Knowing Mathematics as a Teacher

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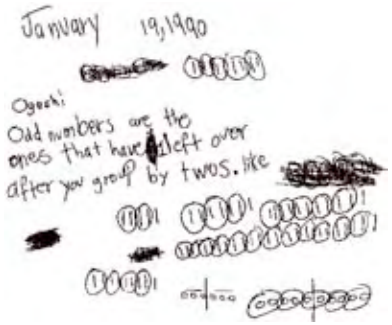

Annual Meeting of

the National Council of Teachers of Mathematics

Anaheim, CA

April 8, 2005

Possible sized squares	Length of side of main square: 1	2	3	4	5	6	7	10	50
1x1	1	4	9	16	25	36	49		
2x2		1	4	9	16	25	36		
3x3			1	4	9	16	25		
4x4				1	4	9	16		
5x5					1	4	9		
6x6						1	4		
7x7							1		
8x8									
9x9									
10x10									
Total	1	5	14	30	55	91	140		

(B)

$$\begin{array}{r} {}^{1}{}_{2}35 \\ \times\ 25 \\ \hline 255 \\ 80 \\ \hline 1055 \end{array}$$

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Overview of Session

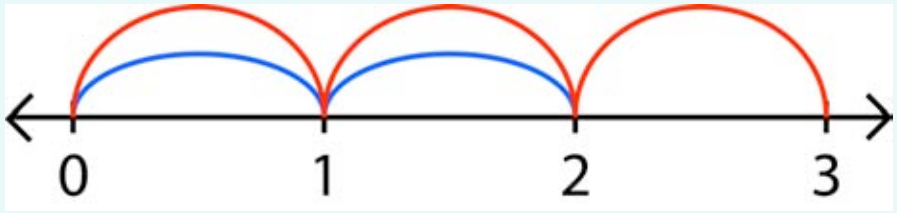
1. Setting the policy context
2. Mathematical knowledge for teaching: A theory in progress
3. Developing and using measures of mathematical knowledge for teaching

