1 Preliminaries: Hyperbolic Geometry

Euclidean Geometry has several very nice and intuitive features: the shortest path between any two points follows a line, a point and a line uniquely determine a parallel line, distance is visually intuitive, etc. In hyperbolic geometry none of these statements are true. Shortest paths follow semi-circles, a point and a line determine many other non-intersecting lines (parallel carries little meaning when most geodesics don’t intersect one another), and visual cues on distance are completely false.

This is all a result from the metric placed on hyperbolic space. The metric placed on Euclidean space is of the form $\sqrt{dx^2 + dy^2}$; or for any given two points $(a,b)$ and $(c,d)$, the distances is given by $\sqrt{(a-c)^2 + (b-d)^2}$. Roughly speaking, the hyperbolic metric is given by dividing by the height of the segment we are trying to find the distance of - for two points $(a,b), (c,d)$ arbitrarily close, the distance is given by $\sqrt{(a-c)^2 + (b-d)^2 \over y}$. Precisely, the hyperbolic distance between two points is found by integrating the form $\sqrt{dx^2 + dy^2 \over y}$ over a path. (This metric extends to any dimension where it’s given by dividing the euclidean metric by one of the coordinates; however in this paper we will only be concerned about the 2-dimensional case.)

Because of the non-intuitive formulation of distance, the Isometry Group of this metric space is particularly interesting. In this paper we explore Isometries of 2-dimensional hyperbolic space and some of their applications.

1.1 The Metric and Geodesics

First, we define the metric space we are working in: $\mathcal{H} := \{(x, y) \in \mathbb{R}^2 | y > 0\}$ (equivalently, the upper half of the complex plane), equipped with the hyperbolic metric: $\sqrt{dx^2 + dy^2 \over y}$.

Necessary Fact. $PSL(2, \mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | ad - bc = 1 \right\}$ is the group of orientation-preserving isometries of $\mathcal{H}$. (See [1], Theorem 2.8 for this proof, which is simply computation.) Precisely, this means that $PSL(2, \mathbb{R})$ acts invariantly on distances. One important use of this fact will be the use of translations in our proof; ie. isometries of the form $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$. We will return to a more detailed discussion on the isometries and algebraic structure of this group action on $\mathcal{H}$ in the next section.

Proposition 1.1. Geodesics of $\mathcal{H}$ are vertical lines and semi-circles resting on the Real Axis.

Proof. First, it is necessary to show that the imaginary line is a geodesic. Let $f : [0, 1] \rightarrow \mathcal{H}$ be a path which connects points $pi, qi$, where $p, q \in \mathbb{R}$ (and $p \leq q$.) Let $f(t) := x(t) + iy(t)$.

$$\int_0^1 f'(t)dt = \int_0^1 \sqrt{dx^2 + dy^2 \over y(t)} dt \geq \int_0^1 {dy \over y(t)} dt = \int_p^q {dy \over y(t)} dt$$
So a path \( f \) from \( p_i \) to \( q_i \) such that \( x(t) = 0, \forall t \in [0, 1] \) has a shorter hyperbolic distance than all other paths; i.e. the quickest path is one which is contained completely on the imaginary line. It follows that the imaginary line (and consequently so are all vertical lines, using appropriate translations \( \tau \in PSL(2, \mathbb{R}) \)) is a geodesic.

Suppose we are given a half circle in \( \mathcal{H} \) resting on the real axis. Move this circle using a proper translation \( \tau \in PSL(2, \mathbb{R}) \) (that is, an isometry) so that the center of this circle becomes \((r, 0)\). Note that since these circles are isometric, so one is a geodesic iff the other is. In Euclidean coordinates, the equation of a circle centered at \((r, 0)\) with radius \( r \) (call this \( C_r \)) is given by:

\[
(x - r)^2 + y^2 = r^2
\]

\[
x^2 - 2xr + r^2 + y^2 = r^2
\]

\[
x^2 + y^2 = 2xr
\]

Now, interpreting \((x, y) \in \mathbb{R}^2\) as \(z \in \mathbb{C}\), we have

\[
|z|^2 = 2xr
\]

\[
2r = \frac{|z|^2}{x}
\]

