


Knowing Mathematics Well Enough to Teach It: From Teachers' Knowledge to Knowledge for Teaching

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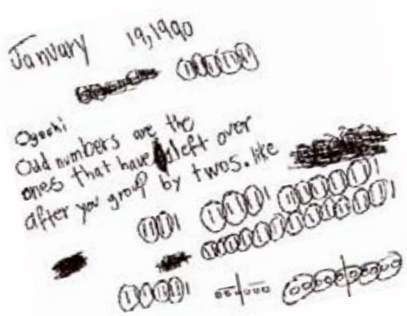
SCHOOL OF EDUCATION UNIVERSITY OF MICHIGAN

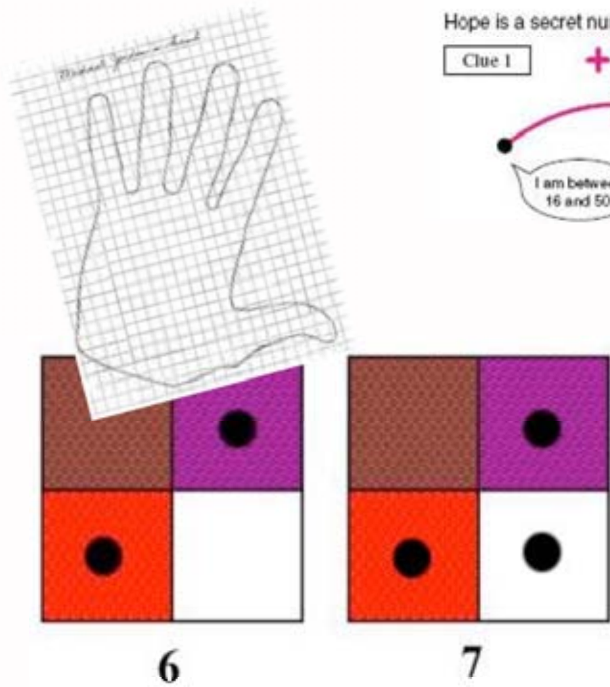
Overview

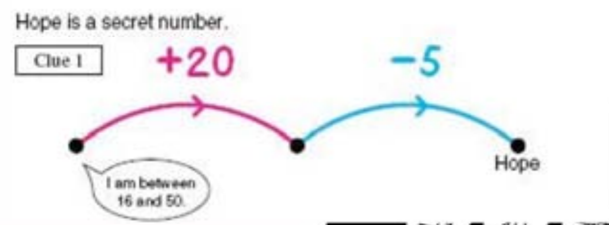
1. Origin of my interest in this question
2. Research on “teacher content knowledge”
3. Shifting the core question
4. Progress toward a “practice-based theory” of mathematical knowledge for teaching
5. Unanswered and new questions, and next steps

