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Learning Mathematics for Teaching: Seeing Teaching as Mathematical Work

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Overview

1. Clarifying the problem of teachers' mathematical knowledge
2. Seeing teaching as mathematical work
3. Developing a practice-based approach to learning mathematics for teaching

1. Clarifying the Problem of Teachers' Mathematical Knowledge

Shifting the Focus — From Knowing to Knowing and Using

- Many criticisms of teachers for not knowing mathematics
- Evidence: teachers' scores, research on teacher knowledge, anecdotes, mathematicians' shock
- Remedy: increase and make more rigorous the mathematics requirements for teachers
- This remedy is likely inadequate to solving the main problem

What is the actual problem that we are trying to address?

The quality of mathematics teaching and learning

Teachers' knowledge of mathematics and their ability to use it in their teaching

2. Seeing Teaching as Mathematical Work: What Do Teachers DO with Mathematics?

Tasks of Teaching Mathematics: Some Examples

- Analyzing errors
- Evaluating procedures
- Building correspondences between a model and a concept or procedure
- Judging representations
- Choosing and using representations
- Evaluating students' work and statements (often quickly)
- Producing and evaluating mathematical explanations
- Choosing and using definitions
- Explaining goals and mathematical purposes to others
- Designing homework and quizzes

Analyzing Errors

Student A

$$\begin{array}{r} .75 \\ \times .50 \\ \hline 00 \\ 455 \\ \hline .4550 \end{array}$$

Student B

$$\begin{array}{r} 2.75 \\ \times .24 \\ \hline 1100 \\ 550 \\ \hline .1650 \end{array}$$

Are both students making the same types of errors? What are they doing? Why might they do these things?

Evaluating Mathematical Procedures

$$\begin{array}{r} {}^{2}\cancel{3}\,{}^{9}0\,{}^{1}7 \\ -\ 1\,6\,8 \\ \hline 1\,3\,9 \end{array}$$

$$\begin{array}{r} 3\,0\,7 \\ -\ 1\,6\,8 \\ \hline 2\text{-}6\text{-}1 \\ \hline 1\,3\,9 \end{array}$$

What is going on in each case?

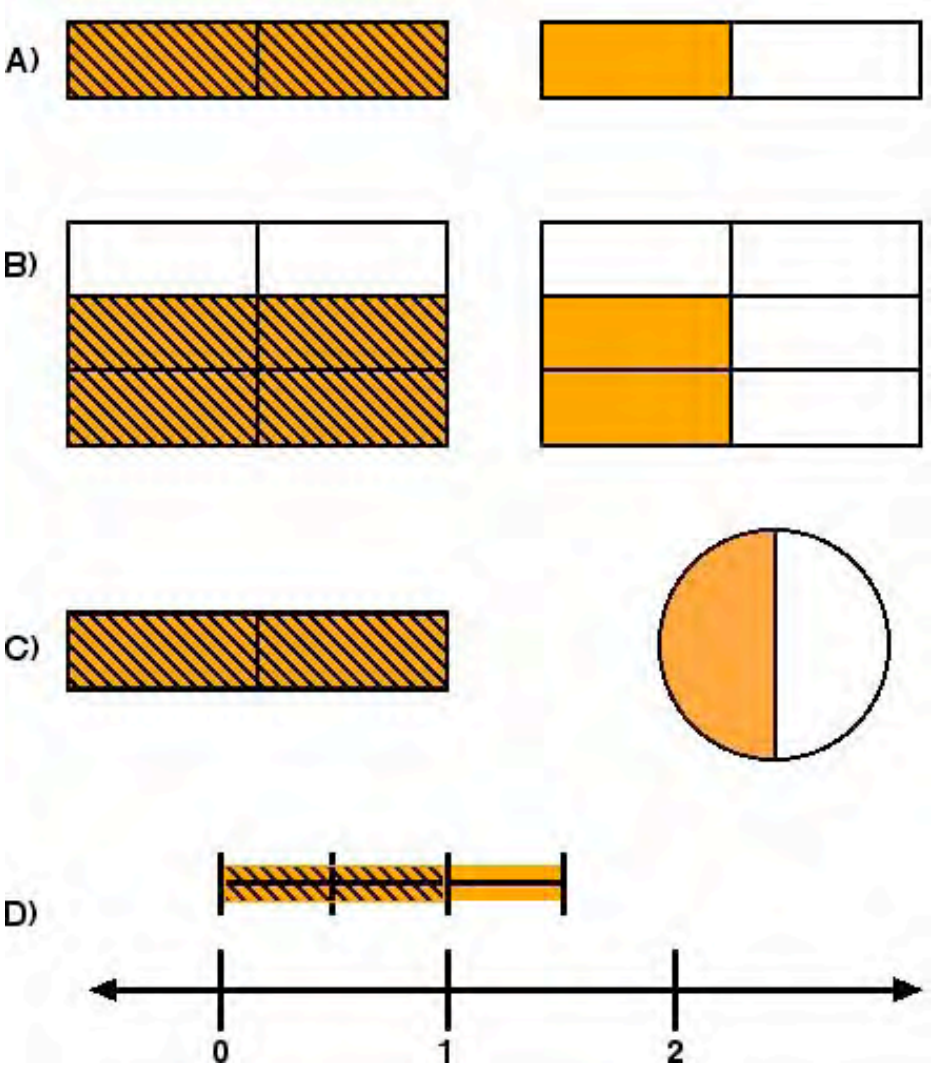
Will either method work to subtract any two whole numbers?