Now consider the transformation \( T \in PSL(2, \mathbb{R}) \) where \( T(z) = -z^{-1} + \frac{1}{2r} \). Examine how \( T \) acts on \( C_r \) by imposing the simplified equation for \( C_r \):

\[
T(z) = -\frac{1}{z} + \frac{x}{|z|^2}
\]

\[
T(z) = -\frac{\bar{z}}{|z|^2} + \frac{x}{|z|^2}
\]

Converting back to Euclidean coordinates, we get

\[
T(z) = \frac{-x + iy + x}{|z|^2} = \frac{iy}{|z|^2}
\]

And so it is clear that the image of \( C_r \) under \( T \) has no \( x \)-coordinate; i.e. \( T \) maps \( C_r \) onto the imaginary line.

### 1.2 Isometries

Every isometry of \( \mathcal{H} \) is either orientation-preserving or orientation-reversing. The orientation-preserving isometries are the \textbf{Möbius transformations} (i.e., the elements of \( PSL(2, \mathbb{R}) \)). (See [1] for proof.) This means every orientation preserving isometry of \( \mathcal{H} \) has the form \( z \mapsto \frac{az + b}{cz + d} \) where \( a, b, c, d \in \mathbb{R} \) such that \( ad - bc = 1 \).

#### 1.2.1 Classifying Isometries

We can classify a given orientation-preserving isometry \( T = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) by looking at the fixed points of \( T \).

To find the fixed points, we simply solve the equation \( T(z) = z \), which gives us

\[
z = \frac{az + b}{cz + d} \implies cz^2 + (d - a)z - b = 0.
\]

Applying the quadratic formula, we get

\[
z = \frac{\frac{a}{c} \pm \sqrt{(\frac{d}{c} - a)^2 + 4bc}}{2c}. \quad \text{But we also have}
\]

\[
(d - a)^2 + 4bc = d^2 + a^2 - 2ad + 4(ad - 1) = d^2 + a^2 + 2ad - 4 = (a + d)^2 - 4.
\]
Thus, we find that the fixed points of $T$ are precisely $u = \frac{(a-d)+\sqrt{(a+d)^2-4}}{2c}$ and $w = \frac{(a-d)-\sqrt{(a+d)^2-4}}{2c}$. This gives us 3 clear cases:

1. If $|\text{tr}(T)| = |a+d| > 2$, then $u$ and $w$ are two distinct points on the real axis. We call such an isometry a hyperbolic isometry.

2. If $|\text{tr}(T)| = |a+d| = 2$, then $u = w$ is a single point on the real axis. We call such an isometry a parabolic isometry.

3. If $|\text{tr}(T)| = |a+d| < 2$, then $u$ and $w$ are two distinct points with imaginary parts not equal to 0. Only one of these points has an imaginary part greater than 0. This point is then the unique fixed of $T$ in $\mathcal{H}$. In this case, we call such an isometry $T$ an elliptic isometry.

Now let’s take a further look at each of these types of isometries.

1.2.2 Hyperbolic Isometries

Let’s take a further look at the hyperbolic isometries. Consider the example $T = \begin{pmatrix} 2 & 0 \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}$. (The following proofs and observations work for any arbitrary hyperbolic element $S \in \text{PSL}(2, \mathbb{R})$, but these facts may be easier to see with this particular example.)

It is immediately seen that $|\text{tr}(T)| = \frac{5}{2} > 2$, $T(0) = 0$ and $T(1) = 1$. Therefore, $T$ is a hyperbolic isometry with fixed points $u = 0$ and $w = 1$. Now consider the geodesic from 0 to 1, which we will call $C(T)$ or the axis of $T$. We already know that every isometry takes geodesics to geodesics and we know $T$ fixes 0 and 1, so $T$ must take $C(T)$ to $C(T)$ by construction. (Note that this is not point-wise invariance, for the only fixed points of $T$ are 0 and 1. We only have that $C(T) \mapsto C(T)$ under $T$.)

Note that, in general, $S'(z) = \frac{a(z+c)+b(cz+d)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}$ for any $S \in \text{PSL}(2, \mathbb{R})$ because $ad-bc = 1$. A fixed point $v$ of $S$ is attracting if $|S'(v)| < 1$ and repelling if $|S'(v)| > 1$. It is easy to see that $|T'(0)| = 4 > 1$ and $|T'(1)| = \frac{1}{4} < 1$, so we know that 0 is a repelling fixed point for $T$ and 1 is an attracting fixed point for $T$. This can be visualized by seeing all the points of $C(T)$ moving rightwards away from 0 towards 1.