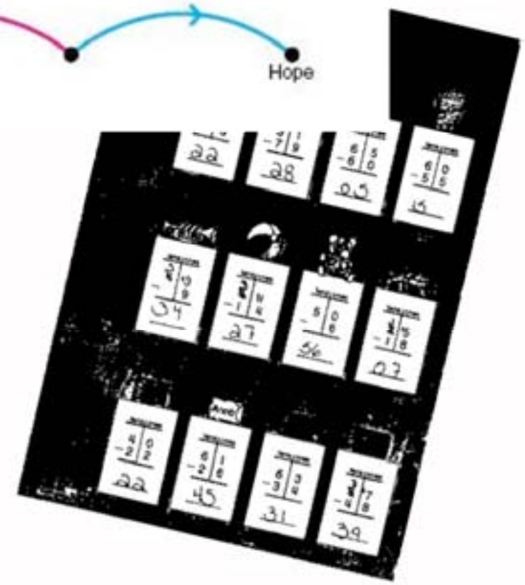
1. Roots in my own history

- My mathematical preparation for teaching, and my experience

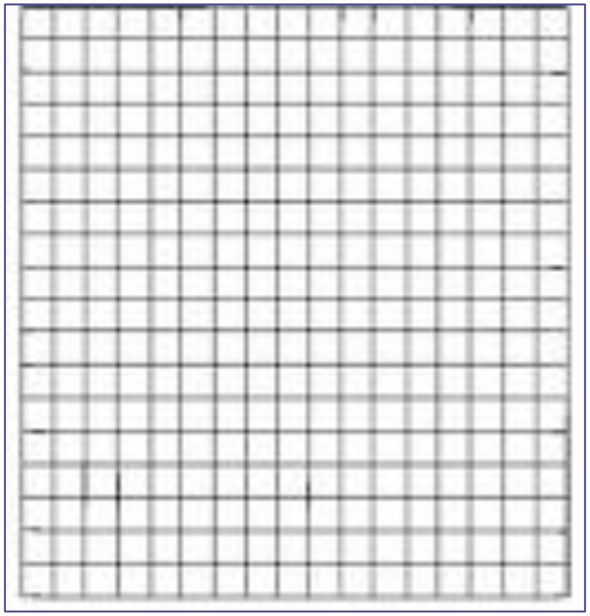


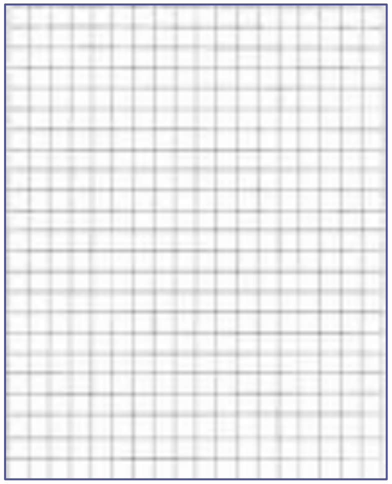






Measuring area in first grade





2. Studying teachers' mathematical knowledge

- Typical approach: certification in math; mathematics degree; course-taking

Across studies[1]:

- At the K-8 level, majoring in mathematics is not associated with greater gains in students' learning
- At the secondary level, uneven results, not powerful; sometimes negative effects

How would you explain these results?

[1]National Mathematics Advisory Panel (2008)

The problem

How can we improve students' learning?

. . .

Teachers' mathematical knowledge is a key factor shaping what they are able do.

What mathematical knowledge do teachers need?

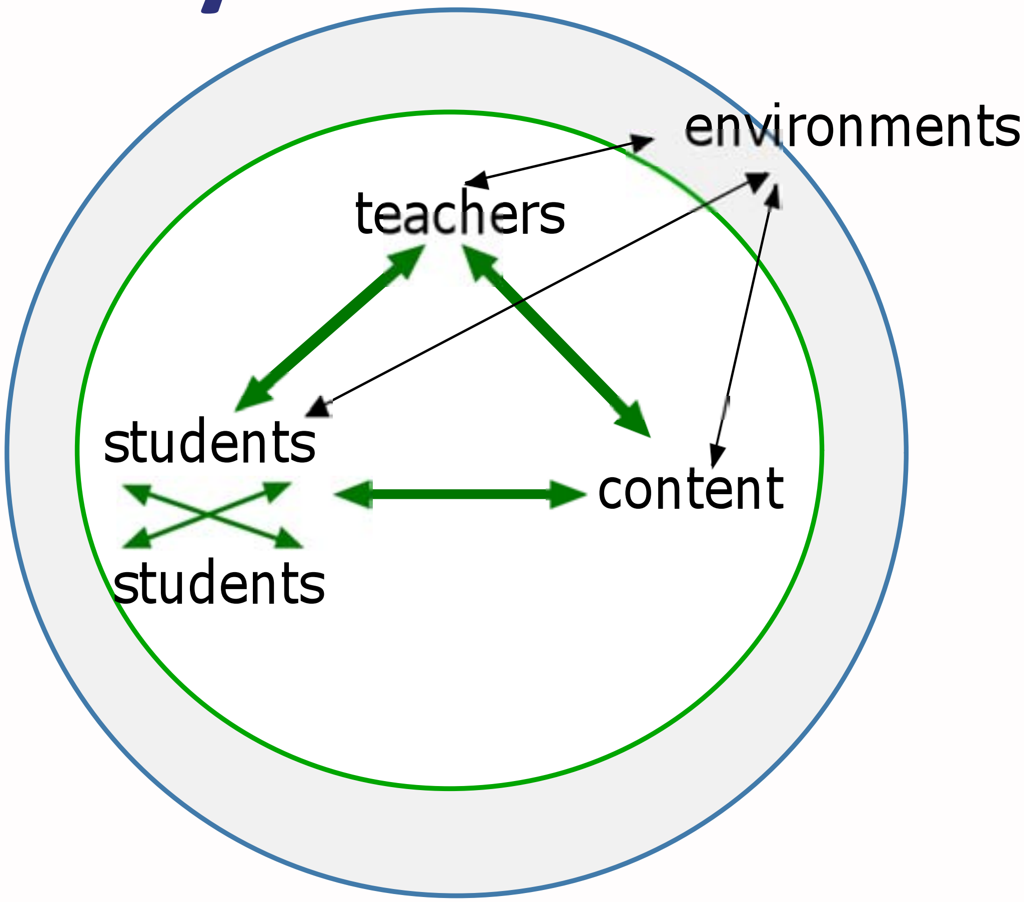
Early research on teacher knowledge

- Efforts to predict student achievement based on resources (education production function studies);
- Teacher resources:
  - Proxy measures: Teachers' professional preparation, characteristics, years of experience;
  - More direct measures: Teachers' performance on exams; verbal ability

Critiques of education production function studies

- Methodological flaws — e.g., the use of cross-sectional rather than longitudinal data, the use of composite measures of both teachers' knowledge and students' achievement
- More pressing: imprecise definition and indirect measurement of teachers' intellectual resources; mis-specification of the causal processes linking teachers' knowledge to students' learning

What are resources, and how do they affect students?



