = p_{12s} > \bar{p}_1$). Next, as in the proof of Claim 3, note that $k \in d_2$ and j playing σ^* implies $AE\,(d_2, \bar{p}_2) \subset \overline{W}_2^{t+1}$. Let t' be the ending time of auction 2. By definition of j, during $[t+1, t']$ no bidder in $S \setminus i$ will displace any bidder $k \in \overline{W}_2^{t+1}$. However, bidder i might do so, but for each such displacement either k itself or some h_i associated to i will end up winning an item in auction 2 (since i plays $\bar{\sigma}$, i might win multiple items). In particular, each bidder in $AE\,(d_2, \bar{p}_2)$ corresponds to a unique bidder in $\overline{W}_2 \setminus (S \setminus i)$. Hence,

$$m_2 = |\overline{W}_2| \geq |AE\,(d_2, \bar{p}_2)\,| + |(S \setminus i) \cap \overline{W}_2|. \tag{A5}$$

Since $\bar{p}_2 = p_{12s}$,

$$\begin{aligned}
m_1 + m_2 \ &\leq\ |AE\,(d_1 \cup d_2 \cup S, p_{12s}) \setminus i| \quad \text{by (A3)}, \\
&=\ |AE\,(d_1 \cup d_2 \cup (S \setminus i), p_{12s}) \setminus i|, \\
&=\ |AE\left(d_2 \cup ((S \setminus i) \cap \overline{W}_2), p_{12s}\right) \setminus i| + |AE\left(d_1 \cup ((S \setminus i) \setminus \overline{W}_2), p_{12s}\right) \setminus i|, \\
&\leq\ m_2 + |AE\left(d_1 \cup ((S \setminus i) \setminus \overline{W}_2), p_{12s}\right) \setminus i| \quad \text{by (A5) and } \bar{p}_2 = p_{12s}.
\end{aligned}$$

Hence, $m_1 \le |AE\left(d_1 \cup ((S \setminus i) \setminus \overline{W}_2), p_{12s}\right) \setminus i|$. But, $j \in AE\left(d_1 \cup ((S \setminus i) \setminus \overline{W}_2), p_{12s}\right) \setminus i$ implies $v_j \ge p_{12s} > \bar{p}_1$, j plays σ^*, and hence $j \in \overline{W}_1 \setminus \overline{W}_{1i}$. Since $|\overline{W}_{1i}| \ge 1$, we have $|\overline{W}_1| \ge m_1 + 1$, a contradiction. Hence, $p_2 = p_{12s}$ implies $\bar{p}_1 \ge p_{12s}$.

Sub-subcase (b3): $\bar{p}_2 < p_{12s}$. For $j \in AE\left(d_1 \cup d_2 \cup S, p_{12s}\right) \setminus i$, $v_j \ge p_{12s} > \bar{p}_1, \bar{p}_2$ implies that by σ^* $j \in (\overline{W}_1 \cup \overline{W}_2) \setminus (\overline{W}_{1i} \cup \overline{W}_{2i})$. Together with equation (A3) this yields $|(\overline{W}_1 \cup \overline{W}_2) \setminus (\overline{W}_{1i} \cup \overline{W}_{2i})| \ge m_1 + m_2$; because $|\overline{W}_{1i}| \ge 1$, we have a contradiction to $|\overline{W}_1 \cup \overline{W}_2| = m_1 + m_2$, and Lemma 2 is proved. □

We now prove Proposition 10, merely a recasting of Proposition 1 from the main text.

Proposition 10. *No bidder i can increase her payoff by unilaterally deviating from strategy σ^* during the game.*

Proof. If i does not transact under $\bar{\sigma}$, i is clearly no better off since under σ^* i's payoff is non-negative (under σ^* i never bids above her valuation). Suppose w.o.l.o.g. that $|\overline{W}_{1i}| \ge 1$. If $v_i \le \bar{p}_1$ then i has non-positive payoff when playing $\bar{\sigma}$; since i's payoff under σ^* is non-negative, we are done. If $v_i > \bar{p}_1$, then i's payoff under $\bar{\sigma}$ is bounded above by $v_i - \bar{p}_1$, where the payoff could be lower since i can win multiple items. Since under σ^* i does not win multiple items (when holding a winning bid, i never submits another winning bid under σ^*), we have that i's payoff under σ^* is equal to v_i minus i's transaction price if i transacts; thus, we need only show that under σ^* i transacts and pays at most $\bar{p}_1$. Note that either $i \in d_1$ or $i \in S$, so i has access to auction 1. Thus under σ^*, $v_i > \bar{p}_1 \ge p_1^*$ (where the second inequality follows from Lemma 2) implies i transacts. In particular, $i \in d_1$ implies $i \in \overline{W}_1$ and by Lemma 2 i's transaction price is no greater under σ^* than under $\bar{\sigma}$. For $i \in S$, Claim 3 implies i transacts at price $\min\{p_1^*, p_2^*\}$, which by applying Lemma 2 is clearly bounded above by $\bar{p}_1$, i's transaction price under $\bar{\sigma}$ and we are done. □

B. Proof of Proposition 3

Proof. To find the moments of the price of auction 1, we multiply equation (1) by π^k, condition on the events U_1, U_2 and E defined in the proof of Proposition 2, and integrate, which yields

$$
\begin{aligned}
&E[\Pi^k(d_1,s,d_2,m_1,m_2)] \\
&= E[X^k_{m_1+m_2+1\,:\,d+s}] \underbrace{\Big(1-P(X_{m_2+1\,:\,d_2+s}<X_{m_1+1\,:\,d_1})-P(X_{m_1+1\,:\,d_1+s}<X_{m_2+1\,:\,d_2})\Big)}_{P(E)} \\
&\quad +E[X^k_{m_1+1\,:\,d_1}\mid \underbrace{X_{m_2+1\,:\,d_2+s}<X_{m_1+1\,:\,d_1}}_{U_1}]\,\underbrace{P(X_{m_2+1\,:\,d_2+s}<X_{m_1+1\,:\,d_1})}_{P(U_1)} \\
&\quad +E[X^k_{m_1+1\,:\,d_1+s}\mid \underbrace{X_{m_1+1\,:\,d_1+s}<X_{m_2+1\,:\,d_2}}_{U_2}]\,\underbrace{P(X_{m_1+1\,:\,d_1+s}<X_{m_2+1\,:\,d_2})}_{P(U_2)}. \qquad \text{(A6)}
\end{aligned}
$$

Turning to the individual terms, we have

$$
\begin{aligned}
P(U_1) &= \int_0^\infty P(X_{m_1+1\,:\,d_1}=\pi,\ X_{m_2+1\,:\,d_2+s}<\pi)\;d\pi, \qquad \text{(A7)}\\
&= \int_0^\infty \Big[d_1\binom{d_1-1}{m_1} f(\pi)F^{d_1-m_1-1}(\pi)(1-F(\pi))^{m_1} \\
&\qquad \cdot\sum_{i=0}^{m_2}\binom{d_2+s}{i}F^{d_2+s-i}(\pi)(1-F(\pi))^i\Big]\;d\pi, \qquad \text{(A8)}\\
&= \sum_{\substack{i=0\\ \text{s.t. } d+s>m_1+i}}^{m_2} \frac{d_1\binom{d_1-1}{m_1}\binom{d_2+s}{i}}{(d+s)\binom{d+s-1}{m_1+i}}. \qquad \text{(A9)}
\end{aligned}
$$

Equation (A8) follows from the identities (see Chapter 2 of Balakrishnan and Cohen 1991)

$$
f_{X_{b\,:\,a}}(X_{b\,:\,a}=x)=a\binom{a-1}{b-1}f(x)F^{a-b}(x)(1-F(x))^{b-1},\quad P(X_{b\,:\,a}<x)=\sum_{i=0}^{b}\binom{a}{i}F^{a-i}(x)(1-F(x))^i, \qquad \text{(A10)}
$$

where $f_{X_{b\,:\,a}}$ is the probability density function of $X_{b\,:\,a}$, F is the valuation cumulative distribution function, and f is the valuation probability density function. Equation (A9) follows from the fact that terms $\binom{d_1-1}{m_1}$ and $\binom{d_2+s}{i}$ are both positive only if $d_1>m_1$ and $d_2+s\geq i$, which implies $d+s>m_1+i$, in which case we can write

$$
\begin{aligned}
1 &= \int_0^\infty f_{X_{m_1+i+1\,:\,d+s}}(X_{m_1+i+1\,:\,d+s}=\pi)d\pi,\\
&= \int_0^\infty (d+s)\binom{d+s-1}{m_1+i}f(\pi)F^{d+s-m_1-i-1}(\pi)(1-F(\pi))^{m_1+i}d\pi.
\end{aligned}
$$

