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Connecting Mathematics Education Research and Classroom Practice

The four articles by mathematics educators in this issue are an outgrowth of the panel discussion of the same name at the special session on Mathematics and Education Reform at the 2002 January Joint Mathematics Meeting in San Diego. In turn, each author— Deborah Loewenberg Ball, Rina Zazkis, Marilyn P. Carlson, and Chris L. Rasmussen— reflects on the dynamical interplay of mathematics education research and teaching mathematics within his or her own classroom. Individual case studies, self-reflections— whatever you choose to call them— these analyses reveal surprising complexities and richness of experience in the classroom.

Knowing Mathematics for Teaching: Relations between Research and Practice

by Deborah Loewenberg Ball, University of Michigan

This special MER panel probed a fundamental question, "How can research in mathematics education relate to practice?" One way to answer this question is to consider how they might be related, to envision possibilities and promise. Another, the one taken by members of this panel, was to examine specific cases in their own work where research and practice intermingle in generative ways. As a case of an area in which disciplined inquiry and practice might inform one another, research and practice on issues of mathematics teacher knowledge offers a useful site to probe how those connections might work productively. To provide a glimpse of these, I trace here my own journey— in the context of the explorations of many others— in pursuit of questions about the mathematics knowledge for teaching. Having begun my work on this problem as a classroom teacher, practice has often been the starting point for me in my work, the site for genesis of problems and ideas. It has also been the site where, in early use, ideas are tested, refined, and sometimes abandoned.

I can trace my first questions about the mathematics knowledge needed for teaching to early in my elementary school teaching career. After about five years in the classroom, I grew aware that my teaching of mathematics was lacking. Although I tried to make the mathematical ideas make sense to my fifth graders, I was frustrated by the fact that they seemed to forget what we worked on as

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fast as I could teach it. Without any clear sense of what was missing, I thought it possible that my own knowledge of mathematics might be a factor. Unlike other areas of the elementary curriculum, mathematics was an area in which I lacked depth and experience. I liked mathematics and felt myself reasonably competent, but I had the sense that I lacked the sort of perspective and connected knowledge that enabled me to move flexibly in other subjects. I had gone through high school at a time when students were allowed to make many decisions about what to take, and I had consequently opted to take many courses in French, Spanish, and German, as well as English and humanities, and only a little mathematics. I suspected that studying some mathematics might help me.

Unwittingly, I was entering as a young teacher into an arena that had already occupied many others. Many had worked hard to develop courses for teachers, courses that were successful within themselves, but disappointing with respect to what teachers were able to *do* with mathematics in their own practice. Edward Begle (1979) had conducted analyses across studies of teacher effects, attempting to identify the trends in how teachers' mathematics study impacted their effectiveness with students. Begle's analysis produced a surprise: Advanced mathematics course-taking produced positive main effects on students' achievement in only 10% of the cases, and— more startling— negative main effects in 8%.* Begle concluded from his analyses that the belief that "the more a teacher knows about his subject matter, the more effective he will be as a teacher" demanded "drastic modification" (Begle, 1979, p. 51). He noted that, although "it seems to be taken for granted that it is important for a teacher to have a thorough understanding of the subject matter being taught," never in these studies was the question of what should be meant by "thorough" examined (p. 28).

Not yet knowing of these startling results, I decided to take some courses at the university. Because I had never taken any mathematics courses in college other than an independent study that substituted for the standard mathematics course for elementary teachers (I placed out of this course by virtue of scoring very high on a placement exam!), I launched in with a review course in algebra, and then continued with four terms of calculus and number theory. I was teaching fifth grade, and then first grade during this time. Although none of these courses dealt with the mathematics of the primary grades, I noticed that I was learning things that affected how I worked on mathematics with my students. For example, one day in first grade we were using graph paper to measure shapes to provide some early experience with the concept of area measure. Some shapes were regular, some irregular. The book suggested asking the children to trace their hands, and to come up with a measurement for the area. When some children, dissatisfied with the inexactness of the one-inch grid graph paper we were using, asked to go get some of the finer-grid graph paper that the fifth graders used, I heard this as a significant question. I realized that their interest in using the smaller grid graph paper represented an early sensibility for precision and an instinct about "getting closer" without being exact, a notion that would later underlie an appreciation of limits, and specifically the approach to area in integral calculus. Had I not recently studied calculus, I might have heard their desire to get the finer graph paper as unimportant. Put more accurately, I would not really have "heard" it. I surely did not have "thorough" knowledge of mathematics. But I was also clearly learning something useful to my work with children.

Over the ensuing years, I continued to teach elementary school and to take university mathematics courses. I enjoyed them for the most part. A significant course was one on number theory, taught by Joe Adney, then the chair of the Michigan State mathematics department. The course was different from anything I had taken up to that point, because for the first time we were being expected to prove claims, not just engage as spectators. My induction into the world of conjectures and proofs, of lemmas and theorems, fascinated me, and sparked my imagination as a teacher. I suddenly saw the connections to ways in which my students pursued questions and investigations in science, developed interpretations of poems and books, and examined artifacts in social studies. I also saw ways in which proof in mathematics was unlike producing conviction in these other subjects. For example, I saw that examples did not constitute sufficient support for an argument.