Approaches to measuring teachers' mathematical knowledge

1. Proxy measures
2. Mathematics tests
3. Interviews and tasks
4. Observations of classroom teaching

Improving the measurement of teachers' mathematical knowledge

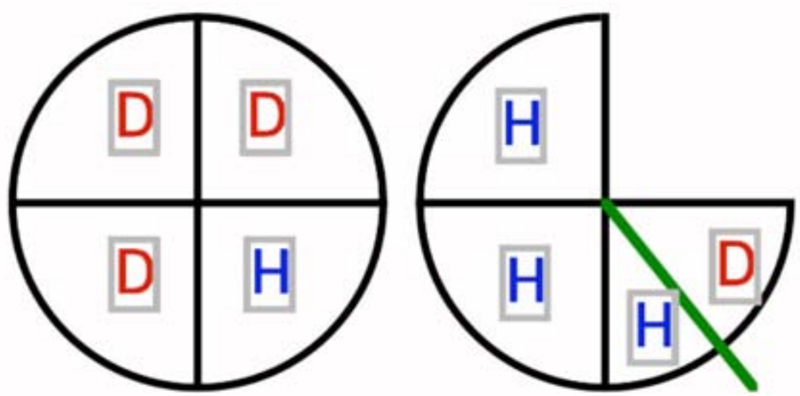
- Questions that asked for reasoning, representing, meanings

Calculate $1\frac{3}{4} \div \frac{1}{2}$

Write a story problem for $1\frac{3}{4} \div \frac{1}{2}$

Most frequent answer:

I have two pizzas. My friend eats one quarter of one of the pizzas . I have one and three quarters pizzas left. Then I split it evenly between two of my other friends. Each person gets three and a half pieces of pizza.



Other "better" questions

- How would you explain 7 ÷ 0?
- How is the concept of the derivative related to slope?
- Why do you "move the numbers over" on the second line of a multiplication problem?

```
   49
x  25
-----
  225
  100
-----
  325
```

By early 1990s . . .

1. Many studies claiming that U.S. teachers lacked mathematical knowledge (but sampling problems)
2. Evidence that course-taking or majoring in math did not provide “deep” or “flexible” knowledge, nor that it was correlated with student achievement gains
3. Same items used across studies, but with little or no investigation of their psychometric properties

3. Shifting the question

From teacher knowledge to knowledge for teaching:

1. What mathematics do teachers need to know?
2. What mathematics do teachers know?
3. What mathematics do teachers use?
4. What mathematics does teaching entail?

Starting with practice: What mathematics does teaching entail?

1. Study instruction and identify the mathematical work of teaching
2. Analyze what mathematical knowledge is entailed by the work (MKT)
3. Test the working hypotheses based on these analyses by developing measures of MKT, validating teacher scores against practice and against student achievement gains
4. Develop and evaluate approaches to helping teachers learn mathematical knowledge for teaching

Taking a step back:
What is involved in teaching mathematics?

Viewing a video segment from a third grade class

- Diverse class
- Working on even and odd numbers
- Can identify even and numbers, but no definition yet
- Student proposes that "6 can be an even number and it can be an odd number"

Viewing focus

- What mathematical issues arise?
- What mathematical work is involved for the teacher?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Viewing focus

- What mathematical issues arise?
- What mathematical work is involved for the teacher?

Examples of the work of teaching

- Choosing tasks
- Using mathematical language both precisely and accessibly
- Listening to and interpreting students' talk and written work
- Posing strategic questions
- Giving explanations
- Making judgments about what is important
- Evaluating and modifying textbook treatments of a topic
- Choosing and using representations, examples, metaphors
- Identifying and working toward the mathematical goal of the lesson
- Teaching students what counts as "mathematics" and mathematical practice
- Making error a fruitful site for mathematical work
- Deciding what to clarify, what to make more precise, what to leave in student's own language
- Managing instructional conversation and discussion, including deciding when to step in and clarify and when to let students work further themselves

Teaching mathematics involves special kinds of mathematical work

1. Solving special kinds of mathematical problems
2. Engaging in specialized mathematical reasoning
3. Using mathematical language precisely but accessibly