We can also see from the formula for the $T$ a circle, which we will call $I(T)$ or the isometric circle of $T$, where the Euclidean lengths will not be altered. Because $|T'(z)| = |\frac{3}{2}z + \frac{1}{2}|^{-2}$, Euclidean lengths are multiplied by $|\frac{3}{2}z + \frac{1}{2}|^{-2}$ and are unaltered if and only if $|\frac{3}{2}z + \frac{1}{2}| = 1$. Because $c = \frac{3}{2} \neq 0$, the locus of such points is a circle with center at $-\frac{d}{c} = -\frac{1}{3}$ and radius $\frac{1}{|c|} = \frac{2}{3}$.

1.2.3 Parabolic Transformations

Parabolic Transformations are transformations $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a + d = 2$, so that there is exactly one fixed point on the boundary of the Hyperbolic Plane (particularly, $\partial \mathcal{H} \cong \mathbb{R} \cup \{\infty\}$). Understanding the case in which the fixed point is $\infty$ gives insight into the structure of these transformations.

If $c \neq 0$, then clearly the fixed point (see earlier section) is a Real number. If $c = 0$, then the fixed point is exactly $\infty$. So parabolic transformations in which the fixed point $= \infty$ are of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ such that $ad-bc = ad = 1$ and $a + d = 2$. This is true only when $a = d = 1$; ie. all parabolic transformations with fixed point $\infty$ are of the form $x+b$, $b \in \mathbb{R}$.

This aligns with our expectation, because certainly translations are parabolic transformations, and their unique fixed point is indeed $\infty$.

**Fact.** All Parabolics are conjugate to translations; ie. when $c \neq 0$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = S\tau S^{-1}$, for $S, \tau \in \text{PSL}(2, \mathbb{R})$, $\tau(z) = z+b$.

**Notation.** Let $G^b_a$ denote the geodesic with endpoints $a, b \in \mathbb{R} \cup \{\infty\}$. 

3
Here, we will examine the key properties of a translation which we can abstract to an arbitrary Parabolic Transformation. Take the translation \( \tau(z) = z + b \). Draw geodesics of the form \( G_{kb}^\infty \) on the plane, for \( k \in \mathbb{N} \). It is easy to see that \( \tau \) acts transitively on these lines purely geometrically, as these translations are Euclidean and so the geometry is simple. This is a simple consequence of the properties of geodesics. Since \( \infty \) is fixed and \( \tau(kb) = (k + 1)b \) (and the fact that \( \tau \) maps geodesics to geodesics), we get that each vertical line gets mapped to the next vertical line.

(An important geometric property to note here is that each orbit in of the action \( < \tau > \) is connected by a horizontal line.)

Now, let’s extend this concept to Parabolic transformations \( \rho \) with fixed point \( s \in \mathbb{R} \). Draw geodesics of the form \( G_s^\rho(r) \) where \( n \in \mathbb{N} \), \( r \in \mathbb{R} \) (and being careful to pick \( r \) so that \( \rho^n(r) \neq s, \forall n \in \mathbb{N} \)). This should look something like an infinitely tall flower with its stem base at \( s \) and semi-circular petals emanating from the stem. By the same logic as in the previous case, each geodesic is mapped to the next by \( \rho \). Now, if we connect points in an orbit by \( < \rho > \)-action, we see that in this case they are connected by a circle(!) tangent to the real axis at \( s \). So we see a similar geometric action by \( \rho \) as we did with \( \tau \) - however, points are “translated” along circles in one direction, rather than on horizontal lines.

### 1.2.4 Elliptic Isometries

We turn our attention to Elliptic transformations - ie. transformations of the form \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) where \( a + d < 2 \). First, we need some tools that will allow us to visualize these transformations.