U.S. DEPARTMENT OF EDUCATION
Secretary's No Child Left Behind

1. Setting the Policy Context

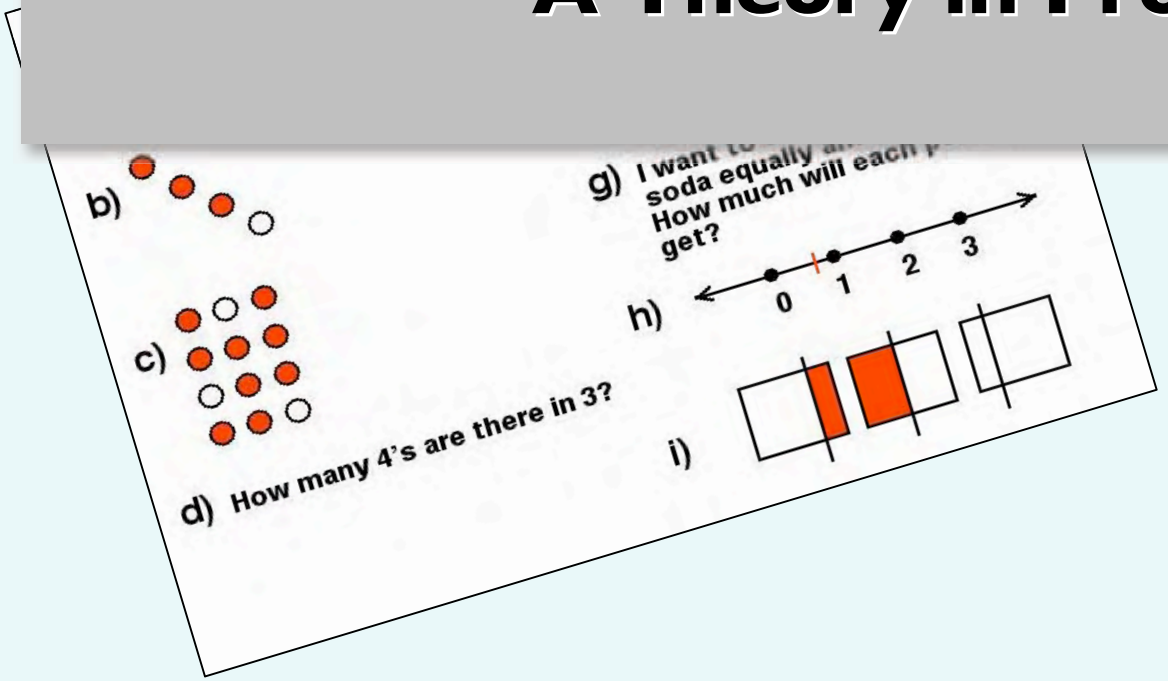
Challenges to the Education Profession

1. To claim professional jurisdiction for the preparation and development of teachers (Grossman, 2005)
2. To articulate characteristics of "high quality" teaching, and the knowledge and skill demands for teachers
3. To seek evidence on effects



$$2 \div \frac{2}{3}$$

2. Mathematical Knowledge for Teaching: A Theory in Progress



Common Framing of the Problem → Failure to Solve the Problem

- Teachers do not know mathematics well enough (evidence and causes vary)
- Concentration of under-qualified teachers in urban and high poverty schools

- Increase number and rigor of mathematics requirements for teacher certification
- Recruit mathematically-trained people into teaching

Clarifying the Problem

The quality of mathematics teaching and learning

Teachers' knowledge of mathematics

and their ability to use it in teaching

What mathematical resources do teachers need in order to teach mathematics to all students effectively?

A Practice-Based Approach to Answering the Question

1. **Study instruction**, and identify the mathematical work of teaching
2. Analyze **what mathematical knowledge** is needed to do that work effectively, and **how** it must be understood in order to be useful for the work
3. Develop, test, and refine **approaches to helping teachers** develop and use mathematical knowledge for teaching, study results

Mathematical Knowledge for Teaching

What do we mean when we use this term, "**mathematical knowledge for teaching**"?

- Mathematical knowledge, skill, habits of mind that are entailed by the work of teaching

What do we mean by the "**work of teaching**"?

- The tasks in which teachers engage, and the responsibilities they have, to teach mathematics, both inside and outside of the classroom

What *is* "Mathematical Knowledge for Teaching"? An Example from Multiplication of Decimals

Multiply:

$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline \end{array}$$

Analyzing Incorrect Answers for $\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline \end{array}$

Why do you multiply before you add?

(a)
$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline 255 \\ 80 \\ \hline 10.55 \end{array}$$

(b)
$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline 62.5 \end{array}$$

Why do you have as many decimal places in the answer as the total number of decimal places in the problem?

Analyzing Correct Answers for

$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline \end{array}$$

(a)

$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline .25 \\ 1.5 \\ 1 \\ 6 \\ \hline 8.75 \end{array}$$

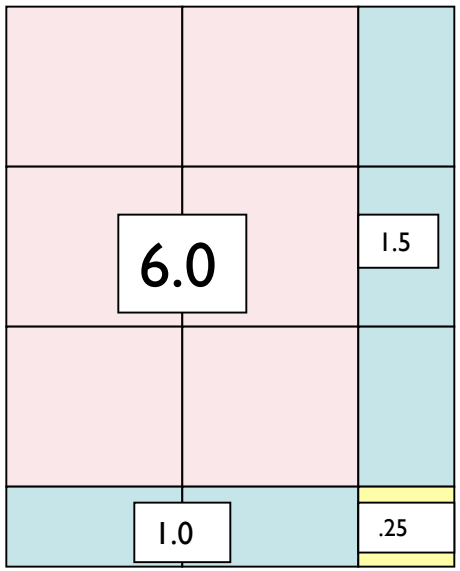
(b)

$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline 1.25 \\ 7.5 \\ \hline 8.75 \end{array}$$

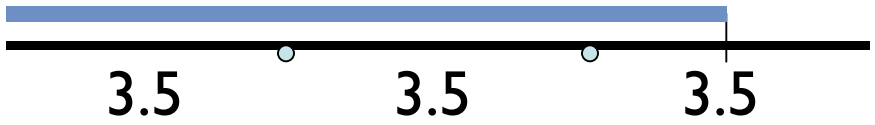
Is there a method?
Does it work for
all decimal numbers?
How do you know?

