Developing Measures of Mathematical Knowledge for Teaching

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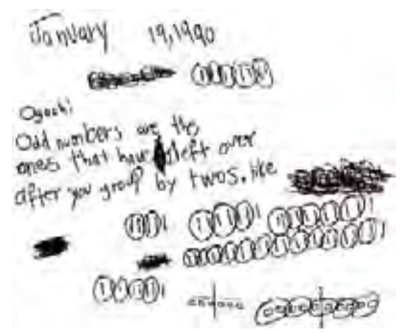
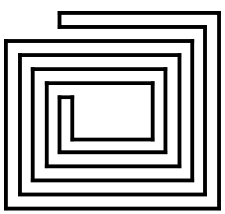
University of Michigan
Learning Mathematics for Teaching/Study of Instructional Improvement

Teachers Development Group Leadership Seminar
Mathematics Professional Development
February 18, 2005



Possible sized squares	Length of side of main square 1	2	3	4	5	6	7	10	50
1x1	1	4	9	16	25	36	49		
2x2		1	4	9	16	25	36		
3x3			1	4	9	16	25		
4x4				1	4	9	16		
5x5					1	4	9		
6x6						1	4		
7x7							1		
8x8									
9x9									
10x10									
Total	1	5	14	30	55	91	140		



B

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 1
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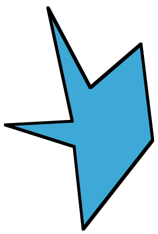

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Overview

1. "Mathematical knowledge for teaching"
2. Efforts to "measure" teachers' mathematical knowledge: Purposes, history
3. Our measures development approach
4. Using data to test and improve theory: factor analysis, reliability, validity
5. Trying your hand at writing items

Hypotheses About Knowledge of Mathematics for Teaching

PREVALENT HYPOTHESES

1. Teachers need knowledge of the curriculum, plus N levels more knowledge
2. Pedagogical content knowledge, curricular knowledge

OUR CURRENT HYPOTHESES

- **Common** content knowledge
- **Specialized** content knowledge
- Knowledge of content and **students**
- Knowledge of content and **teaching and curriculum**

What does this have to do with "pedagogical content knowledge"?

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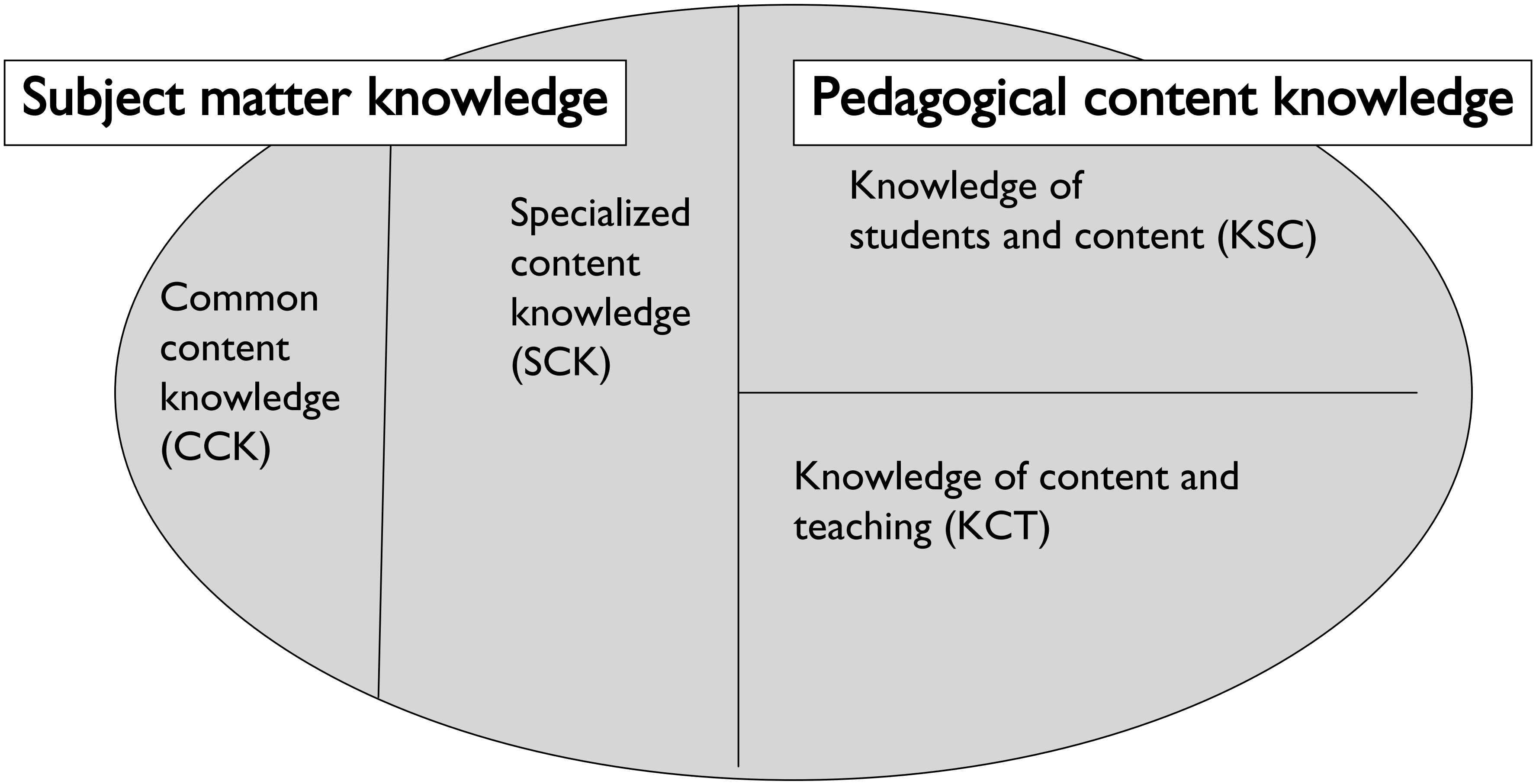
Shulman's Original Category Scheme (1985) Compared with Ours