Moreover, equations (A7)-(A10) imply that

$$
\begin{aligned}
E[X^k_{m_1+1\,:\,d_1}|U_1]P(U_1) &= \int_0^\infty \pi^k P(X_{m_1+1\,:\,d_1}=\pi,\ X_{m_2+1\,:\,d_2+s}<\pi)\;d\pi,\\
&= \sum_{\substack{i=0\\ \text{s.t. } d+s>m_1+i}}^{m_2} \frac{d_1\binom{d_1-1}{m_1}\binom{d_2+s}{i}}{(d+s)\binom{d+s-1}{m_1+i}}E[X^k_{m_1+1+i\,:\,d+s}]. \qquad \text{(A11)}
\end{aligned}
$$

Similar reasoning yields

$$P(U_2) = 1 - \sum_{\substack{i=0 \\ \text{s.t. } d+s>m_1+i}}^{m_2} \frac{(d_1+s)\binom{d_1+s-1}{m_1}\binom{d_2}{i}}{(d+s)\binom{d+s-1}{m_1+i}}, \tag{A12}$$

$$E[X^k_{m_1+1\,:\,d_1+s}|U_2]P(U_2) = E[X^k_{m_1+1\,:\,d_1+s}] \tag{A13}$$

$$- \sum_{\substack{i=0 \\ \text{s.t. } d+s>m_1+i}}^{m_2} \frac{(d_1+s)\binom{d_1+s-1}{m_1}\binom{d_2}{i}}{(d+s)\binom{d+s-1}{m_1+i}} E[X^k_{m_1+1+i\,:\,d+s}]. \tag{A14}$$

Substituting (A9) and (A11)-(A14) into (A6) gives the final result. □

C. Proof of Proposition 4

Proof. Using equation (3) and simplifying we find that

$$\begin{aligned} & E[\Pi(n-s,s)] - E[\Pi(n-s+2,s-2)] \\ &= 2(n-1)\Big[\frac{1}{2}E[X_{1\,:\,n}] - E[X_{1\,:\,n-1}] + \frac{1}{2}E[X_{1\,:\,n-2}]\Big] + E[X_{2\,:\,\frac{n-s}{2}+s}] - E[X_{2\,:\,\frac{n-s}{2}+s-1}]. \end{aligned} \tag{A15}$$

The first term on the right side of (A15) is independent of s, and is a constant we denote by $-C$. This constant is negative because the bracketed portion of this term can be multiplied by the positive value $s(2d_1+s-1)$ to yield precisely the term in (3) that adjusts the mean price downward from its idealized value of $E[X_{2:d_1+s}]$.

Defining $\delta_s \triangleq E[X_{2\,:\,\frac{n-s}{2}+s}] - E[X_{2\,:\,\frac{n-s}{2}+s-1}]$, we can express (5) as $\delta_s \geq C$ for $2 \leq s \leq n-2$, and $\delta_n = C$. We begin by showing $\delta_n = C$. This result follows by expressing the expectations of second-highest valuations in terms of expectations of highest valuations. In particular, we use the fact that $E[X_{2\,:\,j}] = jE[X_{1\,:\,j-1}] - (j-1)E[X_{1\,:\,j}]$ (David 1970, page 38) to write

$$\begin{aligned} -C + \delta_n &= -C + \big(nE[X_{1\,:\,n-1}] - (n-1)E[X_{1\,:\,n}]\big) - \big((n-1)E[X_{1\,:\,n-2}] - (n-2)E[X_{1\,:\,n-1}]\big), \\ &= 2(n-1)\Big[\frac{1}{2}E[X_{1\,:\,n}] - E[X_{1\,:\,n-1}] + \frac{1}{2}E[X_{1\,:\,n-2}]\Big] \\ &\quad + nE[X_{1\,:\,n-1}] - (n-1)E[X_{1\,:\,n}] - (n-1)E[X_{1\,:\,n-2}] + (n-2)E[X_{1\,:\,n-1}], \\ &= (n-1)E[X_{1\,:\,n}] - 2(n-1)E[X_{1\,:\,n-1}] + (n-1)E[X_{1\,:\,n-2}] \\ &\quad + nE[X_{1\,:\,n-1}] - (n-1)E[X_{1\,:\,n}] - (n-1)E[X_{1\,:\,n-2}] + (n-2)E[X_{1\,:\,n-1}], \\ &= 0, \end{aligned}$$

where the second equality follows by substituting in for $-C$. For $2 \leq s \leq n-2$, the inequalities $\delta_s \geq C$ follow from $\delta_n = C$ and our assumption that $\delta_s \geq \delta_{s+2} \geq \cdots \geq \delta_n$.

Finally, (6) can be expressed via (A15) as $\delta_s - C$ is nonincreasing for $2 \leq s \leq n$, which follows directly from (5). $\square$

D. Proof of Proposition 5

Proof. For the $U[a,b]$ case, $E[X_{i:j}] = a + (b-a)\frac{j-i+1}{j+1}$, $i = 1, \ldots, j$ (David 1970, page 27), and hence

$$\begin{aligned}\delta_s \equiv E\big[X_{2:\frac{n-s}{2}+s} - X_{2:\frac{n-s}{2}+s-1}\big] &= (b-a)\left[\frac{n+s-2}{n+s+2} - \frac{n+s-4}{n+s}\right], \\ &= (b-a)\frac{8}{(n+s+2)(n+s)}, \qquad \text{(A16)}\end{aligned}$$

which decreases with s. For the $\exp(\lambda)$ case,

$$E[X_{i:j}] = \frac{1}{\lambda}\sum_{l=i}^{j}\frac{1}{l}, \qquad i = 1, \ldots, j \quad \text{(see David, page 17), which implies that} \qquad \text{(A17)}$$

$$\delta_s = \frac{1}{\lambda}\sum_{l=2}^{\frac{n-s}{2}+s}\frac{1}{l} - \frac{1}{\lambda}\sum_{l=2}^{\frac{n-s}{2}+s-1}\frac{1}{l} = \frac{1}{\lambda}\frac{2}{n+s}, \quad \text{which also decreases in } s.$$

Finally, for Pareto(α, k), we have (Johnson and Kotz 1970, page 241)

$$E[X_{i:j}] = \frac{\Gamma(j+1)\Gamma(i-\alpha^{-1})}{\Gamma(j+1-\alpha^{-1})\Gamma(i)}k, \quad i = 1, \ldots, j. \qquad \text{(A18)}$$

Using the fact that equation (A18) implies

$$E[X_{i:j}] = \frac{\alpha j}{\alpha j - 1}E[X_{i:j-1}], \quad \text{we get} \qquad \text{(A19)}$$

$$\delta_s \equiv E\big[X_{2:\frac{n-s}{2}+s} - X_{2:\frac{n-s}{2}+s-1}\big] = E\big[X_{2:\frac{n-s}{2}+s-1}\big]\frac{1}{\alpha(\frac{n-s}{2}+s)-1},$$

and

$$\begin{aligned}\delta_{s+2} - \delta_s &= E\big[X_{2:\frac{n-s}{2}+s}\big]\frac{1}{\alpha(\frac{n-s}{2}+s+1)-1} - E\big[X_{2:\frac{n-s}{2}+s-1}\big]\frac{1}{\alpha(\frac{n-s}{2}+s)-1}, \\ &= E\big[X_{2:\frac{n-s}{2}+s-1}\big]\left(\frac{\alpha(\frac{n-s}{2}+s)}{\alpha(\frac{n-s}{2}+s)-1}\cdot\frac{1}{\alpha(\frac{n-s}{2}+s+1)-1} - \frac{1}{\alpha(\frac{n-s}{2}+s)-1}\right), \\ &= E\big[X_{2:\frac{n-s}{2}+s-1}\big]\left(\frac{1-\alpha}{(\alpha(\frac{n-s}{2}+s+1)-1)(\alpha(\frac{n-s}{2}+s)-1)}\right),\end{aligned}$$

which is negative for $\alpha > 1$. $\square$

E. Heuristic condition for equation (4)

An alternative approach to characterizing the valuation distributions that satisfy condition (4) relies on the asymptotic approximation (15). By approximation (15), condition (4) holds as long as

$$F^{-1}\left(\frac{j-1}{j}\right)-F^{-1}\left(\frac{j-2}{j-1}\right) \;\geq\; F^{-1}\left(\frac{j}{j+1}\right)-F^{-1}\left(\frac{j-1}{j}\right) \tag{A20}$$

for $j=\lceil\frac{n}{2}\rceil+1,\ldots,n-1$ (here j is shorthand for $\frac{n+s}{2}$ in Proposition 4). By treating j as a continuous variable, (A20) is approximated by

$$\frac{f(x)}{1-F(x)} \geq -\frac{1}{2}\frac{d}{dx}\ln f(x) \quad \text{for} \quad x\in\left[F^{-1}\left(\frac{\lceil\frac{n}{2}\rceil-1}{\lceil\frac{n}{2}\rceil}\right),\ F^{-1}\left(\frac{n-2}{n-1}\right)\right]; \tag{A21}$$

see the next paragraph for a derivation of (A21). In words, (A21) implies that along the domain's tail, the log of the pdf decreases no more than twice as fast as the hazard rate; if this condition is violated at some $x=F^{-1}\left(\frac{j-1}{j}\right)$, the relative steepness of F to the left of x implies moving from $j-1$ to j increases the idealized price only a small amount, compared to the more sizeable increase from j to $j+1$ effected by the relative flatness of F to the right of x. The $U[a,b]$, $\exp(\lambda)$, and Pareto(α,k) distributions can all be shown to strictly satisfy (A21) (we omit the calculations), as do Weibull distributions provided n and the Weibull shape parameter c satisfy $n>2\exp(1/c-1)$, suggesting that (A21) has merit as a proxy for the condition of Proposition 4 (simple numerical observations not included in the present study verify the usefulness of (A21) in the Weibull case).

