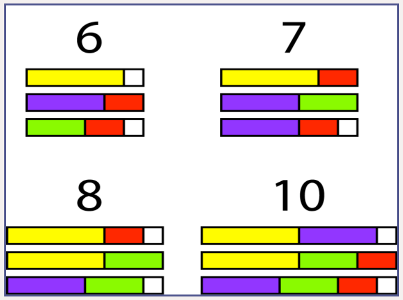
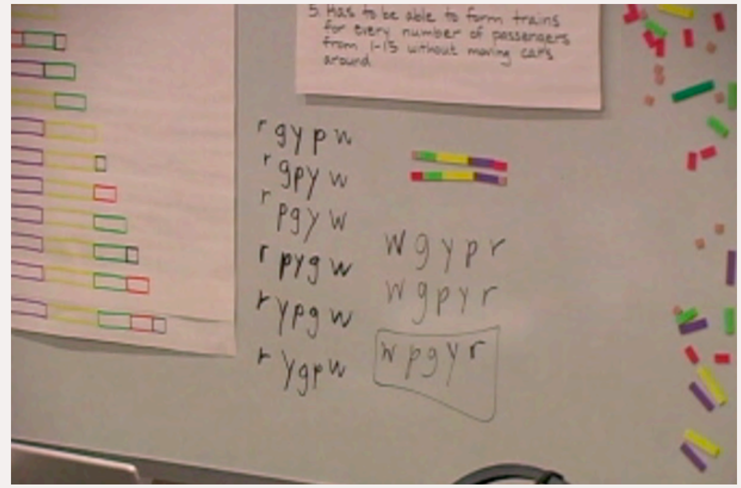
Proving the Impossible





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Two senses of "proving the impossible"

1. Proving impossibility in mathematics: A problem with no solution
2. Proving possibility in teaching: "My students could never do this."

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Overview

1. What is mathematical proof, and why does it matter?
2. What is involved in teaching proving in school?
3. Scaffolding in teaching proving: three key resources:
   - the tasks
   - mathematical language
   - the social and intellectual culture

1. What is proving, and why does it matter?

Proving is central to mathematics

- The basis of knowing that something is true
- Search for proof can help determine what is plausible
- A basic mathematical skill and disposition

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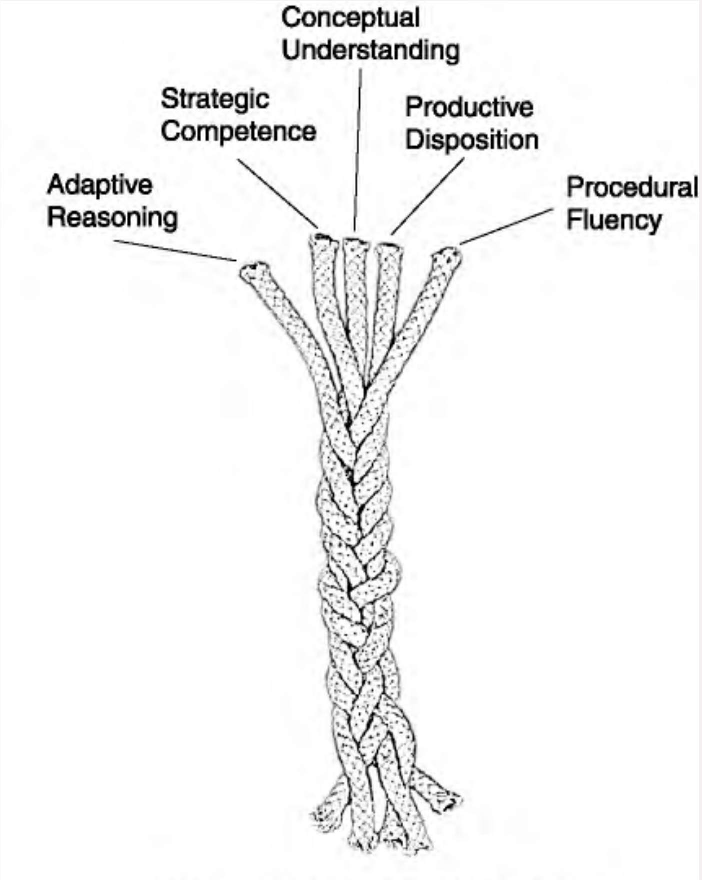
Proving is central to mathematical proficiency