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Why Would We Want to "Measure" Teachers' Content Knowledge for Teaching?

- To understand role of teachers' content knowledge in students' performance
- To study and compare outcomes of professional development and teacher education
- To inform design of teachers' opportunities to learn content knowledge

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Study of Instructional Improvement

- Study of three Comprehensive School Reforms; teacher knowledge a key variable
- Instrument development goals:
  - Develop measures of content knowledge teachers *use* in teaching – not just *what* they teach
  - Develop measures that discriminate among teachers (not criterion referenced)
  - Non-partisan

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Problems As We Began This Work

- No way to measure teachers' content knowledge for teaching on a large scale
  - Small number of items, many written by Ball, Post, others appeared on every instrument
  - Nothing known about the statistical qualities of those items (difficulty, reliability)
  - Studies relying on single items -- single items unlikely valid or reliable measures of teacher knowledge

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Early Decisions and Activity

- Survey-based measure of content knowledge for teaching mathematics
  - 3000 teachers participating in SII
  - Multiple choice
- Specified domain map
- 5 people + 5 lbs cheese + 5 weeks = 150 items (May 2001)
- Large-scale piloting, summer 2001

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Original Sampling Frame

	Types of knowledge		
Mathematical content		Common and specialized content knowledge	Using knowledge of students and content
	Number		
	Operations		
	Patterns, functions, and algebra		

Sample Items
K-5 Number and Operations

- Common knowledge

  What number is halfway between 1.1 and 1.11?

- Specialized knowledge
  1. Representing mathematical ideas and operations
  2. Providing explanations for mathematical ideas and procedures
  3. Evaluating mathematical methods, claims, or solutions

Definitions

- Instrument
- Stem
- Item
- Distractor
- Measure (*n.)*, scale

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Representing Number Concepts

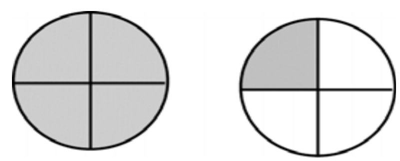
Mrs. Johnson thinks it is important to vary the whole when she teaches fractions. For example, she might use five dollars to be the whole, or ten students, or a single rectangle. On one particular day, she uses as the whole a picture of two pizzas. What fraction of the two pizzas is she illustrating below? (Mark ONE answer.)