Using Representations for $\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline \end{array}$

(a)



(b)



Explain:
Where is the 3.5?
Where is the 2.5?
Where is 8.75?

Hypotheses About Knowledge of Mathematics for Teaching

PREVALENT HYPOTHESES

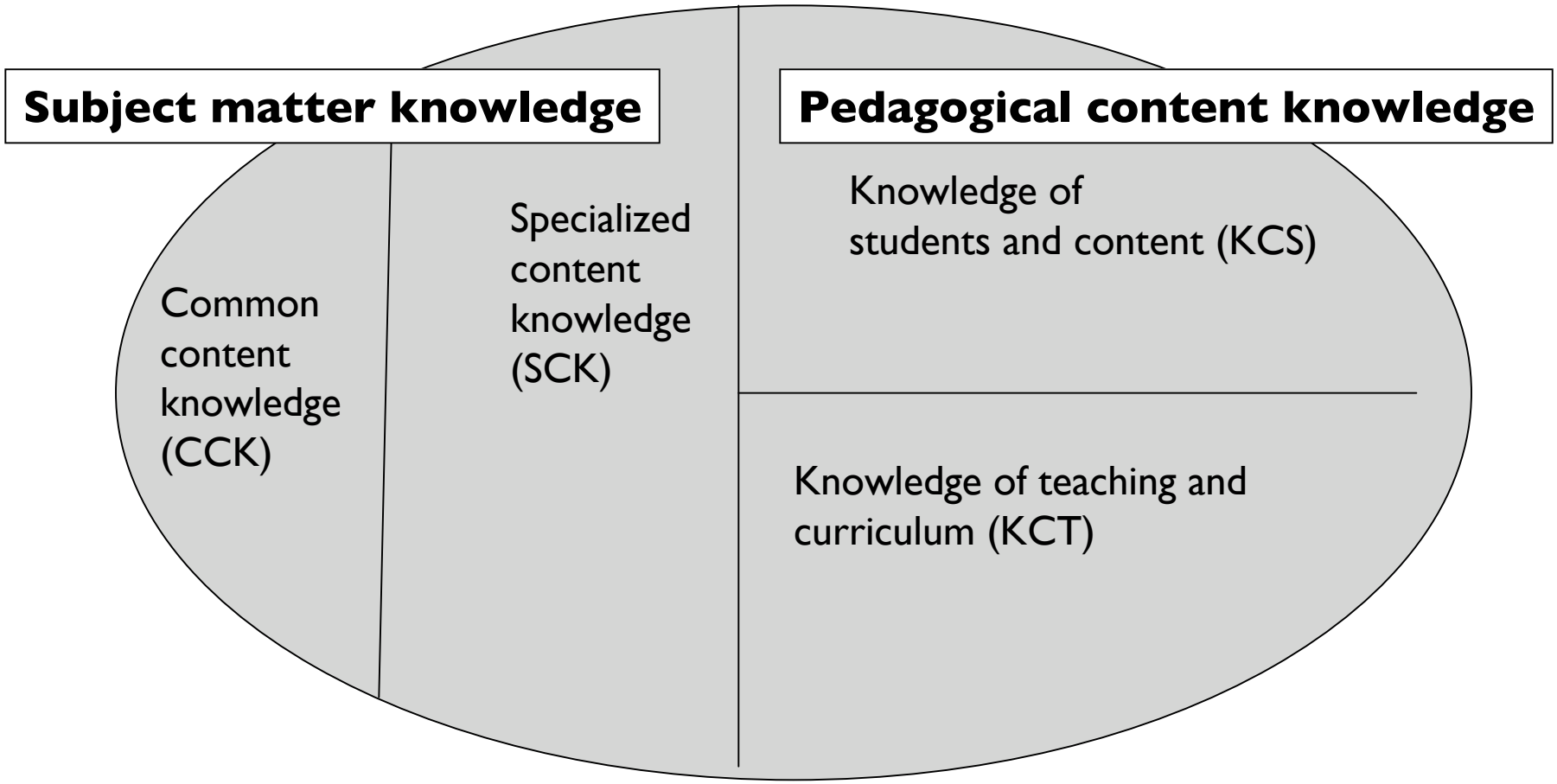
1. Teachers need knowledge of the curriculum, plus N levels more knowledge
2. Pedagogical content knowledge, curricular knowledge