What are the mathematical advantages and pitfalls of each method?

Judging Representations

Which of the following represents the meaning of

$$1\frac{1}{2} \times \frac{2}{3}$$



Designing Quiz Questions

A. .5 7 .01 11.4

B. .60 2.53 3.12 .45

C. .6 4.25 .565 2.5

D. These lists are all equally good for assessing whether students understand how to order decimal numbers.

Building Correspondences Between a Mathematical Idea and a Model

$38 \div 4$

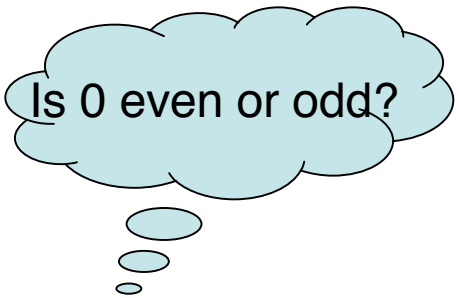
Write three different stories that correspond to this division expression and that reveal different interpretations of the meaning of division or its mapping to specific situations.

Defining Concepts and Terms

Examining Textbook Definitions

How well does any of these help to answer the the question?

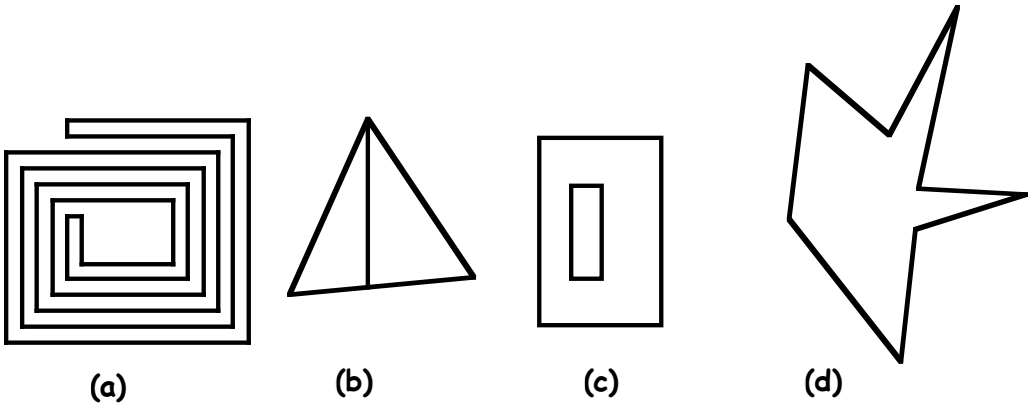
1. An even number is a number of the form 2k, where k is an integer.
2. An even number is a natural number that is divisible by 2.
3. An even number is any multiple of 2.
4. An even number is a number that has 0, 2, 4, 6, or 8 in the ones place.



What is a mathematically appropriate and usable definition of "even number" for third graders?

Choosing and Developing Usable Definitions

Which of these shapes are polygons?



Definition: A *polygon* is a simple closed plane curve formed by straight line segments.

What is a mathematically appropriate and usable definition of “polygon” for fifth graders?

3. Developing a Practice-Based Approach to Learning Mathematics for Teaching

Learning Mathematics for Teaching

How can we help teachers learn mathematics in ways that equip them for the mathematical work of teaching?

Mathematical Proficiencies for Teaching

- Learn to unpack and explain mathematical ideas
- Practice examining, analyzing, comparing alternative mathematical solutions, representations, ideas
- Develop skill with mathematical practices such as explaining, reasoning, representation, examining equivalences, careful use of language
- Practice solving and justifying solutions to mathematical problems of teaching

Investigation:
The Work of Teaching Third Grade Mathematics

- Third graders (8 year-olds) working on even and odd numbers
- Culturally and linguistically diverse
- Orientation of the instruction:
  - To work on substantial mathematics and treat the mathematics with integrity
  - To take students' thinking seriously
  - To treat the construction of mathematical knowledge as the work of an intellectual collective, with justification crucial

From a mathematical perspective, what is important to notice about what the children are doing?