a) 5/4

b) 5/3

c) 5/8

d) 1/4



Providing Mathematical Explanations: Divisibility Rules

Ms. Harris was working with her class on divisibility rules. She told her class that a number is divisible by 4 if and only if the last two digits of the number are divisible by 4. One of her students asked her why the rule for 4 worked. She asked the other students if they could come up with a reason, and several possible reasons were proposed. Which of the following statements comes closest to explaining the reason for the divisibility rule for 4? (Mark ONE answer.)

a) Four is an even number, and odd numbers are not divisible by even numbers.

b) The number 100 is divisible by 4 (and also 1000, 10,000, etc.).

c) Every other even number is divisible by 4, for example, 24 and 28 but not 26.

d) It only works when the sum of the last two digits is an even number.

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Evaluating Solution Approaches

Student A	Student B	Student C
35	35	35
x25	x25	x25
125	175	25
+75	+700	150
875	875	100
		+600
		875

Which of these students is using a method that could be used to multiply any two whole numbers?

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Using Data to Test and Improve Theory

- Factor analyses
- Analyses of reliability
- Analyses of validity

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Factor Analysis

- Enables analyses of the number of underlying factors in a data set
- Can link items to constructs
- Ask similar question of our data:
  - How many factors? What are they named?
  - How do items group?

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Overarching Findings: Factor Analyses

Multidimensionality of CKT

- Number and and operations CK
  - Specialized
  - Common
- Patterns, functions & algebra CK
- Geometry CK
- Knowledge of students and content

Analyses of Reliability

- What is “reliability”?
  - How well a set of items discriminates among individuals of different ability levels
  - Ratio of “signal” in the data to “signal + noise”
- Item response theory (IRT) enables analyses of:
  - “Difficulty” of an item relative to other items for a particular population
  - Whether items items discriminate well among individuals of different ability

Reliability Results

	Number of items	1 pl reliability	2 pl reliability	max info at
Number & Operations	18	.76	.79	-.25
Patterns, functions, algebra	15	.84	.87	-1.00
Geometry	43	.91	.92	-.62

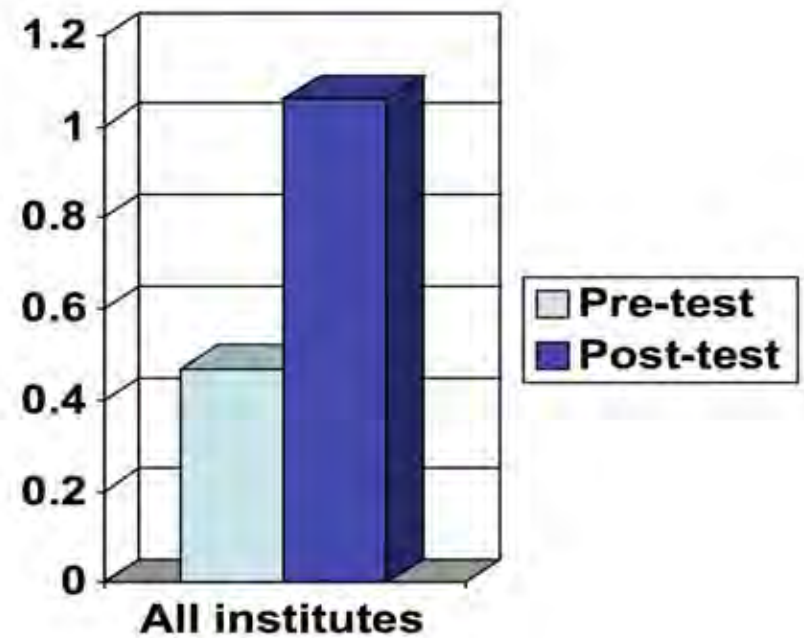
An Example:
Evaluating Teacher Professional Development