- Supports sense-making and understanding
- Eases retention and retrieval
- Affords intellectual authority and independence



Example: odd + odd = even

7 + 9 = 16

1 + 5 = 6

3 + 11 = 14

5 + 13 = 18

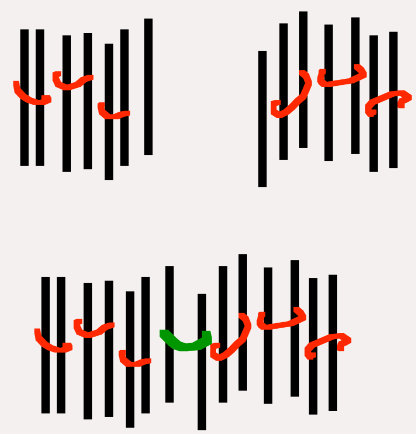
7 + 3 = 10

2467 + 1 = 2468

-3 + 5 = 2

-3 + 3 = 0

$(n2 + 1) + (j2 + 1) = (n + j + 1)2$



What kinds of claims do proofs prove?

1. **Generality**: That something is true for all cases.
2. **Properties**: Features or properties of certain mathematical objects, possibly to characterize them uniquely.
3. **Classification**: Finding all examples of a class.
4. **Equivalence**: That two or more things are the "same."
5. **Existential**: That something exists or does not exist.

1. odd + odd = even.
   V - E + F = 2
2. Every positive integer is a sum of four squares (e.g., 5 = 0 + 0 + 1 + 4; 9 = 0 + 1 + 4 + 4). Sum of angles of a triangle = 180º
3. All solutions to a problem, "I have pennies, nickels, and dimes in my pocket. If I pull out two coins, how much money might I have?" All regular solids.
4. $1+3+5+...+(2n-1) = n^2$. A teacup and a donut are (topologically) the same.
5. There is no solution to $b \div 0$. Fermat's Last Theorem (Wiles)

Foundations of proof

1. Common knowledge
   – Already-established concepts, definitions, theorems
2. Mathematical language
   – Definitions, precision
   – Logic and syntax
3. Intellectual community
   – Social imperative, goal to convince others

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Our claim

- Young students—even those who are not succeeding in school mathematics—can learn to prove complex mathematical claims.
- Making this possible entails intricate instructional work.

2. What is involved in teaching proving in school?

The setting:
Elementary Mathematics Laboratory (EML)

- A live laboratory for the design and study of teaching
  - Direct summer program for children in local school district (working class community, under-resourced district, falling achievement)
  - Live setting for the study of teaching, learning, and mathematics
  - Source of records of practice

2007 EML

- 27 students, entering fifth grade, unsuccessful in school math
- 2½ hours per day
- Mathematical content: Fractions (definitions, representations), proving an impossible combinatorics problem
- Mathematical skills: explaining, representing, proving

A glimpse of the classroom

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The observers in the laboratory classroom

The work of teaching proving

Scaffolding students' mathematical work by—

1. Choosing and using an appropriate task
2. The development of specialized mathematical language
3. The construction of the social and intellectual culture