We let j be a continuous variable in $\left[\lceil\frac{n}{2}\rceil,n\right]$. Then (A20) is satisfied if $F^{-1}(1-1/j)$ is concave over j. Since $\frac{dF^{-1}(x)}{dx}|_{x=y}=1/[\frac{dF(x)}{dx}|_{x=F^{-1}(y)}]$, letting f denote the pdf of the valuation distribution yields

$$\frac{d}{dj}F^{-1}\left(1-\frac{1}{j}\right) \;=\; \frac{1}{f\left(F^{-1}\left(1-\frac{1}{j}\right)\right)j^2}.$$

Taking the second derivative gives

$$\frac{d^2}{dj^2}F^{-1}\left(1-\frac{1}{j}\right) \;=\; -\frac{f\left(F^{-1}\left(1-\frac{1}{j}\right)\right)2j+\frac{f'\left(F^{-1}\left(1-\frac{1}{j}\right)\right)}{f\left(F^{-1}\left(1-\frac{1}{j}\right)\right)j^2}j^2}{\left(f\left(F^{-1}\left(1-\frac{1}{j}\right)\right)j^2\right)^2},$$

so $F^{-1}(1-1/j)$ is concave over j if

$$f^2\left(F^{-1}\left(1-\frac{1}{j}\right)\right)2j+f'\left(F^{-1}\left(1-\frac{1}{j}\right)\right) \;\geq\; 0.$$

Making the change-of-variable $x = F^{-1}(1 - 1/j)$ and simplifying completes the derivation of equation (A21).

F. Proving right hand side of Pareto case in (7) converges to $\frac{2\alpha-1}{2\alpha}2^{1/\alpha} - 1$

The proof relies on bounding the product on the right side of (7). Our approach uses logarithms to split the product into a sum. First, the upper bound is

$$\begin{aligned} \ln \prod_{j=\frac{n}{2}+1}^{n} \left(1 + \frac{1}{\alpha j - 1}\right) &= \sum_{j=\frac{n}{2}+1}^{n} \ln\left(1 + \frac{1}{\alpha j - 1}\right), \\ &\leq \sum_{j=\frac{n}{2}+1}^{n} \frac{1}{\alpha j - 1} < \frac{1}{\alpha} \sum_{j=\frac{n}{2}+1}^{n} \frac{1}{j-1} \leq \frac{1}{\alpha} \ln\left(\frac{n-1}{\frac{n}{2}-1}\right), \end{aligned}$$

where the first inequality follows by $\ln(1 + 1/x) \leq 1/x$, the second by $\alpha > 1$, and the third by (8). Next, the lower bound is

$$\sum_{j=\frac{n}{2}+1}^{n} \ln\left(1 + \frac{1}{\alpha j - 1}\right) \geq \sum_{j=\frac{n}{2}+1}^{n} \frac{1}{\alpha j} \geq \frac{1}{\alpha} \ln\left(\frac{n+1}{\frac{n}{2}+1}\right),$$

where the first inequality is by $\ln(1 + 1/x) \geq 1/(x + 1)$ and the second by (8). Hence,

$$\frac{2\alpha-1}{2\alpha}\left(\frac{n+1}{\frac{n}{2}+1}\right)^{1/\alpha} - 1 \leq \frac{2\alpha-1}{2\alpha} \prod_{j=\frac{n}{2}+1}^{n} \left(1 + \frac{1}{\alpha j - 1}\right) - 1 \leq \frac{2\alpha-1}{2\alpha}\left(\frac{n-1}{\frac{n}{2}-1}\right)^{1/\alpha} - 1,$$

and taking $n \to \infty$ proves the result.

G. Proof of Proposition 6

Proof. Since n_2^* is invariant under linear transformations of the underlying valuation distribution, we prove the proposition for $U[a, b]$ valuations by way of characterizing n_2^* for $U[0, 1]$ valuations. While this proposition assumes $pn \in \mathbb{Z}_+$, there is no guarantee that $pn_2^*(p, n_1) \in \mathbb{Z}_+$. Finding $n_2^*(p, n_1)$ amounts to solving for n_2 such that

$$\begin{aligned} E[\Pi(n_1(1-p), n_2 - \lfloor n_2 p \rfloor, n_1 p + \lfloor n_2 p \rfloor)] - E[X_{2:n_1}] &\geq 0 \quad \text{and} \\ E[\Pi(n_1(1-p), (n_2-1) - \lfloor (n_2-1)p \rfloor, n_1 p + \lfloor (n_2-1)p \rfloor)] - E[X_{2:n_1}] &< 0. \end{aligned}$$

To consider an equation that we can solve, we for the moment ignore the $\lfloor \cdot \rfloor$ operator and instead find $\hat{n}_2(p, n_1)$ that solves

$$E[\Pi(n_1(1-p), n_2(1-p)), n_1 p + n_2 p] - E[X_{2:n_1}] = 0. \tag{A22}$$

Note that we are abusing notation in that the revenue Π has no realistic interpretation as an operator on non-integers. Therefore, for the purposes of this proof it may help to think of Π simply as a function over $\mathbb{R}^3_+$.

We prove Proposition 6 using two lemmas. The first lemma ignores integrality (i.e., looks at Π over $\mathbb{R}^3_+$) and lays the groundwork for the bounds in (9); the second lemma permits these bounds to be modified for integer arguments.

Lemma 3.

$$\frac{n_1}{2.3} + \frac{1}{2} \le \hat{n}_2(p, n_1) \le \frac{n_1}{2} + \frac{1}{2},$$

where $\hat{n}_2(p, n_1)$ is the real, positive root of equation (A22).

Proof. By (3) and $E[X_{i+1:j}] = \frac{j-i}{j+1}$, $i = 0, \ldots, j-1$, the left side of (A22) can be expressed as

$$\begin{aligned} & p\big(2n_2^3 + (-p^2 n_1 + 4n_1 - p^2)n_2^2 + (2n_1^2 - 3pn_1 - 3n_1^2 p + p^2 n_1 + p^2 n_1^2 - 2)n_2 \\ & + pn_1 + 1 - 3n_1^2 + pn_1 + 1 - 3n_1^2 - 2n_1^3 \\ & + 2n_1^2 p + n_1^3 p\ \big) \,/\, \big((n_1 + pn_2 + 1)(n_1 + n_2 - 1)(n_1 + n_2 + 1)(n_1 + 1)\big), \qquad \text{(A23)} \\ \triangleq\ & \frac{ph(n_2)}{(n_1 + pn_2 + 1)(n_1 + n_2 - 1)(n_1 + n_2 + 1)(n_1 + 1)}, \end{aligned}$$

where $h(n_2)$ is defined to be (A23)'s right side's numerator divided by p. Equation (A23) is zero where the numerator is zero; since by assumption $p \ge 1/n_1 > 0$, this occurs when $h(n_2)$ is zero. As $h(n_2)$ is a third-degree polynomial in n_2, its real positive root $\hat{n}_2(p, n_1)$ is complicated and difficult to extract insight from. For this reason we resort to proving bounds on $\hat{n}_2(p, n_1)$. Even this is a little tricky, since the bounds we derive are independent of p, and hence we must take care to show that the bounds are valid for all p such that $n_1 p \in \{1, 2 \ldots, n_1\}$.