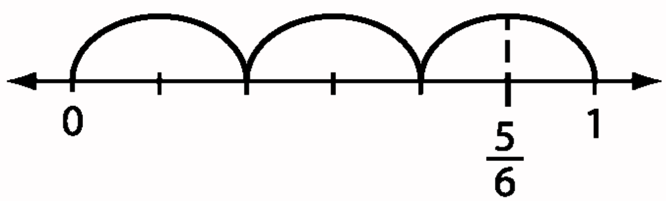
Choosing pedagogically-strategic examples

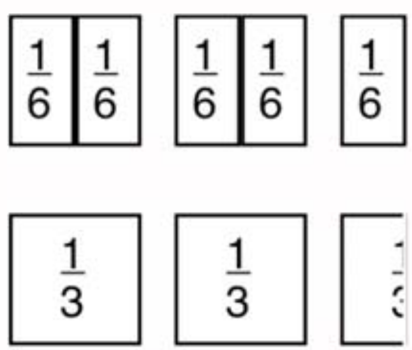
Which of the following lists would be best for assessing whether your students understand decimal ordering? Justify your choice.

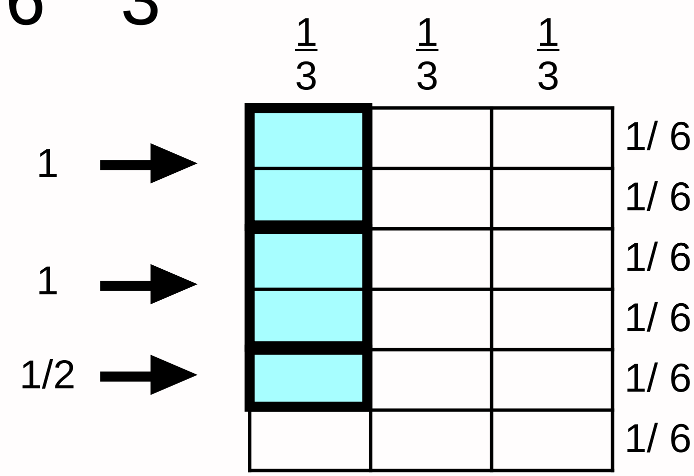
A.	.5	7	.01	11.4
B.	.60	2.53	3.12	.45
C.	.6	4.25	.565	2.5

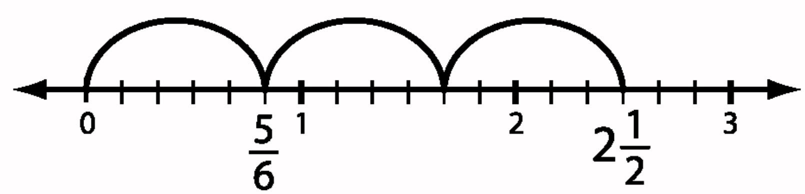
Analyzing — and "talking" — representations

Which of these can be used to represent $\frac{5}{6} \div \frac{1}{3}$? Explain with reference to all parts of the expression.









Analyzing errors

What mathematical steps could have produced this answer?

(a)	(b)	(c)
49	49	49
x 25	x 25	x 25
405	225	1250
108	100	25
1485	325	1275

Using language precisely and accessibly

a) An even number is a number that can be divided into two equal parts.
b) An even number is any multiple of 2.
c) An even number is any integer multiple of 2.
d) An even number is any number whose unit digit is 0, 2, 4, 6, or 8.
e) A whole number is even if it is the sum of a whole number with itself.

Are these definitions okay? What are the consequences of any of these?

a) An even number is a number that can be divided into two equal parts.

b) An even number is any multiple of 2.

All numbers, for example 7, 3/5, √2, π, are even!

c) An even number is any integer multiple of 2.

This is a correct definition of even number.

d) An even number is any number whose unit digit is 0, 2, 4, 6, or 8.

In this case, 36.7 is an even number!

e) A whole number is even if it is the sum of a whole number with itself.

This is a correct definition of evenness for whole numbers, and is consistent with the general definition for integers that will arrive later.