I began to draw my students out a little more. When they noticed patterns, or had tentative ideas, I asked them to explain their reasoning, and to give evidence for their assertions. I asked students to respond to other's ideas, as I had seen Professor Adney do with us. The more I pulled them into reasoning, the better they seemed to understand the ideas we were working on. They also surprised me more and more often by noticing things I would not have expected elementary students to think about. For example, they wondered whether 0 was even or not, and they developed a method of subtracting multi-digit numbers that seemed more efficient than the one we knew

*A positive main effect would mean that more credits of mathematics were associated with greater student performance; a negative main effect meant that more credits of mathematics were associated with lower levels of achievement.

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and had always taught.

Teaching demanded a kind of mathematical flexibility and appreciation, as well as knowledge and skill. I wanted to learn more, but was finding it difficult to identify precisely what I needed to learn as well as where I could learn it. I did see that I was learning a great deal from listening closely to my students, and to seeing what they did with interesting mathematics tasks. I tried my hand at making other tasks, and was often rewarded by my own learning as I explored a set of ideas and developed problems for the children to do.

I also realized, however, that my capacity plummeted when I myself was unclear about the mathematics, or when I had mistaken ideas. My ability to steer such tasks well, to hear what the students were thinking, and bring a discussion to a clear forward motion, was slowed. I could see that my improvement as a mathematics teacher was interactive with my mathematical knowledge, but I was unsure about how to describe to others the sort of mathematical knowledge that seemed to help me navigate the complexities of my work.

To explore the nature of this mathematics knowledge, I designed some questions that involved mathematical insight and understanding. And I used these to interview prospective teachers. I wanted to see what beginning teachers– including some who had already studied a lot of college-level mathematics and others who had not– brought with them to their formal preparation for teaching mathematics. In one question, I posed a common error that upper elementary students make when they multiply multi-digit numbers:

> Suppose you are trying to help some of your students learn to multiply large numbers. You notice that when they try to calculate
>
> $$\begin{array}{r} 123 \\ \underline{\times\ 645} \end{array}$$
>
> the students seemed to be forgetting to "move the numbers" (i.e., the partial products) over on each line.
> They are doing this instead:
>
> $$\begin{array}{r} 123 \\ \underline{\times\ 645} \\ 615 \\ 492 \\ \underline{738} \\ 1845 \end{array}$$

I was astonished that only about one out of four prospective teachers talked explicitly about place value when they explained the difficulty that the children might be having, or how they might try to "clear up" the problem. For example, Rachel said:

"You would take the last number and multiply it by all three of the top numbers and you put those underneath and then you start with the next one. You'd want to put it underneath the number that you are using. They aren't understanding that they need to be underneath of that instead of just down in one straight row."

And Pam said she would show pupils to "*physically* put a zero every time you moved down a line." She explained that "zero doesn't add anything more to the problem. It's just empty. But instead of having an empty space, you have something to fill in the space so that you can use it as a guideline."

And there were other questions. One of my questions, well known by now, was to ask prospective teachers to calculate 1 3/4 ÷ 1/2 and then to write a story problem that matched the arithmetic calculation. Again, most were able to compute the correct answer, but to produce an appropriate story proved difficult for a majority of the students.

That such questions were challenging for prospective teachers was important. But that they were challenging even for those prospective teachers who had already almost completed majors in mathematics was even more significant. These results helped to provide a clue to the decades-old mystery of why achievement is not positively correlated with higher levels of mathematics study. As I had often wondered, what sort of mathematical knowledge is required in teaching? And how is it related to the mathematical knowledge developed in particular courses? I had myself found the mathematics I learned in calculus, number theory, and probability useful to me as an elementary school teacher. But how had that knowledge been transformed? To what uses was it put to make it usable for teaching? The unprepared reactions of the mathematics majors I interviewed suggested that the connection between advanced mathematics study and knowing mathematics for teaching was not yet well understood. As an elementary teacher who had come to these issues along a different path, I found these insights fascinating, if challenging.

As a research community began to tackle the thorny question of mathematical knowledge for teaching, simple formulations began to recede. No longer did scholars attempt to describe the mathematics that teachers needed

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to know in terms of the number of courses they should take, and although policymakers yearned for a degree or coursework specification, most realized that the question was more complicated.

In the broader community of researchers on teaching, Lee Shulman and his colleagues introduced the notion of "pedagogical content knowledge" (1986, 1987a, 1987b) to capture the special ways in which teachers need to know subject matter. This knowledge, they claimed, consisted of the important ideas and procedures of a field, but also the aspects most difficult to learn, and the range of representations useful for making the ideas accessible. Their work, conducted in English, physics, biology, history, and mathematics, provided helpful insights into the nature of the subject matter knowledge needed for the work of teaching.