We begin by showing the upper bound. The inequalities

$$\begin{aligned} \frac{dh(n_2)}{dn_2}\Big|_{n_2=1} &= (1-p)(2-p)n_1^2 + (8 - p^2 - 3p)n_1 + 4 - 2p^2 \ > \ 0 \qquad \text{for } n_1 \ \ge \ 2, \\ \frac{d^2 h(n_2)}{dn_2^2} &= (8 - 2p^2)n_1 + 12n_2 - 2p^2 \ > \ 0 \qquad \text{for } n_1, \ n_2 \ \ge \ 2, \text{ and} \\ h\left(\frac{n_1}{2} + \frac{1}{2}\right) &= \frac{1}{4}(1-p)(n_1+1)^2((1-p)n_1 + 1 + p) \ \ge \ 0 \qquad \text{for } n_1 \ge 2, \end{aligned} \tag{A24}$$

show that $h(n_2)$ is positive for $n_2 \geq \frac{n_1}{2} + \frac{1}{2} > 1$. We conclude that $\frac{n_1}{2} + \frac{1}{2}$ bounds $\hat{n}_2(p, n_1)$ from above for all p such that $n_1 p \in \{1, 2 \ldots, n_1\}$; equation (A24) implies that this bound is tight when $p = 1$.

Proving the lower bound is more involved. Substituting $n_2 = \frac{10n_1}{23} + \frac{1}{2}$ into $h(n_2)$ and dividing out the common denominator $4 \cdot 23^3$ yields

$$(11960\, n_1^3 + 15134\, n_1^2 - 8993 n_1 - 12167)\, p^2 - (14812\, n_1^3 + 39146\, n_1^2 + 24334\, n_1)\, p \\ - 10216\, n_1^3 + 14904\, n_1^2 + 38088\, n_1 + 12167. \quad \text{(A25)}$$

To show that (A25) is nonpositive for $n_1 \geq 2$ and p such that $n_1 p \in \{1, 2 \ldots, n_1\}$, we begin with the case $n_1 = 2$. When $n_1 = 2$, for $p = 1/2$ and $p = 1$, the quantity in equation (A25) equals $-256509/4$ and -131454, respectively. To show that (A25) is also non-positive for $n_1 > 2$, we first argue that the terms involving p and p^2 sum to a negative value, and then finish by showing that the constant terms also sum to a negative value.

Isolating and rearranging the p and p^2 terms in (A25), and using the fact that $p \in (0, 1]$, we see that

$$\begin{aligned} &(11960\, n_1^3 + 15134\, n_1^2 - 8993\, n_1 - 12167)\, p^2 + (-14812\, n_1^3 - 39146\, n_1^2 - 24334\, n_1)\, p \\ &= (11960\, p^2 - 14812\, p)\, n_1^3 + (15134\, p^2 - 39146\, p)\, n_1^2 + (-8993\, p^2 - 24334\, p)\, n_1 - 12167\, p^2 \; < \; 0. \end{aligned}$$

It remains to show that the constant term in (A25) is negative. To see this, note that

$$\frac{d}{dn_1}\left(-10216\, n_1^3 + 12167 + 38088\, n_1 + 14904\, n_1^2\right) \;=\; -30648 n_1^2 + 29808 n_1 + 38088$$

implies that the constant term is decreasing for $n_1 > 2$. Thus, it only remains to show that the constant term is negative for $n_1 = 3$; by direct computation,

$$-10216\, n_1^3 + 12167 + 38088\, n_1 + 14904\, n_1^2\Big|_{n_1=3} = -15265,$$

completing the argument that (A25) is negative for all $n_1 \geq 2$ and p such that $n_1 p \in \{1, 2 \ldots, n_1\}$. This allows us to conclude that $\frac{n_1}{2.3} + \frac{1}{2} \leq \hat{n}_2(p, n_1)$. □

To put Lemma 3 to work in the context of integer arguments for Π, we now prove the following.

Lemma 4. *Let $a, b, c \in \mathbb{Z}_+$, and $0 \leq g < b$. Then, for U[0,1] valuations,*

$$E[\Pi(a, b - g, c + g)] \geq E[\Pi(a, b, c)]. \quad \text{(A26)}$$

I.e., the revenue of auctioneer 1 increases by sharing bidders previously dedicated to auctioneer 2.

Proof. $E[\Pi(a,b-g,c+g)] - E[\Pi(a,b,c)] =$

$$-g\big(-1-12abc+3c^2+3cg+3c^2g+cg^2+3a^2g+ag^2+6acg-6a^2b-6ab^2 \\ +3a^2+3ag+2a-2b^3+2b+g^2+2c-6bc^2-6b^2c+6ac\big) \\ /\big((a+b+c+1)\,(a+c+g+1)\,(a+b+c-1)\,(a+b+c)\,(a+c+1)\big). \quad \text{(A27)}$$

The denominator of the right side of (A27) is positive. The numerator is nonnegative if

$$(-a-c-1)g^2+(-6ac-3a-3c-3c^2-3a^2)g+1+12abc-3c^2-6ac-2c \\ +6bc^2+6b^2c-3a^2+6a^2b+6ab^2-2b-2a+2b^3 \geq 0. \quad \text{(A28)}$$

Since (A28) is decreasing in g, we can prove inequality (A26) by showing that (A28) is nonnegative for $g=b$. Substituting $g=b$ into equation (A28) and simplifying yields

$(b-1)(a+b+c+1)(3a+3c+2b-1)$, which is nonnegative because $a,b,c \in \mathbb{Z}_+$. □

We now use Lemmas 3 and 4 to derive Proposition 6. We first note that Lemma 3 implies

$$E\left[\Pi\left(n_1(1-p),\ (\frac{n_1}{2}+\frac{1}{2})(1-p),\ n_1p+p(\frac{n_1}{2}+\frac{1}{2})\right)\right] \geq E[X_{2:n_1}], \quad \text{(A29)}$$

and

$$E\left[\Pi\left(n_1(1-p),\ (\frac{n_1}{2.3}+\frac{1}{2})(1-p),\ n_1p+p(\frac{n_1}{2.3}+\frac{1}{2})\right)\right] \leq E[X_{2:n_1}]. \quad \text{(A30)}$$

We now show that if n_2 is large enough such that

$$\lfloor n_2p \rfloor \geq \left\lceil p(\frac{n_1}{2}+\frac{1}{2}) \right\rceil, \quad \text{then} \quad E[\Pi(n_1(1-p),\ n_2-\lfloor pn_2 \rfloor,\ pn_1+\lfloor pn_2 \rfloor)] \ \geq\ E[X_{2:n_1}]. \quad \text{(A31)}$$

By taking $g = \lfloor n_2p \rfloor - \lceil p(\frac{n_1}{2}+\frac{1}{2}) \rceil$, we get

$$\begin{aligned} & E\left[\Pi\left(n_1(1-p),\ n_2-\lfloor pn_2 \rfloor,\ pn_1+\lfloor pn_2 \rfloor\right)\right] \\ &= E\left[\Pi\left(n_1(1-p),\ n_2-\left\lceil p(\frac{n_1}{2}+\frac{1}{2})\right\rceil-g,\ pn_1+\left\lceil p(\frac{n_1}{2}+\frac{1}{2})\right\rceil+g\right)\right] \\ &\geq E\left[\Pi\left(n_1(1-p),\ n_2-\left\lceil p(\frac{n_1}{2}+\frac{1}{2})\right\rceil,\ pn_1+\left\lceil p(\frac{n_1}{2}+\frac{1}{2})\right\rceil\right)\right], \end{aligned}$$

where the inequality comes from applying Lemma 4 (where g is as defined above, $a = n_1(1-p)$, $b = n_2 - \lceil p(\frac{n_1}{2}+\frac{1}{2})\rceil$, and $c = pn_1 + \lceil p(\frac{n_1}{2}+\frac{1}{2})\rceil$). Continuing in a similar fashion,

$$\begin{aligned}
& E\left[\Pi\left(n_1(1-p),\ n_2 - \left\lceil p(\frac{n_1}{2}+\frac{1}{2})\right\rceil,\ pn_1 + \left\lceil p(\frac{n_1}{2}+\frac{1}{2})\right\rceil\right)\right] \\
\geq\ & E\left[\Pi\left(n_1(1-p),\ n_2 - p(\frac{n_1}{2}+\frac{1}{2}),\ pn_1 + p(\frac{n_1}{2}+\frac{1}{2})\right)\right] \\
& \qquad \text{by applying Lemma 4 again, this time with } g = \left\lceil p(\frac{n_1}{2}+\frac{1}{2})\right\rceil - p(\frac{n_1}{2}+\frac{1}{2}), \\
\geq\ & E\left[\Pi\left(n_1(1-p),\ \frac{n_1}{2}+\frac{1}{2} - p(\frac{n_1}{2}+\frac{1}{2}),\ pn_1 + p(\frac{n_1}{2}+\frac{1}{2})\right)\right], \qquad \text{(A32)} \\
=\ & E\left[\Pi\left(n_1(1-p),\ (\frac{n_1}{2}+\frac{1}{2})(1-p),\ pn_1 + p(\frac{n_1}{2}+\frac{1}{2})\right)\right] \\
\geq\ & E[X_{2:n_1}] \quad \text{by equation (A29).}
\end{aligned}$$