The Poincaré Disk \( \mathcal{U} := \{ x \in \mathbb{C}||x| < 1 \} \) is a different interpretation of the topologically equivalent Hyperbolic Plane \( \mathcal{H} \). The explicit homeomorphism \( \varphi : \mathcal{H} \to \mathcal{U} \) is given by \( \varphi(z) = \frac{z + i}{z + i} \). Vaguely, this map curls the real line, with the point at \( \infty \) compactifying the real line into the circular boundary of \( \mathcal{U} \). The center of this disc is \( i \in \mathbb{C} \) and geodesics through \( i \) are diameters of the disc. All other geodesics are arcs which meet the boundary at right angles.

Before we begin, note that Elliptical transformations have no real fixed points; however, if we solve for the fixed points (given in beginning of 2.2), we get 2 conjugate complex values, one of which will be contained in \( \mathcal{H} \). Thus, elliptical transformations have exactly 1 fixed point in the interior of the hyperbolic plane.

To start our discussion of Elliptical transformations, let’s first focus on Elliptical transformations of the form \( \zeta \in PSL(2, \mathbb{R}) \) such that \( \zeta(i) = i \). This allows us to put heavy restrictions on the possible matrices:

\[
\zeta(i) = i = \frac{ai + b}{ci + d}
\]

\[
ai + b = i(ci + d)
\]

\[
ai + b = di - c
\]

And so it follows that \( a = d \) and \( b = -c \). So we have \( \zeta = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \). Given that this is a Möbius Transformation (ie. \( \in PSL(2, \mathbb{R}) \)), we also have that \( a^2 + b^2 = 1 \). This allows us to parametrize our Elliptic transformations in the form \( \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \), \( \theta \in \mathbb{R} \); these might look familiar as rotation matrices of the vector space \( \mathbb{R}^2 \). In fact, this matrix acts on \( \mathcal{U} \) by rotating the disc about the fixed point \( i \) by \( \theta \) radians (with the appropriate conjugation by \( \varphi \) so we have a well-defined action on \( \mathcal{U} \)). This can be checked with explicit computation, or we can just note that rotation matrices multiply by adding the angles of each rotation; approximate a rotation arbitrarily closely by a fraction multiple of \( \pi \), ie. \( \frac{n}{k0} \pi < \epsilon \) for any \( \epsilon \). Taking this rotation \( 2\pi \) times will give us something arbitrarily close to the identity rotation, which shows us that these rotations act as they should.

### 1.3 Fuchsian Groups

**Definition 1.3.1.** We call a subgroup \( \Gamma \subset Isom(\mathcal{H}) \) discrete if for every sequence \( (T_n)_{n=0}^\infty \) with \( T_n \in \Gamma \) we have \( T_n = 1 \) for sufficiently large \( n \).
Definition 1.3.2. A discrete subgroups of $\text{PSL}(2, \mathbb{R})$ is called a Fuchsian group.

1.4 Fundamental Regions

Let $G \subset \text{Isom}(\mathcal{H})$ be a subgroup. Then, $F \subset \mathcal{H}$ is a fundamental region of $G$ if and only if all of the following are true:

1. $F$ is closed in $\mathcal{H}$ and bounded by a finite number of geodesics.
2. The images of $F$ under the isometries of $G$ cover $\mathcal{H}$.
3. If $T, S \in G$ and $T \neq S$ then the interiors of $T(F)$ and $S(F)$ do not intersect.

Conceptually, a fundamental region is a single tile that gets moved around to cover $\mathcal{H}$. Note that not every subgroup of $\text{Isom}(\mathcal{H})$ has a fundamental region (e.g., the whole group $\text{Isom}(\mathcal{H})$ has no fundamental region.) Also, if a subgroup $G$ has a fundamental region $F$, then $T(F)$ is clearly a fundamental region for all $T \in G$ and $T(F) \neq F$ unless $T = \text{id}$ by construction. Thus, the fundamental region is not unique even if it exists (unless $G = \{\text{id}\}$ is the trivial subgroup). There are three types of fundamental regions that we can look at: infinite volume, noncompact with finite volume and compact with finite volume. Note that if one fundamental region of a subgroup $G$ has one of these properties then every fundamental region of $G$ has that same property.

Example 1:

If $G = \langle T \rangle$ where $T(z) = \frac{1}{2} z$, then one fundamental region for $G$ is the semi-annulus centered at 0 and filled between $\frac{1}{2}$ and 1. This fundamental region clearly has infinite volume (and is thus not compact.)