OUR CURRENT HYPOTHESES

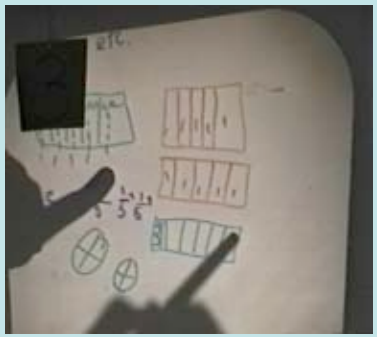
- **Common** content knowledge
- **Specialized** content knowledge
- Knowledge of content and **students**
- Knowledge of content and **teaching and curriculum**

What does this have to do with "pedagogical content knowledge"?

Shulman's Original Category Scheme (1985) Compared with Ours



3. Developing and Using Measures of Mathematical Knowledge for Teaching



Opportunity: Study of Instructional Improvement

- Study of three Comprehensive School Reforms; teacher knowledge a key variable
- Instrument development goals:
  - Develop measures of content knowledge teachers *use* in teaching – not just *what* they teach
  - Develop measures that discriminate among teachers (not criterion referenced)
  - Non-partisan

Early Decisions and Activity

- Survey-based measure of content knowledge for teaching mathematics
  - 5000 teachers participating in SII
  - Multiple choice
- Specified domain map
- 5 people + 5 lbs. cheese + 5 weeks = 150 items (May 2001)
- Large-scale piloting, summer 2001

Original Sampling Frame

	Types of knowledge		
Mathematical content		Common and specialized content knowledge	Using knowledge of students and content
	Number		
	Operations		
	Patterns, functions, and algebra		

Sample Items
Number Concepts and Operations

- **Common content knowledge**

  What number is halfway between 1.1 and 1.11?

- **Specialized content knowledge**

  a) Representing mathematical ideas and operations

  b) Providing **explanations** for mathematical ideas and procedures

  c) Evaluating mathematical methods, claims, or solutions

Representing Number Concepts

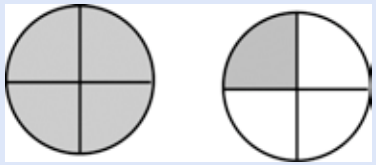
Mrs. Johnson thinks it is important to vary the whole when she teaches fractions. For example, she might use five dollars to be the whole, or ten students, or a single rectangle. On one particular day, she uses as the whole a picture of two pizzas. What fraction of the two pizzas is she illustrating below? (Mark ONE answer.)

a) 5/4

b) 5/3

c) 5/8

d) 1/4



Providing Mathematical Explanations Number Concepts

Ms. Harris was working with her class on divisibility rules. She told her class that a number is divisible by 4 if and only if the last two digits of the number are divisible by 4. One of her students asked her why the rule for 4 worked. She asked the other students if they could come up with a reason, and several possible reasons were proposed. Which of the following statements comes closest to explaining the reason for the divisibility rule for 4? (Mark ONE answer.)

a) Four is an even number, and odd numbers are not divisible by even numbers.

b) The number 100 is divisible by 4 (and also 1000, 10,000, etc.).

c) Every other even number is divisible by 4, for example, 24 and 28 but not 26.

d) It only works when the sum of the last two digits is an even number.