Equation (A32) follows from two facts; first, again allowing non-integer arguments for Π, we get

$$\frac{\partial E[\Pi(a,b,c)]}{\partial b} = \frac{2}{(a+b+1)^2} + \frac{c(2a+c-1)(3b^2+6cb+6ab+3a^2+3c^2+6ac-1)}{(a+b+c+1)^2(a+b+c)^2(a+b+c-1)^2} \geq 0$$

for $a \geq 1$ (which is true for $a = n_1(1-p) \in \mathbb{Z}_+$), that is, all other things left fixed, increasing the number of bidders dedicated to auctioneer 2 can only increase the expected revenue to auctioneer 1; and second, $\lfloor pn_2 \rfloor \geq \lceil p(\frac{n_1}{2}+\frac{1}{2})\rceil \geq p(\frac{n_1}{2}+\frac{1}{2})$, which implies

$$pn_2 - (pn_2 - \lfloor pn_2 \rfloor) \geq p(\frac{n_1}{2}+\frac{1}{2}) \qquad \Rightarrow \qquad pn_2 \geq p(\frac{n_1}{2}+\frac{1}{2}) \qquad \Rightarrow \qquad n_2 \geq \frac{n_1}{2}+\frac{1}{2}.$$

Following the same steps for the remaining inequality, we now show that if n_2 is so small that

$$\lfloor n_2 p \rfloor < \left\lfloor p(\frac{n_1}{2.3}+\frac{1}{2})\right\rfloor, \quad \text{then} \quad E\left[\Pi\left(n_1(1-p),\ n_2 - \lfloor n_2 p\rfloor,\ pn_1 + \lfloor n_2 p \rfloor\right)\right] \leq E[X_{2:n_1}]. \qquad \text{(A33)}$$

Inequality (A33) follows from

$$\begin{aligned}
& E\left[\Pi\left(n_1(1-p),\ n_2 - \lfloor pn_2\rfloor,\ pn_1 + \lfloor pn_2\rfloor\right)\right] \\
\leq\ & E\left[\Pi\left(n_1(1-p),\ n_2 - \left\lfloor p(\frac{n_1}{2.3}+\frac{1}{2})\right\rfloor,\ pn_1 + \left\lfloor p(\frac{n_1}{2.3}+\frac{1}{2})\right\rfloor\right)\right] \quad \text{by Lemma 4,} \\
\leq\ & E\left[\Pi\left(n_1(1-p),\ \frac{n_1}{2.3}+\frac{1}{2} - \left\lfloor p(\frac{n_1}{2.3}+\frac{1}{2})\right\rfloor,\ pn_1 + \left\lfloor p(\frac{n_1}{2.3}+\frac{1}{2})\right\rfloor\right)\right], \qquad \text{(A34)} \\
\leq\ & E\left[\Pi\left(n_1(1-p),\ (\frac{n_1}{2.3}+\frac{1}{2})(1-p),\ pn_1 + p(\frac{n_1}{2.3}+\frac{1}{2})\right)\right] \quad \text{by Lemma 4 again,} \\
\leq\ & E[X_{2:n_1}] \quad \text{by equation (A30),}
\end{aligned}$$

where, similar to the previous analysis, inequality (A34) follows from the fact that

$$\lfloor pn_2 \rfloor < \left\lfloor p(\frac{n_1}{2.3}+\frac{1}{2}) \right\rfloor \Rightarrow \lfloor pn_2 \rfloor + 1 \leq \left\lfloor p(\frac{n_1}{2.3}+\frac{1}{2}) \right\rfloor \Rightarrow pn_2 - (pn_2 - \lfloor pn_2 \rfloor) + 1 \leq \left\lfloor p(\frac{n_1}{2.3}+\frac{1}{2}) \right\rfloor$$

$$\Rightarrow \quad pn_2 \leq \left\lfloor p(\frac{n_1}{2.3}+\frac{1}{2}) \right\rfloor \Rightarrow \quad pn_2 \leq p(\frac{n_1}{2.3}+\frac{1}{2}) \Rightarrow \quad n_2 \leq \frac{n_1}{2.3}+\frac{1}{2}.$$

Taken together, equations (A31) and (A33) prove the proposition. □

H. Proof of Proposition 7

Proof. As in the proof of Proposition 6, we prove Proposition 7 for the simpler case exp(1), with the proof for general λ following by n_2^* invariant under linear transformations of the underlying valuations. We first prove the proposition for the case $p = 1$. We derive $n_2^*(1, n_1)$ by finding the value of n_2 at which auctioneer 1 is indifferent between pooling or not pooling with auctioneer 2, i.e.,

$$E[\Pi(0,0,n_1+n_2)] - E[X_{2:n_1}] = 0. \tag{A35}$$

Plugging (A17) into (3) yields

$$E[\Pi(d_1,d_2,s)] = -\frac{s(2d_1+s-1)}{2(d+s)(d+s-1)} + \sum_{j=2}^{d_1+s}\frac{1}{j},$$

and hence

$$E[\Pi(0,0,n_1+n_2)] = -\frac{1}{2} + \sum_{j=2}^{n_1+n_2}\frac{1}{j}. \tag{A36}$$

By (A17) and (A36), equation (A35) is given by

$$-\frac{1}{2} + \sum_{j=n_1+1}^{n_1+n_2}\frac{1}{j} = 0.$$

Equation (8) provides the bounds

$$-\frac{1}{2} + \ln\left(\frac{n_1+n_2+1}{n_1+1}\right) \;\leq\; -\frac{1}{2} + \sum_{j=n_1+1}^{n_1+n_2}\frac{1}{j} \;\leq\; -\frac{1}{2} + \ln\left(\frac{n_1+n_2}{n_1}\right).$$

The upper bound on $n_2^*(1, n_1)$ follows by $n_2^*(1, n_1)$'s integrality, the definition of $\theta_u(1)$, and the fact that

$$\begin{aligned} -\frac{1}{2} + \ln\left(\frac{n_1+n_2+1}{n_1+1}\right) > 0 \quad &\Longleftrightarrow \quad \ln\left(\frac{n_1+1}{n_1+n_2+1}\right) \;<\; -\frac{1}{2}, \\ &\Longleftrightarrow \quad \frac{n_1+1}{n_1+n_2+1} < e^{-1/2}, \\ &\Longleftrightarrow \quad (n_1+1)(e^{1/2}-1) \;<\; n_2, \end{aligned}$$

which implies $n_2^*(1,n_1) \le (n_1+1)(e^{1/2}-1)$. The lower bound follows analogously:

$$-\frac{1}{2} + \ln\left(\frac{n_1+n_2}{n_1}\right) < 0 \iff \ln\left(\frac{n_1}{n_1+n_2}\right) > -\frac{1}{2},$$
$$\iff \frac{n_1}{n_1+n_2} > e^{-1/2},$$
$$\iff n_1(e^{1/2}-1) > n_2,$$

which implies $n_2^*(1,n_1) \ge n_1(e^{1/2}-1)$. This concludes the proof for the case $p=1$.

For the remainder of the proof we assume $p<1$. We begin by deriving the lower bound,

$$\left\lfloor n_1\left(e^{p/2}-1\right)\right\rfloor \le \lfloor p n_2^*(p,n_1)\rfloor. \tag{A37}$$

More specifically, we show that for n_2 violating (A37), the difference between the partial-pooling revenue and the non-pooling revenue,

$$E\Big[\Pi\left(\lceil n_1(1-p)\rceil, \lceil n_2(1-p)\rceil, \lfloor n_1p\rfloor + \lfloor n_2p\rfloor\right)\Big] - E\left[X_{2:n_1}\right]$$
$$= \sum_{j=n_1+1}^{n_1+\lfloor n_2p\rfloor} \frac{1}{j} - \frac{\left(\lfloor n_1p\rfloor + \lfloor n_2p\rfloor\right)\left(2\lceil n_1(1-p)\rceil + \lfloor n_1p\rfloor + \lfloor n_2p\rfloor - 1\right)}{2(n_1+n_2)(n_1+n_2-1)}, \tag{A38}$$

is negative, meaning that (n_1,n_2) would not be in the mutually-feasible pooling region. We first derive the following lemma, which will be needed to prove (A37).