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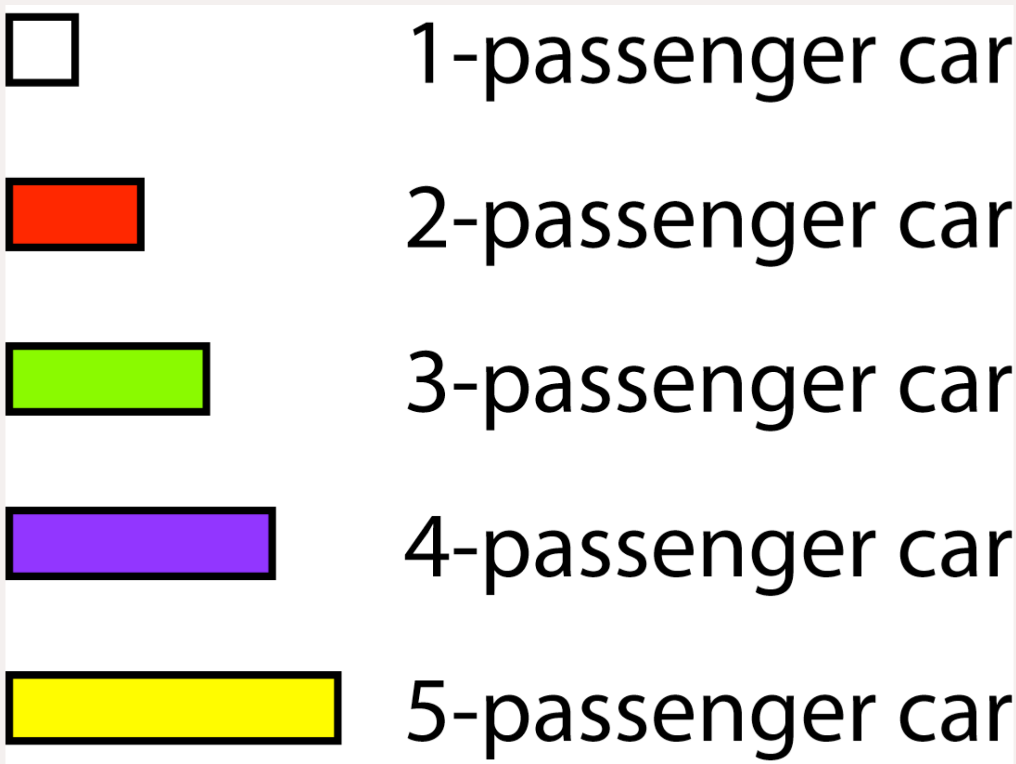
The EML Train Company



The task: The EML Train Problem

Mr. Howe wants to order a special five-car train that uses one of each of the different-sized cars. He wants to be able to break apart his 5-car train to form smaller trains that hold exactly 1 to 15 people. In addition, he wants to be able to form these smaller trains using cars that are next to each other in the larger train.

Can the EML Train Company fill Mr. Howe's order? Explain how you know.

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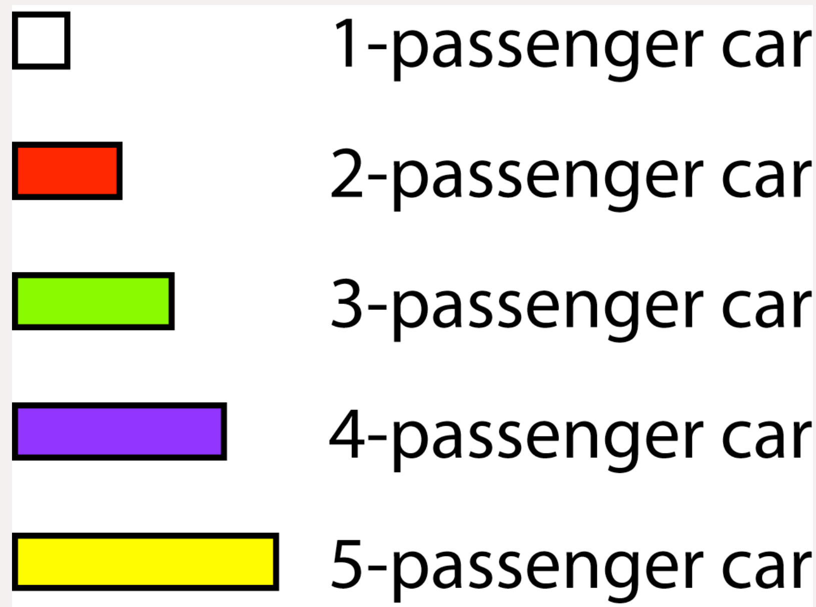
Problem representation: Cuisenaire rods