Example: Tracking Teacher Growth

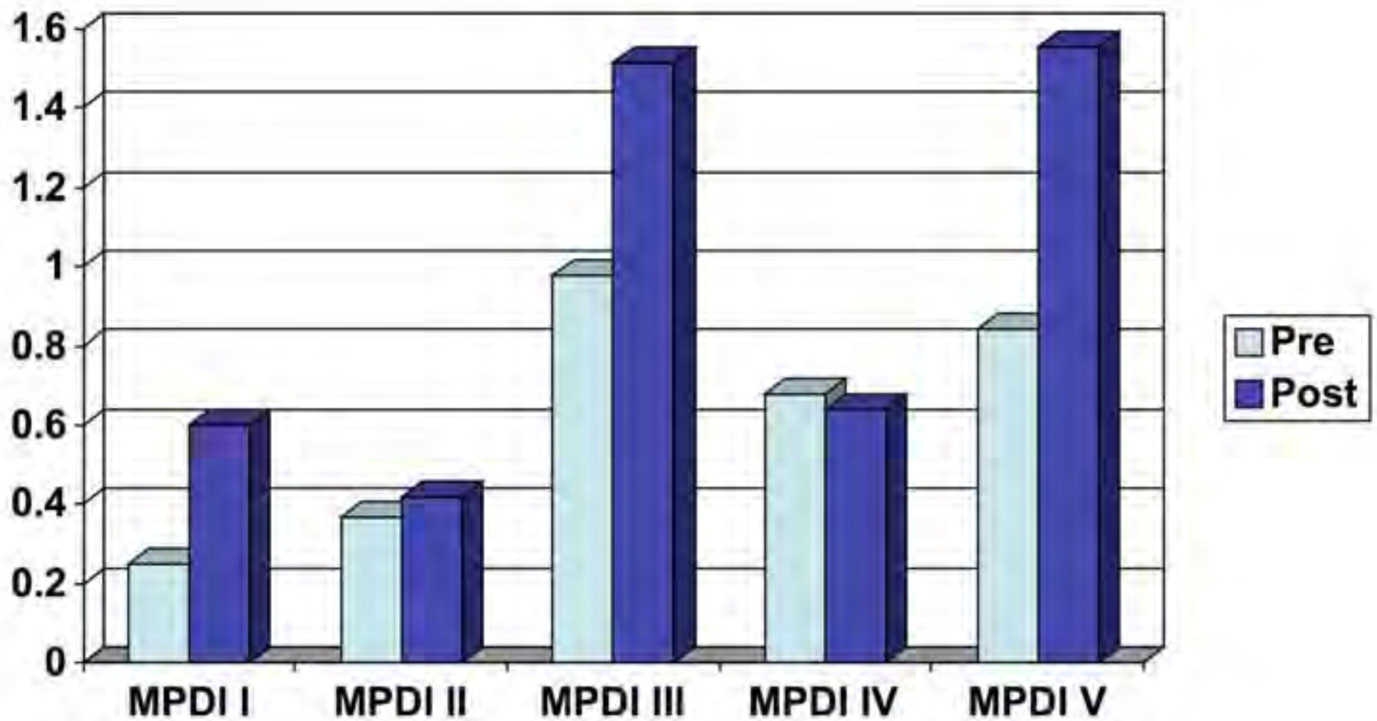
- Items piloted in California's Mathematics Professional Development Institutes (MPDI)
  - Instructors: Mathematicians and mathematics educators
  - 40-120 hours of professional development
  - Focus is squarely on mathematics content
  - Summer 2001
  - Pre/post assessment format (parallel forms)

MPDI Teacher Growth

- For all institutes for which we have data, teachers gained .48 logits, or roughly ½ standard deviation
- Translates to 2-3 item increase on assessment
- Considered substantial gain



Results from Sample Institutes



MPDI Evaluation: Other Findings

- Length of institute predicts teacher gains
  - 120-hour institutes most effective, on average
  - But some 40-hour institutes very effective
- Focus on mathematical analysis, proof, and communication leads to higher gains
- Many questions remain
  - Effects of content (e.g., mathematics vs. student thinking)
  - Treatment of content: “packed”/”unpacked”
  - Effects of teacher motivation

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Validating the Measures

1. Validation theory

2. Validation results
   a) Cognitive interviews
   b) Videotape validation study
   c) Linking measures to student achievement
   d) Mathematician interviews

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Assessing Validity: Our Approach

- Constructing a validity argument (Kane)
  - How do we interpret teachers' performance on our assessment?
    - Reflects their mathematical thinking
    - Higher scores mean higher-quality mathematics instruction
    - Higher scores related to improved student learning
    - Scores reflect common and specialized knowledge of content
  - Investigate interpretation with multiple sources of evidence