Lemma 5. *For $p<1$ and $n_2 \ge n_1/2$,*

$$\ln\left(\frac{n_1+n_2p}{n_1+\lfloor n_2p\rfloor}\right) \ge \frac{(2n_1-1)(n_2p-\lfloor n_2p\rfloor) + (n_2p)^2 - \lfloor n_2p\rfloor^2}{2(n_1+n_2)(n_1+n_2-1)}. \tag{A39}$$

Proof. Equation (A39) is established by

$$\ln\left(\frac{n_1+n_2p}{n_1+\lfloor n_2p\rfloor}\right) = \ln\left(1+\frac{r}{n_1+\lfloor n_2p\rfloor}\right) \quad \text{where } r \triangleq n_2p - \lfloor n_2p\rfloor,$$
$$\ge \left(\frac{r}{n_1+\lfloor n_2p\rfloor}\right)\left(\frac{n_1+\lfloor n_2p\rfloor}{n_1+n_2p}\right) \quad \text{by } \ln(1+x) \ge \frac{x}{1+x},$$
$$= \frac{r}{n_1+n_2p},$$

and the fact that for $p<1$ and $n_2 \ge n_1/2$,

$$\frac{1}{n_1+n_2p} \ge \frac{2n_1-1+n_2p+\lfloor n_2p\rfloor}{2(n_1+n_2)(n_1+n_2-1)}. \tag{A40}$$

For $p<1$ and $n_2 \ge n_1/2$, to prove (A40) we show that

$$2(n_1+n_2)(n_1+n_2-1) \ge (n_1+n_2p)(2n_1+n_2p+\lfloor n_2p\rfloor - 1). \tag{A41}$$

Noticing that $p < 1$ and $n_1 p \in \mathbb{Z}_+$ implies $p \leq (n_1 - 1)/n_1$, we bound (A41)'s right side by

$$\begin{aligned} (n_1 + n_2 p)(2n_1 + n_2 p + \lfloor n_2 p \rfloor - 1) &\leq (n_1 + n_2 p)(2n_1 + 2n_2 p - 1) \qquad \text{since } n_2 p \geq \lfloor n_2 p \rfloor, \\ &\leq 2\left(n_1 + n_2 \frac{(n_1 - 1)}{n_1}\right)\left(n_1 + n_2 \frac{(n_1 - 1)}{n_1} - \frac{1}{2}\right). \end{aligned}$$

To complete the proof of this lemma, we show that the left side of (A41) minus this bound is nonnegative for $p < 1$ and $n_2 \geq n_1/2$. This fact is derived in two steps:

$$\begin{aligned} &\frac{d}{dn_2}\left[2(n_1 + n_2)(n_1 + n_2 - 1) - 2\left(n_1 + n_2 \frac{(n_1 - 1)}{n_1}\right)\left(n_1 + n_2 \frac{(n_1 - 1)}{n_1} - \frac{1}{2}\right)\right] \\ &\qquad = \frac{n_1(3n_1 - 1) + n_2(8n_1 - 4)}{n_1^2} \geq 0, \quad \text{and} \end{aligned}$$

$$2(n_1+n_2)(n_1+n_2-1) - 2\left(n_1 + n_2 \frac{(n_1 - 1)}{n_1}\right)\left(n_1 + n_2 \frac{(n_1 - 1)}{n_1} - \frac{1}{2}\right)\Bigg|_{n_2 = \frac{n_1}{2}} = \frac{3n_1}{2} - 1 \geq 0.$$

□

With Lemma 5 in hand, we first derive an upper bound on (A38) for $n_2 \geq n_1/2$; using the fact that $n_1 p \in \mathbb{Z}_+$, we get

$$\sum_{j=n_1+1}^{n_1+\lfloor n_2p\rfloor} \frac{1}{j} - \frac{(n_1p+\lfloor n_2p\rfloor)(2n_1-n_1p+\lfloor n_2p\rfloor-1)}{2(n_1+n_2)(n_1+n_2-1)} \tag{A42}$$

$$\begin{aligned}
&\leq \ln\left(\frac{n_1+\lfloor n_2p\rfloor}{n_1}\right) - \frac{(n_1p+\lfloor n_2p\rfloor)(2n_1-n_1p+\lfloor n_2p\rfloor-1)}{2(n_1+n_2)(n_1+n_2-1)} \qquad \text{since } \ln\left(\frac{a}{b}\right) \geq \sum_{j=b+1}^{a}\frac{1}{j}, \\
&\leq \ln\left(\frac{n_1+\lfloor n_2p\rfloor}{n_1}\right) + \ln\left(\frac{n_1+n_2p}{n_1+\lfloor n_2p\rfloor}\right) - \frac{(2n_1-1)(n_2p-\lfloor n_2p\rfloor)+(n_2p)^2-\lfloor n_2p\rfloor^2}{2(n_1+n_2)(n_1+n_2-1)} \\
&\quad - \frac{(n_1p+\lfloor n_2p\rfloor)(2n_1-n_1p+\lfloor n_2p\rfloor-1)}{2(n_1+n_2)(n_1+n_2-1)} \qquad \text{by equation (A39)}, \\
&= \ln\left(\frac{n_1+n_2p}{n_1}\right) \\
&\quad + \frac{-(2n_1-1)(n_2p-\lfloor n_2p\rfloor+n_1p+\lfloor n_2p\rfloor)-(n_1p+\lfloor n_2p\rfloor)(-n_1p+\lfloor n_2p\rfloor)-(n_2p)^2+\lfloor n_2p\rfloor^2}{2(n_1+n_2)(n_1+n_2-1)}, \\
&= \ln\left(\frac{n_1+n_2p}{n_1}\right) + \frac{-(2n_1-1)(n_2p+n_1p)+(n_1p)^2-\lfloor n_2p\rfloor^2-(n_2p)^2+\lfloor n_2p\rfloor^2}{2(n_1+n_2)(n_1+n_2-1)}, \\
&= \ln\left(\frac{n_1+n_2p}{n_1}\right) + \frac{-(2n_1-1)p(n_1+n_2)-(n_1p+n_2p)(-n_1p+n_2p)}{2(n_1+n_2)(n_1+n_2-1)}, \\
&= \ln\left(\frac{n_1+n_2p}{n_1}\right) - \frac{p(2n_1-p(n_1-n_2)-1)}{2(n_1+n_2-1)}, \\
&\leq \ln\left(\frac{n_1+n_2p}{n_1}\right) - \frac{p(2n_1-(n_1-n_2)-1)}{2(n_1+n_2-1)} \qquad \text{since } p<1, \\
&= \ln\left(\frac{n_1+n_2p}{n_1}\right) - \frac{p}{2}.
\end{aligned} \tag{A43}$$

We can characterize when this upper bound will be nonnegative:

$$\begin{aligned}
\ln\left(\frac{n_1+n_2p}{n_1}\right) - \frac{p}{2} \geq 0 \quad &\iff \quad \frac{n_1+n_2p}{n_1} \geq e^{p/2}, \\
&\iff \quad n_2p \geq n_1\left(\exp\left(\frac{p}{2}\right)-1\right) > n_1\frac{p}{2}, && \text{(A44)} \\
&\Rightarrow \quad n_2 > \frac{n_1}{2}, && \text{(A45)}
\end{aligned}$$

where (A44) follows from $e^x - 1 > x$ for $x > 0$.

By (A45), we have that for $n_2 = n_1/2$, the right side of equation (A43) is less than zero, implying the same for (A42), and hence (A38). The difference between partial pooling and no pooling – i.e., the quantity in (A38) – grows (becomes more positive or less negative) with increasing n_2. This follows because for any d_1, d_2, and s, increasing d_2 to d_2+1 or s to $s+1$ (weakly) increases auctioneer 1's expected revenue. (This can be easily shown by a sample path type argument (i.e., the result can be shown to hold for any bidder valuation realizations), which we omit for brevity.) Hence, (A38) negative for $n_2 = n_1/2$ implies that $n_2^*(p, n_1) > n_1/2$. Since $n_2^*(p, n_1) > n_1/2$, (A43) is an upper bound on (A38) at $n_2 = n_2^*(p, n_1)$. Finally, because (A44) implies that (A43) is nonnegative only if $pn_2 \geq n_1(e^{p/2}-1)$, we have $pn_2^*(p, n_1) \geq n_1(e^{p/2} - 1)$, and applying the lower-integer operator to both sides of this inequality yields the lower bound in (A37).

We now turn to the upper bound of this proposition,

$$\lfloor pn_2^*(p, n_1) \rfloor \;\leq\; \left\lceil (n_1 + 1)\left(\exp\left(\frac{p(4n_1 - n_1 p - 2)}{6n_1 - 4}\right) - 1\right)\right\rceil . \tag{A46}$$

Our approach is to bound (A38) from below, and show that (A46) is a sufficient condition for this lower bound to be positive.