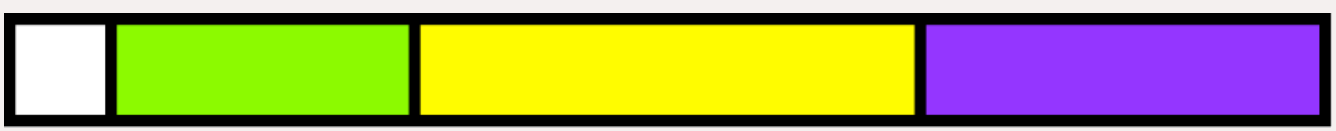
The five cars used:



The basic rule:

All trains must use
only these cars, and
none more than once.

Example: A 13-passenger train.



The enormity of the problem

- The number of possible orderings of 1,2,3,4,5 is 5! = 120

. . . but there is **no** solution

How to prove this without trying every case?

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Mathematical background

Consider the numbers 1, 2, 3, 4, 5. What sums can we make using only these numbers, each one at most once?

For example,

$$1 + 2 + 3 + 4 + 5 = 15$$

is the maximum possible.

What other numbers can we get?

Solutions for Train Problem, Part 1

Length of train	All Possible		Numbers To make it		[rods]	
1	1	[w]				
2	2	[r]				
3	3	[g]	1+2	[w+r]		
4	4	[p]	1+3	[w+g]		
5	5	[y]	1+4	[w+p]	2+3	[r+g]
6	1+5	[w+y]	2+4	[r+p]	1+2+3	[w+r+g]
7	2+5	[r+y]	3+4	[g+p]	1+2+4	[w+r+p]
8	3+5	[g+y]	1+2+5	[w+r+y]	1+3+4	[w+g+p]
9	4+5	[p+y]	1+3+5	[w+g+y]	2+3+4	[r+g+p]
10	1+4+5	[w+p+y]	2+3+5	[r+g+y]	1+2+3+4	[w+r+g+p]
11	2+4+5	[r+p+y]	1+2+3+5	[w+r+g+y]		
12	3+4+5	[g+p+y]	1+2+4+5	[w+r+p+y]		
13	1+3+4+5	[w+g+p+y]				
14	2+3+4+5	[r+g+p+y]				
15	1+2+3+4+5	[w+r+g+p+y]				

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The Train Problem's numerical form

Arrange the numbers 1, 2, 3, 4, 5 in some order.

One example:

1	5	3	4	2

Can you find such an ordering of the numbers so that each number from 1 to 15 is a sum of consecutive terms in our list?

For any ordering we can clearly get the sums 1, 2, 3, 4, 5, and 15.

For the example above you can also get:

$6 = 1 + 5, \quad 7 = 3 + 4, \quad 8 = 5 + 3, \quad 9 = 3 + 4 + 2$

What else can you get?