Example 2:

Let $G = \text{PSL}(2, \mathbb{Z})$ (the modular group) act on $\mathcal{H}$ by Moebius transformations. Then $G = \langle T, S \rangle$ where $T = z+1$ and $S = -\frac{1}{z}$. A fundamental region for $G$ called the Dirichlet region (which is a kind of fundamental region constructed by a certain algorithm) looks like the shaded region in the figure below. This region has finite volume, but is not compact.
2 Geometric Coding

2.1 General Principle

If we have a polygonal fundamental region $F$ where each side is identified by a generator $g_i$ of the fundamental group $\Gamma$, then we can write out a bi-infinite code $(x_i)^{\infty}_{-\infty}$ for every geodesic $\gamma$ in the plane by doing the following. First, assume that $\gamma$ intersects $F$. We simply write down a list of the generators corresponding to the sides of the copies of the fundamental region that the geodesic $\gamma$ passes through. So, continuing forward from $F$, $\gamma$ passes through side $x_0$, then $x_1$, etc. Similarly, $\gamma$ passes through sides $x_{-1}, x_{-2}, \ldots$ going backward from $F$.

If $\gamma$ does not intersect $F$, then we translate $\gamma$ by elements of $\Gamma$ so that it intersects $F$. We assign $(x_i)$ as the code to the equivalence class of geodesics that translate by isometries of $\Gamma$ to a geodesic intersecting $F$, as described above.
2.2 Example: The Morse Method and the Modular Group

One important example of coding is the Morse method geometric code of the modular group. This relates to a particular fundamental region called a Dirichlet region. The important thing to see about Dirichlet regions, which is proved in [2] Theorem 11.8 and Theorem 13.1, is that Dirichlet regions have an even number of sides with generators and their inverses pairing sides.

Figure 1.4 shows the fundamental region $F$ of $\Gamma = PSL(2, \mathbb{Z})$. Note that this is a closed region with a cusp at infinity. Note also that we have a surface $M = \mathcal{H}/\Gamma$ that naturally projects onto $F$ in $\mathbb{H}$. According to this, we identify the right vertical line (green) with the element $T^{-1}$ and the left vertical line (pink/red) with $T$. We will actually consider the bottom of $F$ to be two arcs (marked in blue and orange), both identified with $S$. This makes sense because $S = S^{-1}$. Notice that under these identifications the points on the blue arc are identified with the points on the orange arc and the points on the green line are identified with the points on the red line.

Now refer to the pictures of a closed geodesic on $M = \mathcal{H}/\Gamma$, below for the explanation of the geometric code. Recall that every hyperbolic isometry $A \in PSL(2, \mathbb{Z})$ has a corresponding axis $C(A)$. Our examples will be regarding the coding of the geodesics.

If some axis $C(A)$ for a hyperbolic element $A \in PSL(2, \mathbb{Z})$ does not intersect the fundamental region $F$, then we can find some element $B \in PSL(2, \mathbb{Z})$ that translates $C(A)$ so that it intersects $F$ through the arc marked by $S$. Then we have $D = BAB^{-1}$ with $C(D)$ intersecting $F$ through the arc $S$ and we assign the same code to $D$ and $A$. Thus the closed geodesics on $M$ are equivalence classes of the axes of hyperbolic elements of $PSL(2, \mathbb{Z})$. These geodesics can have a finite length code describing what they do in each “period”.

![Fig. 2.2: Geometric code for geodesic $C(A)$](image)

To find the geometric code for the geodesic $C(A)$ above, we record blocks of “hits” on the vertical boundaries of $F$. The geometric code for $C(A)$ is [6] because it “hits” the right side 6 times, then hits the bottom and returns to its starting point.
The geometric code for $C(B)$ is $[4, -5]$ because the geodesic “hits” the right side 4 times (green), then hits the bottom (green), then it turns around and hits the left side 5 times (red), then hits the bottom (red).
and returns to its starting position.

Now why do we do it like this? There are two things we need to notice.

First, it is clear that it is impossible for a geodesic to hit the bottom arc of \( F \) twice in a row without hitting either side. Second, it is impossible for a geodesic to hit the right side, then the left side immediately (or vice versa) without hitting the bottom arc in between. If this were possible, then the geodesic in \( \mathbb{H} \) would change directions in “mid-air”, which is impossible. This means that every geodesic will have the form:

(block of hits on vertical side), (one hit on bottom), (block of hits on vertical side), (one hit on bottom),...