For later use, let

$$a(x) \triangleq -\frac{p(2n_1 - n_1 p + xp - 1)}{2(n_1 + x - 1)}, \quad \text{and notice that} \tag{A47}$$

$$\frac{da(x)}{dx} = \frac{p(1-p)(2n_1 - 1)}{2(n_1 + x - 1)^2} \geq 0 \quad \text{for} \quad n_1 \geq 1, \tag{A48}$$

in which case $a(x)$ is increasing in x. Using the fact that $n_1 p \in \mathbb{Z}_+$, for $n_2 \geq n_1/2$ we get

$$\begin{aligned}
&\sum_{j=n_1+1}^{n_1+\lfloor n_2 p\rfloor} \frac{1}{j} - \frac{(n_1 p + \lfloor n_2 p \rfloor)(2n_1 - n_1 p + \lfloor n_2 p \rfloor - 1)}{2(n_1+n_2)(n_1+n_2-1)} \\
&\geq \sum_{j=n_1+1}^{n_1+\lfloor n_2 p\rfloor} \frac{1}{j} - \frac{(n_1 p + n_2 p)(2n_1 - n_1 p + n_2 p - 1)}{2(n_1+n_2)(n_1+n_2-1)} \qquad \text{since } n_2 p \geq \lfloor n_2 p \rfloor, \\
&= \sum_{j=n_1+1}^{n_1+\lfloor n_2 p\rfloor} \frac{1}{j} + a(n_2) \qquad \text{by (A47)}, \\
&\geq \ln\left(\frac{n_1 + \lfloor n_2 p \rfloor + 1}{n_1 + 1}\right) + a(n_2) \qquad \text{since } \sum_{j=b}^{a} \frac{1}{j} \geq \ln\left(\frac{a+1}{b}\right), \\
&\geq \ln\left(\frac{n_1 + \lfloor n_2 p \rfloor + 1}{n_1 + 1}\right) + a\left(\frac{n_1}{2}\right) \qquad \text{by (A48) and } n_2 \geq \frac{n_1}{2}.
\end{aligned}$$

However, we also have

$$\ln\left(\frac{n_1 + \lfloor n_2 p \rfloor + 1}{n_1 + 1}\right) + a\left(\frac{n_1}{2}\right) \geq 0 \iff \lfloor n_2 p \rfloor \;\geq\; (n_1+1)\left(e^{-a\left(\frac{n_1}{2}\right)} - 1\right). \tag{A49}$$

We conclude that $n_2^*(p, n_1)$ is less than or equal to any n_2 satisfying (A49) and $n_2 \geq n_1/2$ (since any such n_2 makes (A38) nonnegative). However, only (A49) is necessary since we have already shown that $n_2^*(p, n_1) > n_1/2$. Hence, $n_2^*(p, n_1)$ is less than or equal to the smallest n_2 satisfying (A49); in other words, (A46) holds. □

I. Proof of Proposition 8

Proof. The value $n_2^*(p, n_1)$ is defined to be the smallest integer n_2 such that

$$1 \;\leq\; \frac{E[\Pi(\lceil n_1(1-p)\rceil, \lceil n_2(1-p)\rceil, \lfloor n_1 p\rfloor + \lfloor n_2 p\rfloor)]}{E[X_{2\,:\,n_1}]}. \tag{A50}$$

Using (A19), and the fact that $E[X_{1\,:\,j}] = \frac{\alpha}{\alpha-1}E[X_{2\,:\,j}]$ (see equation (A18)), we can rewrite equation (A50) as

$$\begin{aligned}1 \;\leq\; & \frac{E[X_{2\,:\,n_1}]}{E[X_{2\,:\,n_1}]} \\ & \cdot \left[\prod_{j=n_1+1}^{n_1+\lfloor n_2 p\rfloor} \frac{\alpha j}{\alpha j - 1} + (\lfloor n_1 p\rfloor + \lfloor n_2 p\rfloor)(2\lceil n_1(1-p)\rceil + \lfloor n_1 p\rfloor + \lfloor n_2 p\rfloor - 1) \cdot \frac{\alpha}{\alpha - 1}\right. \\ & \left. \cdot \prod_{j=n_1+1}^{n_1+n_2-2} \frac{\alpha j}{\alpha j - 1} \cdot \left(\frac{1}{2}\frac{\alpha(n_1+n_2)}{\alpha(n_1+n_2)-1}\frac{\alpha(n_1+n_2-1)}{\alpha(n_1+n_2-1)-1} - \frac{\alpha(n_1+n_2-1)}{\alpha(n_1+n_2-1)-1} + \frac{1}{2}\right)\right],\end{aligned} \tag{A51}$$

where we have used equation (3) in expanding $E[\Pi(\lceil n_1(1-p)\rceil, \lceil n_2(1-p)\rceil, \lfloor n_1 p\rfloor + \lfloor n_2 p\rfloor)]$. Letting α tend to 1 from above, the right side of equation (A51) converges to

$$\frac{n_1 + \lfloor n_2 p\rfloor}{n_1} - \frac{(\lfloor n_1 p\rfloor + \lfloor n_2 p\rfloor)(2\lceil n_1(1-p)\rceil + \lfloor n_1 p\rfloor + \lfloor n_2 p\rfloor - 1)}{2n_1(n_1+n_2-1)},$$

which is exactly 1 for $n_2 = n_1$. Because the right side of equation (A51) increases with n_2 (for brevity we omit the differentiation that establishes this), we get $n_2^* = n_1$. □

J. Derivation of Equation (13)

Proof. The value $n_2^*(p, n_1, m_1, m_2)$ is defined to be the smallest integer n_2 such that

$$E[X_{m_1+1\,:\,n_1}] \;\leq\; E[X_{m_1+m_2+1\,:\,n_1+n_2}]. \tag{A52}$$

Equation (A52) is equivalent to

$$\begin{aligned}1 \;\geq\; & \frac{E[X_{m_1+1\,:\,n_1}]}{E[X_{m_1+m_2+1\,:\,n_1+n_2}]}, \\ = \; & \frac{E[X_{m_1+1\,:\,n_1}]}{E[X_{m_1+2\,:\,n_1}]} \cdot \frac{E[X_{m_1+2\,:\,n_1}]}{E[X_{m_1+3\,:\,n_1}]} \cdot \;\cdots\; \cdot \frac{E[X_{m_1+m_2\,:\,n_1}]}{E[X_{m_1+m_2+1\,:\,n_1}]} \\ & \cdot \frac{E[X_{m_1+m_2+1\,:\,n_1}]}{E[X_{m_1+m_2+1\,:\,n_1+1}]} \cdot \frac{E[X_{m_1+m_2+1\,:\,n_1+1}]}{E[X_{m_1+m_2+1\,:\,n_1+2}]} \cdot \;\cdots\; \cdot \frac{E[X_{m_1+m_2+1\,:\,n_1+n_2-1}]}{E[X_{m_1+m_2+1\,:\,n_1+n_1}]}, \\ = \; & \prod_{i=m_1+1}^{m_1+m_2} \left(\frac{\alpha i}{\alpha i - 1}\right) \cdot \prod_{i=n_1+1}^{n_1+n_2} \left(\frac{\alpha i - 1}{\alpha i}\right),\end{aligned} \tag{A53}$$

where the final equality follows from equation (A18). As $\alpha \to 1$, terms in equation (A53) cancel, leaving

$$1 \;\geq\; \frac{m_1+m_2}{m_1}\cdot\frac{n_1}{n_1+n_2}, \quad \text{which is equivalent to} \quad n_2 \;\geq\; n_1\frac{m_2}{m_1},$$

proving equation (13). □

K. Proof of Proposition 9

Proof. Using (2) we can write (16) in Proposition 9 as

$$\beta_j = E[X_{1:n+2j}-X_{1:n+j}]-\frac{j}{n}\left(E[X_{1:3n}-X_{3:3n}]-E[X_{1:2n}-X_{2:2n}]\right)-\frac{j(n-j)}{n(3n-1)}E[X_{2:3n}-X_{3:3n}]. \tag{A54}$$

Since the sign of β_j is insensitive to linear transformations of the underlying valuations, we prove Proposition 9 for the more convenient cases of $U[0,1]$ and $\exp(1)$ valuations. Beginning with the $U[0,1]$ case, we substitute $E[X_{i+1:j}] = \frac{j-i}{j+1}$ into (A54) to get

$$\begin{aligned}\beta_j &= \frac{j}{(n+2j+1)(n+j+1)}-\frac{j(5n^2+3n-1-2nj-j)}{n(3n+1)(3n-1)(2n+1)},\\ &= \frac{j\left[(4n+2)j^3-(4n^2-3n-5)j^2-(13n^3+19n^2+2n-4)j+13n^4-4n^3-12n^2-2n+1\right]}{(n+2j+1)(n+j+1)n(3n+1)(3n-1)(2n+1)} \qquad \text{(A55)}\\ &\triangleq \frac{jg(j)}{(n+2j+1)(n+j+1)n(3n+1)(3n-1)(2n+1)}.\end{aligned}$$