Evaluating Alternative Solution Approaches Operations

Student A	Student B	Student C
35 x25 125 +75 875	35 x25 175 +700 875	35 x25 25 150 100 +600 875

Which of these shows a method that could be used to multiply any two whole numbers?

Using Data to Test and Improve Theory

- Factor analyses
- Analyses of validity
- Uses of measures
  - To predict student achievement
  - To evaluate professional development

Factor Analysis

- Enables analyses of the number of underlying factors in a data set
- Can link items to constructs
- Ask similar question of our data:
  - How many factors? What are they named?
  - How do items group?

Overarching Findings: Factor Analyses

- Multidimensionality of mathematical knowledge for teaching
  - Knowledge of students and content different from content knowledge
  - Patterns, functions & algebra content knowledge for teaching
  - Geometry mathematical knowledge for teachings
  - Number and and operations mathematical knowledge for teaching
    - Specialized
    - Common
- Tentative conclusion: presence of professional knowledge for teaching

Validating Our Measures

How do we interpret teachers' performance on our questions?

1. Their score reflects their mathematical thinking
   - **Cognitive interviews**
2. Higher scores mean higher-quality mathematics instruction
   - **Videotape validation study**
3. Scores reflect common and specialized knowledge of content
   - **Mathematician and non-teacher interviews**
4. Higher scores related to improved student learning
   - **Study of Instructional Improvement student gains analysis**

Studying Links Between Teachers' Knowledge of Mathematics for Teaching and Student Achievement

- Question: Do students learn more when teachers perform better on our measures?
  - SII data and analysis – measured CKT on questionnaire
  - SII measures highly reliable, from test construction perspective
- Model: Student Terra Nova gains predicted by:
  - Student descriptors (family SES, absence rate)
  - Teacher characteristics (math methods/content, content knowledge)
- Teacher content knowledge significant
  - Small effect (< 1/10 standard deviation)
  - But student SES is also on same order of magnitude

(Hill, Rowan, & Ball, AERJ, in press)

Studying Teachers' Gains in Mathematics for Teaching from Professional Development

- Items piloted in California's Mathematics Professional Development Institutes (MPDI)
  - Instructors: Mathematicians and mathematics educators;
  - 40-120 hours of professional development
  - Focus was squarely on mathematics content
  - Summer 2001
  - Pre/post assessment format (parallel forms)

Findings from MPDI Evaluation(2001)

- Length of institute predicts teacher gains
  - 120-hour institutes most effective, on average
  - But some 40-hour institutes very effective
- Focus on mathematical analysis, proof, and communication leads to higher gains
- Many questions remain
  - Effects of content (e.g., mathematics vs. student thinking)
  - Treatment of content: "packed"/"unpacked"
  - Effects of teacher motivation

(Hill & Ball, JRME, 2004)

Conclusion:
Who Knows Mathematics Well Enough to Teach It, and How Do We Make the Argument?

- Mathematical knowledge for teaching is the knowledge needed for the work of teaching. That work entails specialized content knowledge distinct from that required by other mathematically-intensive professions.
- Figuring out what teachers need to know requires multiple kinds of work, including research-based evidence. It is possible to write questions that reliably and validly measure mathematical knowledge for teaching. The work of producing usable items requires multiple kinds of expertise, a good conceptual frame, and many cycles of revision and improvement.
- Such questions are more likely to seem credible to teachers.
- Work on these questions is central to the future of the profession, and to the improvement of mathematics education.

Slides will be available at

http://www-personal.umich.edu/~dball/

http://www.sii.soe.umich.edu

http://www.soe.umich.edu/lmt