4. Progress toward a "practice-based theory" of mathematical knowledge for teaching

1. Developing measures of MKT
2. Factor analyses and validation studies
3. Theory development

Opportunity: Study of Instructional Improvement

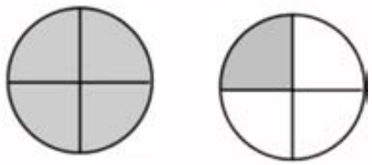
- Study of three Comprehensive School Reforms; teacher knowledge a key variable
- Instrument development goals:
  - Develop measures of content knowledge teachers *use* in teaching – not just *what* they teach
  - Develop measures that discriminate among teachers (not criterion referenced)
  - Non-partisan

Original sampling frame

	Types of knowledge		
Mathematical content		Common and specialized content knowledge	Using knowledge of students and content
	Number		
	Operations		
	Patterns, functions, and algebra		

Representing number concepts

Mrs. Johnson thinks it is important to vary the whole when she teaches fractions. For example, she might use five dollars to be the whole, or ten students, or a single rectangle. On one particular day, she uses as the whole a picture of two pizzas. What fraction of the two pizzas is she illustrating below? (Mark ONE answer.)



a) 5/4

b) 5/3

c) 5/8

d) 1/4

Providing mathematical explanations: Number concepts

Ms. Harris was working with her class on divisibility rules. She told her class that a number is divisible by 4 if and only if the last two digits of the number are divisible by 4. One of her students asked her why the rule for 4 worked. She asked the other students if they could come up with a reason, and several possible reasons were proposed. Which of the following statements comes closest to explaining the reason for the divisibility rule for 4? (Mark ONE answer.)

a) Four is an even number, and odd numbers are not divisible by even numbers.

b) The number 100 is divisible by 4 (and also 1000, 10,000, etc.).

c) Every other even number is divisible by 4, for example, 24 and 28 but not 26.

d) It only works when the sum of the last two digits is an even number.

Analyzing non-standard (but correct) responses

Which student is using a method that could be used to multiply any two whole numbers?

Student A	Student B	Student C
35	35	35
x25	x25	x25
125	175	25
+75	+700	150
875	875	100
		+600
		875

Using data to test and improve theory

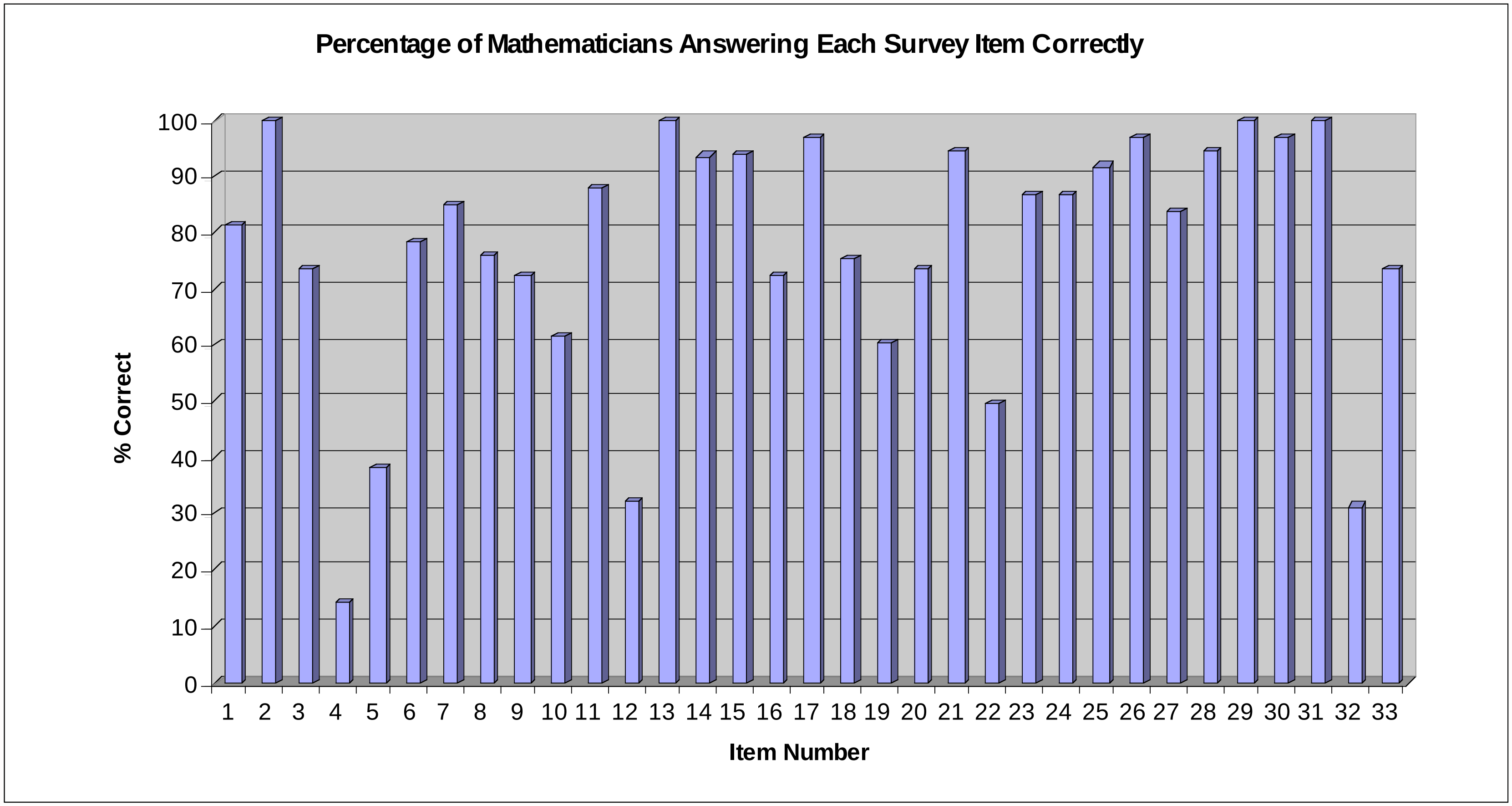
- Factor analyses
- Analyses of validity
- Uses of measures
  - To predict student achievement
  - To evaluate professional development