Staging students' work

Phase I	**The Train Problem, Part 1** **Trains for 1 - 15 passengers** **(Showing what is possible)**	**Practice with the basic elements of the problem, establish common knowledge of the materials and developing norms of explaining solutions**
Phase II	The Train Problem, Part 2	Setting a context for the problem that supports the core mathematical work, captures students' attention, and does not distort or distract
Phase III	White-red ends (David's Conjecture)	Focusing the students' work from empirical experimentation to more systematic inquiry
Phase IV	The only way to get 13 and 14 (The clue from Mrs. Howe)	Provoking the need to prove and further focusing the students' work
Phase V	Six white-red end trains (Reducing the size of the problem)	Building a key sub-part of the proof
Phase VI	Consolidating and assembling the proofs (Showing impossibility)	Organizing the argument
Phase VII	Final report to Mr. Howe	Experiencing the sense of conviction

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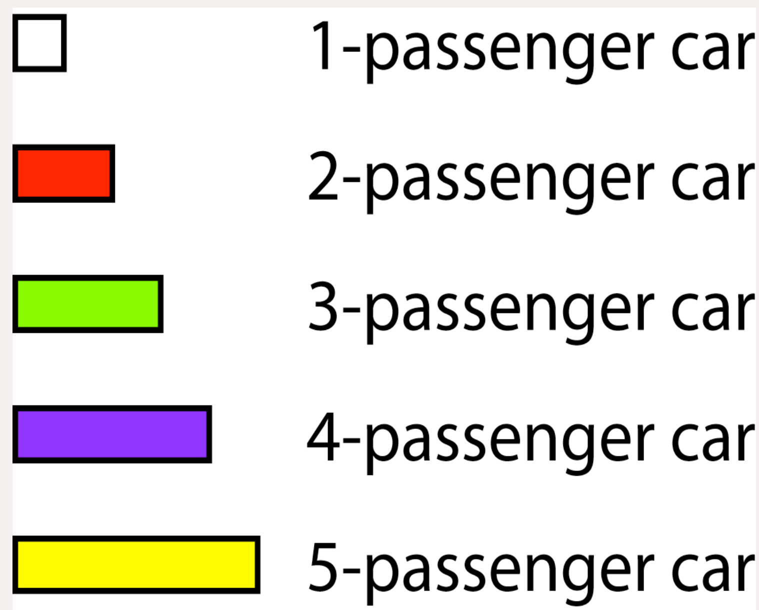
Using mathematical resources in teaching proof

SCAFFOLD	RESOURCE
1. Designing Part 1 of the problem	TASK
2. Conditions	MATHEMATICAL LANGUAGE
3. “at most one”	MATHEMATICAL LANGUAGE
4. Permutations of three cars	TASK
5. “start with the big numbers”	TASK
6. Mr. Howe’s wife	INTELLECTUAL CULTURE
7. Preparing the final report	INTELLECTUAL CULTURE

Scaffolding #1: Train Problem, Part 1

Use at most one of each car.

What are the different numbers of passengers you can build trains to hold?



Scaffolding #1: Simpler problem

- videoclip

Staging students' work

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Scaffolding #2: Conditions

1. Only use w, r, g, p, y
2. Must use each rod exactly once
3. Can't use the same car twice
4. Has to be able to form trains for every number of passengers from 1-15 without moving cars around. These smaller trains must be built from connected cars

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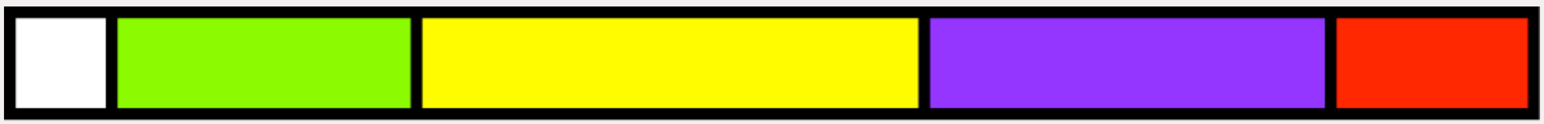
Scaffolding #3: "at most one"

- videoclip

A key insight:

Red and white <u>must</u> be on the ends



Scaffolding #4: "start with the big numbers"