Answering Interpretive Questions

- How do we interpret teachers' performance on our assessment?
  - Reflects their mathematical thinking
    - *Cognitive interviews*
  - Higher scores mean higher-quality mathematics instruction
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  - Higher scores related to improved student learning
    - *Study of Instructional Improvement student gains analysis*
  - Scores reflect common and specialized knowledge of content
    - *Mathematician and laypeople interviews*

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1. Cognitive Interviews

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Answering Interpretive Questions

- How do we interpret teachers' performance on our assessment?
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Cognitive Interviews

- Recruited K-6 teachers via email, flyers
- Teachers completed surveys, then:
  - Selected 23 lowest- and highest-performing teachers for interviews
  - Sent interview items, asked teachers to answer
  - Interviewed around 36 number & operations and geometry items (45-60 minutes)
- Question: What made you choose (answer)? What process did you go through to decide?
- Limited cognitive interview

Cognitive Interviews

Mr. Lewis asked his students to divide 6/8 by 1/2. Charlie said, "I have an easy method, Mr. Lewis. I just divide numerators and denominators. I get 6/4, which is correct." Mr. Lewis was not surprised by this as he had seen students do this before. What did he know?

a) He knew that Charlie's <u>method</u> was wrong, even though he happened to get the right <u>answer</u> for this problem.

b) He knew that Charlie's <u>answer</u> was actually wrong.

c) He knew that Charlie's <u>method</u> was right, but that for many numbers this would produce a messy <u>answer</u>.

d) He knew that Charlie's <u>method</u> only works for some fractions.

Answering Charlie

$$\frac{6}{8} \div \frac{1}{2} = \frac{6 \div 1}{8 \div 2} = \frac{6}{4}$$

$$\left(\frac{6}{8} \cdot \frac{2}{1} = \frac{12}{8} = \frac{6}{4} \right)$$

Answering Charlie

$$\frac{6}{8} \div \frac{1}{6} = \frac{6 \div 1}{8 \div 6} = \frac{6}{\frac{8}{6}} = \frac{6}{1} \cdot \frac{6}{8} = \frac{36}{8} = 4\frac{1}{2}$$

$$\left(\frac{6}{8} \cdot \frac{6}{1} = \frac{36}{8} = 4\frac{1}{2} \right)$$

Cognitive Interviews

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b)He knew that Charlie’s answer was actually wrong.

c)He knew that Charlie’s method was right, but that for many numbers this would produce a messy answer.

d)He knew that Charlie’s method only works for some fractions.

Cognitive Interviews

Wrong answer, wrong reasoning:

"For 3 I put A ("he knew that Charlie's method was wrong, even though he happened to get the right answer for this problem"), basically because that's not how you do it, or it's not how I learned how to do it. And he did get the right answer but I thought it was just happenchance."

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Cognitive Interviews

Right answer, right reasoning

"I did a whole bunch of figuring on this one. Let's see…I finally went with answer C ("He knew that Charlie's method was right, but that for many numbers this would produce a messy answer") and I was actually kind of surprised when I started doing it that it worked, and it just kept working and then-but then I said I gotta be doing something. So then I picked numbers where I would get a messy answer. So I ended up with decimals and a fraction."

Cognitive Interviews

Inconsistent answer/thinking:

(This respondent answered A: He knew that Charlie's <u>method</u> was wrong, even though he happened to get the right <u>answer</u> for this problem) …And I kind of started from the bottom (answer) working my way up. (D) Charlie's method only works for some fractions so then I started thinking of different numbers of trying it in my head and it was working for all of them I was trying in my head, so I didn't circle that one. Then it said (C) his method was right and I thought well it may be right for him, but it's not the one I learned so I'm not going to circle that one. And then (B) answer was wrong, it wasn't that. And then for this one I thought Mr. Lewis being the teacher has probably taught this method where you invert the fraction. So he's probably going to say that the method is wrong (A) even though he got the right answer for the problem.