Background and Context for Video Clip

Pencils cost 7¢.

Sticks of gum cost 2¢.

What combinations of pencils and gum can you buy and spend exactly 30¢?

Starting point: Recognize and correctly label small cases of even and odd numbers, no precise definitions

Ending point: Precise definitions for even and odd numbers, and ability to reason with the definitions

Thursday, January 18

Meeting with fourth grade class:
Even and odd numbers

Friday, January 19

Debriefing the meeting

(Return to conjectures about even and odd numbers
Odd + odd = even
Even + even = even
Odd + even = odd

Shea brings up idea: 6 could be even or odd

Class attempts to refute, using three different definitions of even numbers: alternating (even, odd, even, odd. . .), fair share (2k or k + k), pairing by twos (k2 or 2 + 2 + . . +2 + 2)

Lin generalizes Shea's claim in an effort to refute it:
"What about 10? Why don't you call 10 an even and an odd number?"

Ogechi develops clearer expression of definition of odd numbers ("numbers with a 1 in the middle 2 + 2 + 2 +. . . +1+ . . .+2 + 2 + 2)

Rania observes pattern: 14, 18, 22, 26 . . . (period four)

Teacher labels the generalized class of even numbers "Shea Numbers"; class defines as " even numbers that have an odd number of groups of twos"

Definitions of even and odd number are reconciled and made more precise: k2 (even) and k2 +1 (odd)

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Friday, January 26

Jillian and Shekira argue that it is not possible to prove Bernadette's conjecture: An odd number plus an odd number equals an even number.

Tuesday, January 30

Bernadette offers a proof of the conjecture.

Designing Opportunities to Learn Mathematics From the Study of a Video Clip

What **mathematical issues**

do you notice in this video clip?

Insert video of Shea (6 can be both even and it can be odd)

Mathematical Issues

- Three different definitions of "even number" in use, unreconciled
- Shea has an idea in need of a name; is instead appropriating "even and odd" to label his idea
- Shea makes a claim about 6; Lin generalizes
- Students expecting to listen, interpret, and respond to others' claims
- Imperative for justification
- What is a mathematically appropriate and usable definition of "even" and "odd" numbers for third graders?
- A small step from mod-two to mod-four arithmetic.

Designing Opportunities for Teachers to Learn Mathematics From the Study of a Video Clip

Task #1

Lin used 10 to challenge Shea's claim. Why was this a good choice?

What other number would be strategic to ask Shea about, and why?

Task #2

Three different definitions of even number are being used in this lesson episode. What are they, and who is using which one?

Are the three definitions equivalent? Prove your answer.

Task #3

Your principal is concerned that you let this discussion go on. You were worried about this, too.

Can you explain what mathematics you were working on with the students by letting this discussion continue?

Solving Problems in Learning Mathematics for Teaching

Task	What opportunities for teachers to learn mathematics for teaching does the task offer?
Task #1: Lin used 10 to challenge Shea's claim. Why was this a good choice? What other number would be strategic to ask Shea about, and why?	
Task #2: Three different definitions of even number are being used in this lesson episode. What are they, and who is using which one? Prove that the three definitions are equivalent.	
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Solving Problems in Learning Mathematics for Teaching

Task	What opportunities for teachers to learn mathematics for teaching does the task offer?
Task #1: Lin used 10 to challenge Shea's claim. Why was this a good choice? What other number would be strategic to ask Shea about, and why?	Seeking counterexamples Factorization Analyzing structure Generalization
Task #2: Three different definitions of even number are being used in this lesson episode. What are they, and who is using which one? Prove that the three definitions are equivalent.	Knowledge of definitions Analyzing and reconciling multiple definitions Justifying claims
Task #3: Your principal is concerned that you let this discussion go on. You were worried about this, too. Can you explain what mathematics you were working on with the students by letting this discussion continue?	Seeing, identifying, and naming mathematical ideas, skills, and habits Communicating about mathematics to others Definitions of even and odd numbers Exploring patterns Importance of naming Evaluating conjectures Justifying claims Connections to modular arithmetic

Conclusions

1. Knowledge needed for teaching is different from what is needed for other occupations or professions where mathematics is used (e.g., physics, mathematics, carpentry, tailoring, business).

2. Knowledge needed for teaching must be usable for the mathematical work that teachers do.

3. Mathematics teaching involves specialized mathematical problem solving.

4. Learning mathematics for teaching can be grounded in practice by designing opportunities for teachers to solve mathematical problems that arise recurrently in teaching.