- videoclip

Staging students' work

Phase I	The Train Problem, Part 1 Trains for 1 - 15 passengers (Showing what is possible)	Practice with the basic elements of the problem, establish common knowledge of the materials and developing norms of explaining solutions
Phase II	The Train Problem, Part 2	Setting a context for the problem that supports the core mathematical work, captures students' attention, and does not distort or distract
Phase III	**White-red ends** **(David's Conjecture)**	**Focusing the students' work from empirical experimentation to more systematic inquiry**
Phase IV	The only way to get 13 and 14 (The clue from Mrs. Howe)	Provoking the need to prove and further focusing the students' work
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Phase VII	Final report to Mr. Howe	Experiencing the sense of conviction

→ David's conjecture

"To build Mr. Howe's train, the red and the white cars should be on the ends."

- "Should" versus "must"

Proof by contradiction remains elusive

Teacher

Did you try to use David's Conjecture?

Student

Yeah, we tried it and it didn't work.

Students interpret as a failure something that is actually mathematical progress.

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Scaffolding #5: Mr. Howe's wife

- videoclip

Staging students' work

Phase I	The Train Problem, Part 1 Trains for 1 - 15 passengers (Showing what is possible)	Practice with the basic elements of the problem, establish common knowledge of the materials and developing norms of explaining solutions
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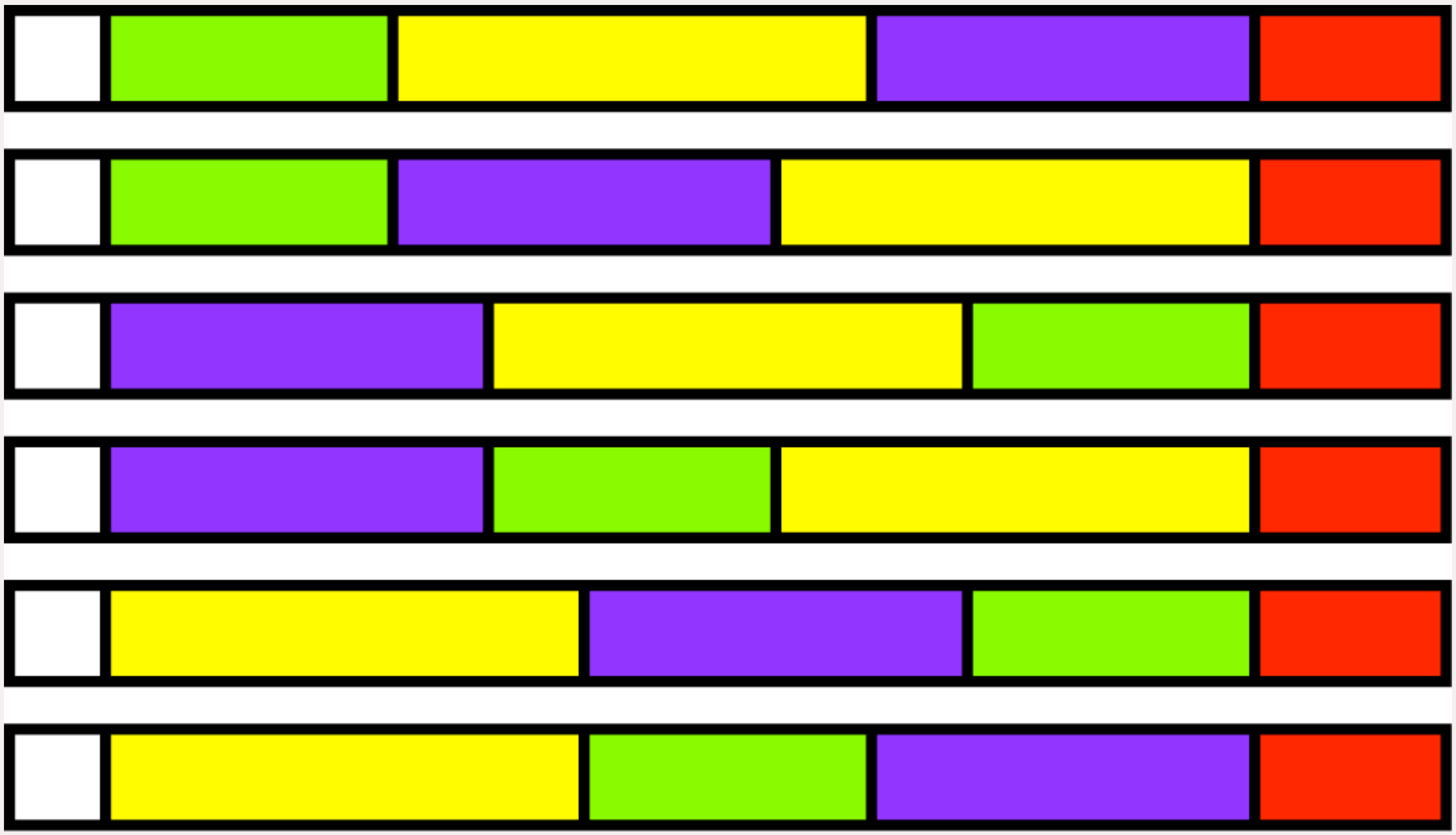
Scaffolding #6: Arranging rods