Validating our measures

How do we interpret teachers' performance on our questions?

1. Their score reflects their mathematical thinking
   – Cognitive interviews
2. Higher scores mean higher-quality mathematics instruction
   – Videotape validation study
3. Scores reflect common and specialized knowledge of content
   – Mathematician and non-teacher interviews
4. Higher scores related to improved student learning
   – Student gains analysis

Percentage of mathematicians answering each survey item correctly



Why did mathematicians get items wrong?

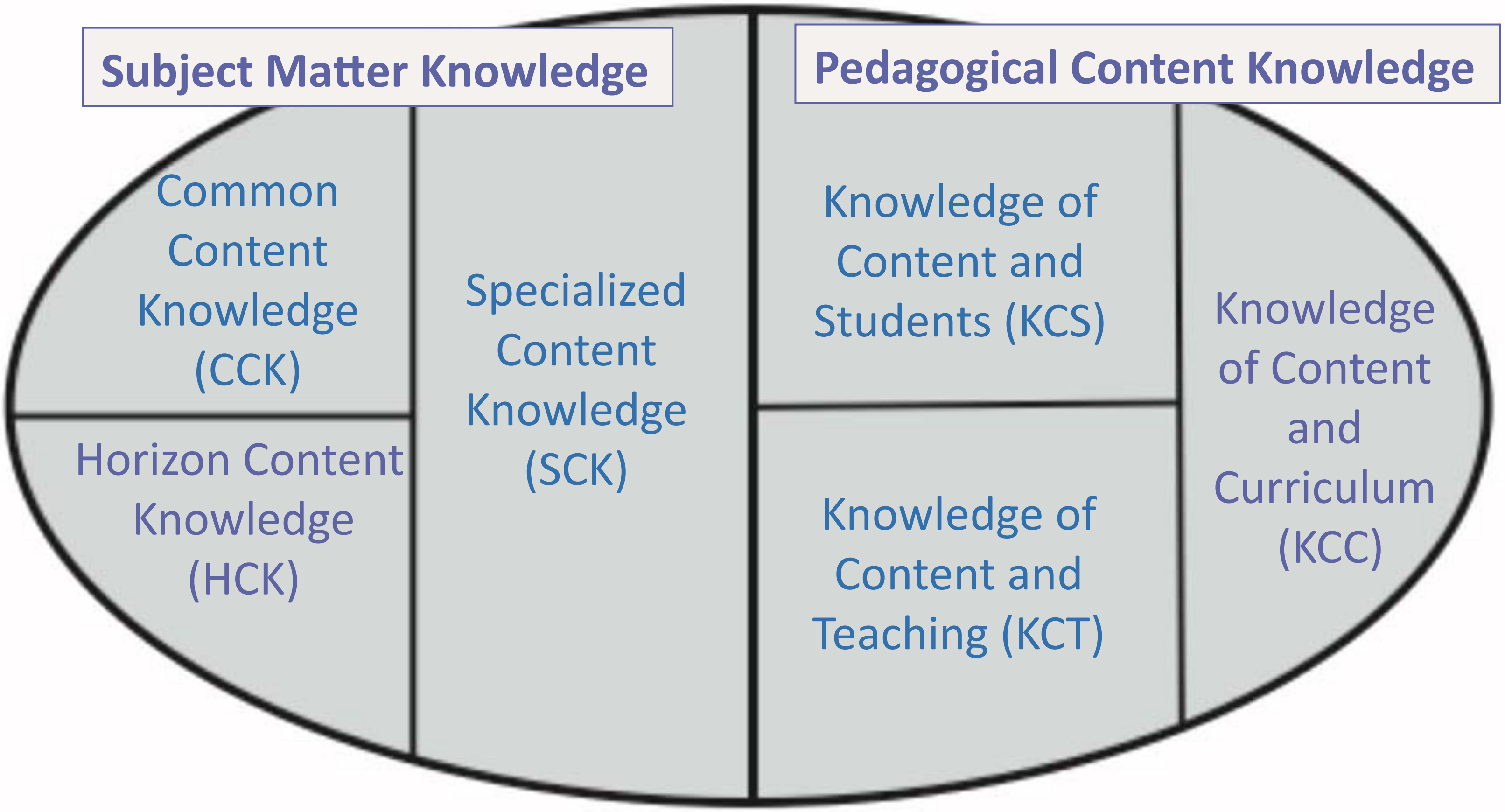
- Items had mathematical flaws.
- Items required knowledge of learners.
- Items demanded mathematical knowledge unique to the work of teaching:
  - Making sense of non-standard solutions or ideas
  - Choosing numerical examples
  - Choosing representations