Other Cognitive Interview Results for Charlie

- Among teachers, 7% for this item (2/27 responses)
  - One response changed as interviewer probed on answer
  - One stem problem (anticipating what Mr. Lewis will say, rather than assessing method)
- Distractors work well in general
- IRT slope .721; difficulty 1.67

Overarching Findings: Cognitive Interviews

- Cognitive interviews
  - 1.85% inconsistency rate across items
  - Half of inconsistents are changed incorrect-correct during interviews
  - KSC problems

2. Videotape Validation Study

Answering Interpretive Questions

- How do we interpret teachers' performance on our assessment?
  - Reflects their mathematical thinking
    - *Cognitive interviews*
  - **Higher scores mean higher-quality mathematics instruction**
    - ***Videotape validation study***
  - Higher scores related to improved student learning
    - *Study of Instructional Improvement student gains analysis*
  - Scores reflect common and specialized knowledge of content
    - *Mathematician and laypeople interviews*

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Videotape Validation Study

- Question: Do higher scores correspond with better classroom mathematics instruction?
- Method
  - 10 K-6 teachers took our survey
  - Videotaped each teacher teaching 9 lessons
  - Developing rubric to quantify quality of mathematical instruction delivered

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Videotape Validation Study

- Rubric grades 5-minute segments as appropriate/inappropriate on:
  - Use of mathematical language
  - Representation of mathematical ideas
  - Presence of mathematical explanation or justification
  - Linkages between classroom task (i.e., what kids are doing) and important mathematical ideas
  - Facility in listening to children's mathematical productions
  - Computational or other mathematical errors (always inappropriate)

Videotape Validation Study

- No findings as of yet
  - Blind to teachers' scores as we code

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3. Links to Student Achievement

Answering Interpretive Questions

- How do we interpret teachers' performance on our assessment?
  - Reflects their mathematical thinking
    - *Cognitive interviews*
  - Higher scores mean higher-quality mathematics instruction
    - *Videotape validation study*
  - **Higher scores related to improved student learning**
    - ***Study of Instructional Improvement student gains analysis***
  - Scores reflect common and specialized knowledge of content
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Links to Study of Instructional Improvement Student Achievement Analysis

- SII measure – 38 items, 6 from Comparison Group
  - SII: .89 IRT reliability
  - 6 items overlap with Comparison B
- Model: Student Terra Nova gains predicted by:
  - Student descriptors (family SES, absence rate)
  - Teacher characteristics (math methods/content, content knowledge)
- Teacher content knowledge significant
  - Small effect (LT 1/10 standard deviation)
  - But student SES is also on same order of magnitude

4. Mathematician Interviews

Answering Interpretive Questions

- How do we interpret teachers' performance on our assessment?
  - Reflects their mathematical thinking
    - *Cognitive interviews*
  - Higher scores mean higher-quality mathematics instruction
    - *Videotape validation study*
  - Higher scores related to improved student learning
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  - **Scores reflect common and specialized knowledge of content**
    - ***Mathematician and laypeople interviews***

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Analyzing the Interview Data: What We Hope to Learn

- How do professional mathematicians answer the items? How do they explain and justify their responses?
- Which items do mathematicians seem to find more difficult, and why? Which items take them longer to answer, and why?

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Why Might Mathematicians Have Trouble with SCK Items?

- Compressed knowledge that is characteristic of post-graduate mathematical expertise
- Tendency to make additional assumptions to remove perceived ambiguity
- Lack of familiarity with the school curriculum
- Flaws in the items

Trying Your Hand at Writing Items

"Item Camp"

When Writing Items (1)

- Try to write in following content areas:
  - Number and operations
  - Patterns, functions, and algebra
  - Geometry
- We can use grades K-8 items; write others for your own use

When Writing Items (2)

- Items must be unambiguously right or wrong
- Think of clever distractors—what would someone with little knowledge think?
- Write easy, medium, and hard items
- Do not cross content (e.g., items requiring both content knowledge and knowledge of student thinking, or both geometry and algebra)

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When Writing Items (3)

- Ignore (or subvert) ideologies about teaching mathematics
- Avoid excess verbiage
- Provide item documentation
  - Author
  - Inspiration (your student teachers; observations of a classroom; curriculum materials; research)
  - Correct answer

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When Writing Items (4)

- Write items that reflect how mathematics is <u>used in teaching</u>
  - Use your knowledge of classrooms and teachers – what issues arise? What do new teachers struggle with?

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