By defining $g(j)$ to be the numerator of the second line's right side divided by j for $j \in [1,n]$, we can calculate the sign of β_j via the simpler function $g(j)$. Since

$$\frac{dg(j)}{dj} \;=\; 3(4n+2)j^2-2(4n^2-3n-5)j-13n^3-19n^2-2n+4,$$

it follows that $\frac{dg(j)}{dj}$ is convex in j. Furthermore, for $n \geq 1$,

$$\begin{aligned}\frac{dg(j)}{dj}|_{j=1} &= -13n^3-27n^2+16n+20 \;<\; 0,\\ \frac{dg(j)}{dj}|_{j=n} &= -9n^3-7n^2+8n+4 \;<\; 0;\end{aligned}$$

that is, $\frac{dg(j)}{dj}$ is negative at both endpoints $j=1,\ n$, implying that $g(j)$ decreases in the interval $[1,n]$. The existence of $\bar{j}$ for $n \geq 3$ can now be proved by showing that $g(1)$ is positive and $g(n)$ is negative. Direct substitution into (A55) yields

$$g(1) \;=\; 13n^4-17n^3-35n^2+3n+12 \;>\; 0 \qquad \text{for all } n \geq 3,$$

$$g(n) = -18n^3 - 9n^2 + 2n + 1 \ < \ 0 \qquad \text{for all } n \geq 1.$$

The bounds on $\bar{j}$ follow from

$$\begin{aligned} g\left(\frac{3n}{5}\right) &= \frac{578}{125}n^4 - \frac{1736}{125}n^3 - \frac{57}{5}n^2 + \frac{2}{5}n + 1 \ > \ 0 \qquad \text{for } n \geq 4, \\ g\left(\frac{4n}{5}\right) &= \frac{261}{125}n^4 - \frac{2032}{125}n^3 - \frac{52}{5}n^2 + \frac{6}{5}n + 1 \ > \ 0 \qquad \text{for } n \geq 9, \end{aligned}$$

coupled with $g(j)$ decreasing over $[1, \ldots, n]$.

Turning to exp(1) valuations, we substitute equation (A17) into (A54) and simplify to yield

$$\beta_j \ = \ \sum_{l=n+j+1}^{n+2j} \frac{1}{l} - \frac{j}{n}\left(\frac{1}{2}\right) - \frac{j(n-j)}{n(3n-1)}\left(\frac{1}{2}\right). \qquad \text{(A56)}$$

The approach to proving the existence of $\bar{j}$ and $\bar{j}/n \leq n/2$ is analogous to that used above for the $U[0,1]$ case, with additional care required by the fact that β_j is only defined over $\mathbb{Z}$, and does not readily extend to $\mathbb{R}$. First, let us define

$$\begin{aligned} \hat{\beta}_j &\triangleq \beta_j - \beta_{j-1}, \\ &= \frac{4j^4 - 2j^3 - (11n^2 - n)j^2 - (3n^3 - 6n^2 + n)j + n^4 + n^3}{(n+2j)(n+2j-1)(n+j)(3n-1)n} \qquad \text{after simplification,} \\ &\triangleq \frac{r(j)}{(n+2j)(n+2j-1)(n+j)(3n-1)n}, \end{aligned}$$

where $r(j)$ is shorthand for the second line's right side's numerator. Next we show $r(j)$ is decreasing for $j \in [1, \ldots, n]$. The first and second derivatives of $r(j)$ are

$$\begin{aligned} \frac{dr(j)}{dj} &= 16j^3 - 6j^2 - (22n^2 - 2n)j - 3n^3 + 6n^2 - n, \\ \frac{d^2r(j)}{dj^2} &= -22n^2 + 2n + 48j^2 - 12j. \end{aligned}$$

Since $\frac{d^2r(j)}{dj^2} = 0$ if and only if $j = 1/8 \pm \sqrt{9 + 264n^2 - 24n}/24$, when $n \geq 2$ we must have at most one first-order condition of $\frac{dr(j)}{dj}$ in $\mathbb{R}^+$, and hence at most one in $[1, n] \subset \mathbb{R}^+$. Since, for $n \geq 2$,

$$\begin{aligned} \frac{dr(j)}{dj}|_{j=1} &= -3n^3 - 16n^2 + n + 10 \ < \ 0, \\ \frac{dr(j)}{dj}|_{j=n} &= -9n^3 + 2n^2 - n \ < \ 0, \\ \frac{d^2r(j)}{dj^2}|_{j=1} &= -22n^2 + 2n + 36 \ < \ 0, \\ \frac{d^2r(j)}{dj^2}|_{j=n} &= 26n^2 - 10n \ > 0, \end{aligned}$$

we have that $\frac{dr(j)}{dj}$ is negative at endpoints $j = 1$ and $j = n$, and has negative and positive slopes at the respective endpoints. Combined with the fact that $\frac{dr(j)}{dj}$ has at most one first-order condition in $[1, n]$, these observations imply that $\frac{dr(j)}{dj}$ is negative for all $j \in [1, n]$ when $n \geq 2$. By the definition of $r(j)$, we have immediately that $\hat{\beta}_j$ must be decreasing for $j \in [1, \ldots, n]$ when $n \geq 2$.

Next, we note that

$$\begin{aligned} r(1) &= n^4 - 2n^3 - 5n^2 + 2 \ > \ 0 \qquad \text{for } n \geq 4, \\ r(n) &= -9n^4 + 6n^3 - n^2 \ < \ 0 \qquad \text{for } n \geq 2. \end{aligned}$$

For $n \geq 4$, this implies that $\hat{\beta}_j$ is positive and negative at the left and right endpoints, respectively, of the interval $[1, \ldots, n]$. Since $\beta_0 \equiv 0$, and, by (8) and (A56),

$$\beta_n = \sum_{l=2n+1}^{3n} \frac{1}{l} - \frac{1}{2} \leq \ln\left(\frac{3}{2}\right) - \frac{1}{2} < 0,$$

we have that β_j is positive over some interval $[1, \ldots, \bar{j}]$, then is negative in the complement $[\bar{j} + 1, \ldots, n]$. That is, $\bar{j}$ exists.

To show that $\bar{j}/n \leq 1/2$, we consider two cases: n even and n odd. In the first case, $n/2$ is the largest j such that $j/n \leq 1/2$. Plugging $j = n/2$ into (A56) yields

$$\beta_{\frac{n}{2}} \ = \ \sum_{l=\frac{3n}{2}+1}^{2n} -\frac{7n-2}{24n-8} \ \leq \ \ln\left(\frac{2n}{\frac{3n}{2}}\right) - \frac{7n-2}{24n-8} \ \triangleq \ u(n).$$

where the inequality follows from (8). Differentiating $u(n)$ gives $\frac{du(n)}{dn} = (3n-1)^{-2}/8 > 0$, implying that $u(n)$ increases with n. Since $\lim_{n \to \infty} u(n) = \ln(4/3) - 7/24 = -0.0039$, we have $u(n) < 0$ for all $n \geq 2$. Hence, if n is even, we have $\beta_{n/2} < 0$, and thus $\bar{j}/n \leq 1/2$.

For the case in which n is odd, let $n = 2k + 1$ and $k \in \mathbb{Z}^+$, so that $j = k$ is the largest j such that $j/n \leq 1/2$. Substituting $n = 2k + 1$ and $j = k$ into (A56) and applying the same bounding argument as above yields

$$\beta_k \ \leq \ \ln\left(\frac{4k+1}{3k+1}\right) - \frac{k(7k+3)}{2(2k+1)(6k+2)} \ \triangleq \ v(k). \quad \text{Differentiating } v(k) \text{ gives}$$

$$\frac{dv(k)}{dk} \ = \ -\frac{20k^3 + 9k^2 - 2k - 1}{4(3k+1)^2(4k+1)(2k+1)^2} \ < \ 0 \qquad \text{for } k \geq 8.$$

Hence, to show $\beta_k < 0$ for $k \geq 8$, we need only show that $\beta_8 < 0$. Evaluating $v(k)$ for $k = 1, \ldots, 8$ gives $\beta_1 = -0.083$, $\beta_2 = -0.0067$, $\beta_3 = -0.006$, $\beta_4 = -0.0055$, $\beta_5 = -0.0053$, $\beta_6 = -0.0051$, $\beta_7 = -0.0049$, $\beta_8 = -0.000015$, thereby showing $\beta_k < 0$ for $k \geq 1$ (i.e., $n \geq 3$ odd). Hence, when $n = 2k + 1$, we have $\beta_k < 0$, and $\bar{j}/n \leq k/n \leq 1/2$. $\square$