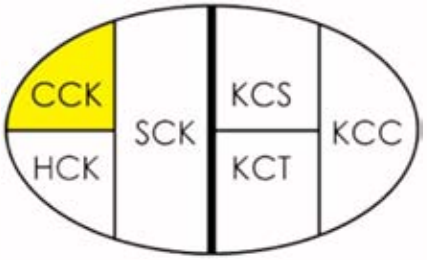
Linking teacher knowledge and student achievement[1]

- Questionnaire consisting of 30 items (scale reliability .88)
- Model: Student Terra Nova gains predicted by:
  - Student descriptors (family SES, absence rate)
  - Teacher characteristics (math methods/content, content knowledge)
- Teacher content knowledge significant
  - Small effect (< 1/10 standard deviation): 2 - 3 weeks of instruction
  - But student SES is also about the same size effect on achievement

[1]Hill, Rowan, and Ball (2005)

Mathematical Knowledge for Teaching (MKT)

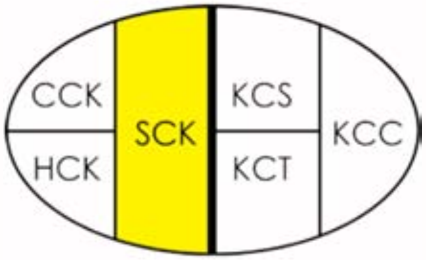




Common content knowledge (CCK)

Calculate:

$$\frac{5}{6} \div \frac{1}{3}$$



Specialized content knowledge (SCK)

$$\frac{5}{6} \div \frac{1}{3} = \frac{10}{12} \div \frac{4}{12} = 10 \div 4 = 2\frac{1}{2}$$

Is this a fluke?

Does it work in general?

If so, why does it work?

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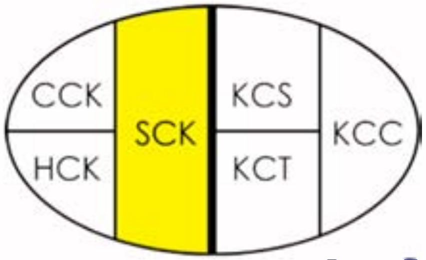
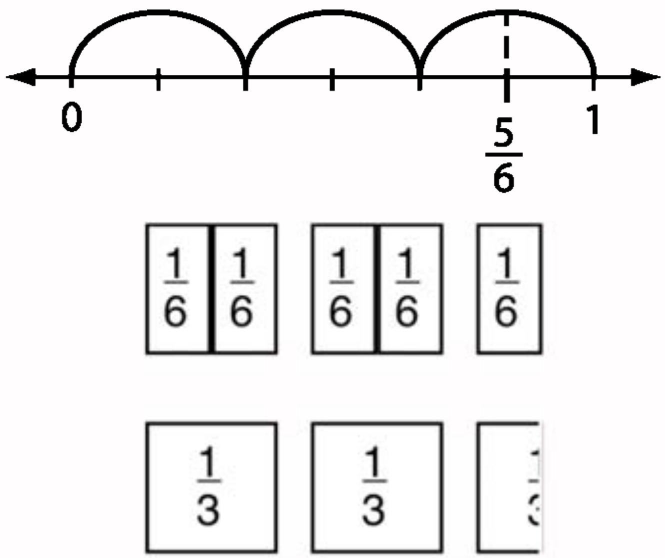
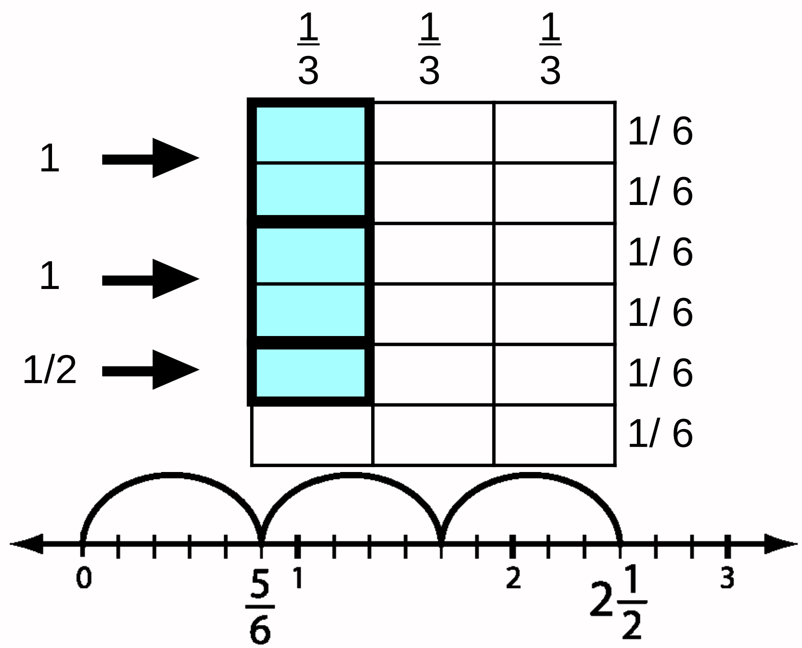
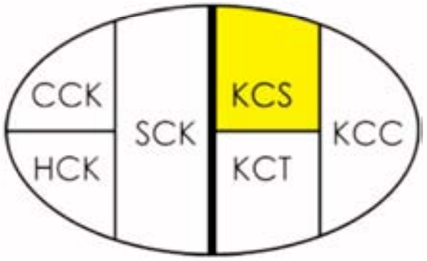