Work on this problem proceeded along a number of avenues. Liping Ma used the same questions that had proved so useful in probing U.S. teachers' mathematical knowledge, and posed them to practicing Chinese elementary teachers (Ma, 1999). Significantly, her results were quite different. Ma used her data to develop a notion of "profound understanding of fundamental mathematics," an argument for a kind of connected, curricularly structured, and longitudinally coherent knowledge of core mathematical ideas. Her notion of "knowledge packages" offered a conceptual structure that could overcome the tendency to make endless disconnected lists of what teachers need to know.

In my own courses for prospective teachers, I sought to select mathematical content that would afford them leverage over the recurrent mathematical issues they would face as teachers. In my teaching of elementary children, I turned increasingly to focus on uncovering these issues: What mathematical questions kept coming up for me as a teacher? What was the nature of the mathematical issues that I confronted? How was the work of teaching third graders mathematical, and not merely, as many assume, pedagogical and rooted in questions of cognitive development?

I noticed that there were several such recurrent problems: One is selecting, adapting, and using representations for mathematical ideas (e.g., Ball, 1993). For example, in teaching arithmetic with integers to my third graders— a topic in my school district's curriculum— I explored several alternative models for negative numbers. I examined the possibilities with money and debt, number lines, checkers that could be used to denote positive and negative values, and a building with many floors above and below the ground floor labeled 0. Each highlighted some key aspects of the mathematics; each offered resources for learning; and each held pitfalls to possible confusions. The problem of selecting and using representations was deeply mathematical, and required the unpacking and close analysis of the mathematical ideas and their relationships to other ideas yet to come in the children's mathematical experience. Another recurrent difficulty is trying to make a judgment about whether an idea is mathematically significant and worth taking up. Still another is sizing up the validity of a child's non-standard procedure: Will it work in all cases, for all numbers? Students are continuously trying their own approaches, inventing their own notation, and trying out their own methods. Not all of these will hold up over time, and some are more good fortune than mathematics. Yet some are potentially robust, and merit analysis, testing, and development. I have several times had students develop methods of subtraction that by using negative numbers avoided the "borrowing" procedure common to standard U.S. curriculum:

$$\begin{array}{r} 36 \\ -19 \\ \hline 2\ {-3} \end{array} \longrightarrow \quad 17$$

This algorithm employed here permits the user to work separately on each column, using integers to keep track of the parts of the calculation. Is this method viable with seven-, twenty-, or one hundred-digit numbers? Teaching requires teachers to be able to understand and appraise such a procedure, and to make a decision about both its significance and generality, as well as what to do with it in class.

Close scrutiny reveals the myriad tasks of teaching that are permeated with mathematical considerations, that require mathematical knowledge and sensibility. While studies of what teachers know (and do not know) are useful for exploring the contours of the problems we face in professional education and in the development of curriculum, such studies do not help in probing the mathematical nature and demands of teaching. They do not help advance our collective understanding of what it is that teachers need to know and how they need to use such knowledge in their work.

My collaboration with Hyman Bass has afforded a means to probe the mathematical work of teaching. With an eye that is intensely mathematical, when he watches a class of children at work, he perceives the emergent ideas in their

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insights and efforts and sees the horizons to which they are headed. Often with a sense of surprise and fascination, he suddenly realizes— and appreciates— the thickets of difficulty they and their teachers meet, even in what many consider to be "elementary" content (Ball, 1999). In our joint work, we have asked the question, what is the mathematical terrain of elementary teaching and learning? What are the mathematical demands of the work of teaching? Where and how do mathematical considerations arise, and what is the mathematics that teachers could profitably use to manage these tasks? (Ball and Bass, 2000)

The work on mathematics knowledge for teaching has come full circle. Originating in questions about how different "amounts" of knowledge affected teachers' effectiveness; researchers soon encountered the surprise that answering this question was far from straightforward. The number of courses a teacher had taken was an inadequate proxy for the mathematical knowledge required for teaching. Yet researchers also began to see up close what many teachers had long felt— that teaching is mathematically demanding work, and that many teachers' mathematical education left them unprepared for its challenge.

While research must continue to contribute to our understanding of what it is that teaching requires from a mathematical perspective, the mathematical education of teachers must continue to develop and test alternative ways to prepare teachers for those mathematical tasks and challenges. The recently released *Mathematical Education of Teachers*, a product of the collaborative work of mathematics educators, mathematicians, and teachers, is an example of this much-needed work (CBMS, 2001). As mathematicians and mathematics educators explore what it takes to be mathematically prepared for teaching mathematics, our understanding of the problem and its solutions will be improved. Working only on the basis of conviction will, however, leave us no better off. Understanding the nature of the mathematical knowledge needed for teaching, and how it can be developed and acquired in usable and useful ways, requires disciplined inquiry. How do different kinds of mathematical experiences affect what teachers know and can do in their classrooms? How do different kinds of mathematical understanding impact their effectiveness? What are the *mathematical* problems they must solve on a daily basis, and how does their mathematical education prepare them for those? The bottom line to this question is the quality and effectiveness of *teaching*, not the improvement of *teachers*. Our favorite notions about teacher preparation require testing and improvement. And when we least expect it, answers we could never have predicted will surprise us if we design ways to intertwine practice and research in the pursuit of these problems.

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