Warm up problem on Day 2:

How many different arrangements can you make for the green, yellow, and purple rods? How do you know you have all of them?

The 6 trains with red and white on the ends



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Staging students' work

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Phase V	**Six white-red end trains** **(Reducing the size of the problem)**	**Building a key sub-part of the proof**
Phase VI	Consolidating and assembling the proofs (Showing impossibility)	Organizing the argument
Phase VII	Final report to Mr. Howe	Experiencing the sense of conviction

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Scaffolding #7: Preparing the final report for Mr. Howe

- videoclip

Staging students' work

Phase I	The Train Problem, Part 1 Trains for 1 - 15 passengers (Showing what is possible)	Practice with the basic elements of the problem, establish common knowledge of the materials and developing norms of explaining solutions
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Phase VII	Final report to Mr. Howe	Experiencing the sense of conviction

3. Scaffolding in teaching proving: Three key resources for teachers' work

Using mathematical resources in teaching proof

SCAFFOLD	RESOURCE
1. Designing Part 1 of the problem	TASK
2. Conditions	MATHEMATICAL LANGUAGE
3. “at most one”	MATHEMATICAL LANGUAGE
4. Permutations of three cars	TASK
5. “start with the big numbers”	TASK
6. Mr. Howe’s wife	INTELLECTUAL CULTURE
7. Preparing the final report	INTELLECTUAL CULTURE

Tasks

Opportunities for learning proof

- Empirical exploration; finding patterns, conjectures
- Structured record keeping
- Representation (physical, written, oral) of mathematical ideas
- Articulation and justification of mathematical claims
- Critical analysis and evaluation of the arguments of others
- Confrontation with an existential claim: discovering and proving impossibility

Housing the mathematics

- A context that can hold the problem and provide language and structure
- An imperative for seeking conviction (proof)
- Tools to represent the mathematics
- Unpacking the problem at key joints (part 1, permuting cars, red/white on ends)

Mathematical language

- Phrasing the task ("be able to form these smaller trains using cars that are next to each other in the larger train")
- Articulate and unpack *conditions* (Lampert, 2001)
- Hearing the difference and unpacking the logical difference between formulations ("should" versus "have to")

Social and intellectual culture

- Students' explicit responsibility to attend to others' ideas
- Active respect for ideas and classmates, learning to critique
- Using Mr. Howe and his wife as clients of the EML Train Company to represent the social imperative for proof

Conclusions

1. Students can be interested in and sustain work on a complex proof
2. Complex work can build students' mathematical proficiency, rather than merely depend on it
3. Three key resources for scaffolding students' learning:
   - the choice and use of the task,
   - attention to mathematical language
   - the development of an intellectual culture,

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Launching the problem