Specialized content knowledge (SCK)

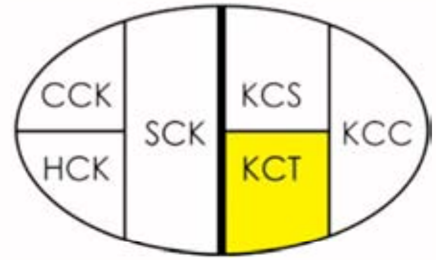
Which of these can be used to represent $\frac{5}{6} \div \frac{1}{3}$?







$$\frac{5}{6} \div \frac{1}{3} = 2\frac{1}{2}$$

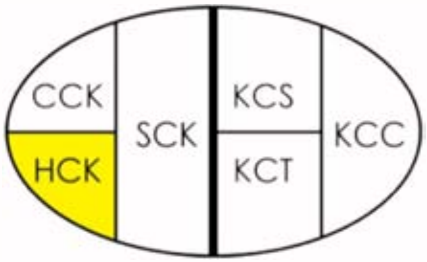


Knowledge of students and content (KCS)

- What are common errors students make when dividing fractions?
- How do students' experiences with division of whole numbers support their understanding of division of fractions? How does it confuse them?
- What difficulties do students typically have interpreting the answer to a division of fractions problem?

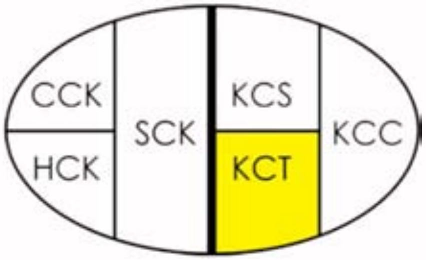
Knowledge of teaching and content (KCT)

- Which representation would you use to introduce the meaning of division of fractions? Or to explain the invert and multiply algorithm?
- What sequence of problems would you use to begin work on division of fractions?
- In a whole-class discussion, what solution methods would you want presented, and in what order?



Horizon content knowledge (HCK)

- A student comments that "if you divide by smaller and smaller fractions, the answers get bigger." Is the student right? Is this mathematically significant or interesting?
- Are there mathematically significant notions that underlie division of fractions?



Knowledge of content and curriculum (KCC)

- At what grade level are students typically taught to divide fractions?
- How is division of fractions related to division of whole numbers in the school curriculum?
- What are the models for fractions and for division with which students would be familiar?

5. Unanswered, partially answered, and new questions, and next steps

1. Studying teaching of other mathematical topics, practices in other settings
2. (How) does MKT differ for teaching high school?
3. How can teachers develop MKT?
4. How might MKT be validly assessed for licensure?
5. Is this theory generalizable to other countries?

Are U.S.-based measures of MKT valid for research in other countries?

THANK YOU!

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