Thus, we simply recall the blocks of hits on the vertical sides and ignore the hits on the bottom arc, which are automatically known. This means that we can write down a code \([n_0, \ldots, n_k]\) of any closed geodesic where \( n_i \in \mathbb{Z}_{\neq 0} \) where each \( n_0 \) corresponds to the number of left or right “hits”. Thus, this condensed form of writing exactly describes the general principle of creating a sequence of generators mentioned in the preceding subsection.

But why do we measure left hits with negative numbers and right hits with positive numbers? If we look at the tiling of \( \mathbb{H} \) with \( F \), then we will see that a “right hit” corresponds to the geodesic traveling from \( F \) into \( T(F) \). Similarly, a “left hit” corresponds to the geodesic traveling from \( F \) into \( T^{-1}(F) \).

Actually, this works out so that, given a hyperbolic isometry \( G \in PSL(2, \mathbb{Z}) \), the geometric coding \( C(G) \) corresponds to the decomposition of \( C(G) \) into the generators \( T \) and \( S \). For instance, the geometric code of \( C(A) \) is [6] and \( A = T^6 \cdot S \). Also, the geometric code of \( C(B) \) is [4, -5] and \( B = T^4 \cdot S \cdot T^{-5} \cdot S \). In general, we will have a code \([n_0, \ldots, n_k]\) corresponding to a hyperbolic transformation \( T^{n_0} \cdot S \cdot \ldots \cdot T^{n_k} \cdot S \).

There are three more things to note quickly. First, by convention, we must always start counting “hits” when we are on the bottom arc boundary of \( F \).

Second, our geodesic must either intersect \( F \) or we will need to conjugate it by elements of \( PSL(2, \mathbb{Z}) \) so that the conjugate intersects \( F \). We consider every geodesic that is conjugate under \( PSL(2, \mathbb{Z}) \) to be equivalent and assign the geometric code described above to this equivalence class.

Finally, if the geodesic is not closed (i.e., any geodesic not corresponding to some element of \( PSL(2, \mathbb{Z}) \)), then we have a bi-infinite, non-repeating sequence as our code. We start at the bottom arc boundary of \( F \), as above, then count the number of “hits” in the same way (except we check the geodesic both forwards and backwards). But, because it is not closed these numbers will never repeat. So the code for a non-closed geodesic will look like \([\ldots, n_{-2}, n_{-1}, n_0, n_1, n_2, \ldots]\). Note that this is actually the same as the code for a closed geodesic because we could write the codes above as \([\ldots, 6, 6, 6, \ldots]\) and \([\ldots, 4, -5, 4, -5, \ldots]\). However, it is more convenient to simply write the least period of these bi-infinite codes, so we simply wrote [6] and [4, -5].

## 3 Billiards

Let’s consider the specific case where the fundamental region is a polygon \( P \) and the corresponding Fuchsian group \( \Gamma \) is generated by all reflections of the polygon. (Note that \( \Gamma \), by definition, is then generated by orientation-reversing isometries.) Then any geodesic on the Poincare disc that intersects the polygon \( P \) can be viewed as a billiards shot “bouncing” off of the sides of \( P/\Gamma \). We will specifically be looking at the case where \( P \) has a right-angle at every corner. This means that the geodesics that make up the sides of the polygons in the tiling of the disc will stay the same geodesic after the reflections. In other words, the tiling of the disc is made by intersecting a collection of geodesics in a “skeleton”.

Geodesics here can be coded using the same cutting-sequence method described above. Some work has been done on describing all admissible codes ([3]), but we have two conjectures that, if true, would give a simple description of all admissible codes.

**Conjecture 3.0.1.** A geometric code \((\gamma_t)\) is inadmissible if and only if it necessitates that any corresponding geodesic \( \gamma \) crosses a geodesic \( \gamma_0 \) within the 1-skeleton of the tiling polygons twice.
Note that the “if” here is clear. If \( \gamma \) crosses \( \gamma_0 \) twice, then either \( \gamma = \gamma_0 \) or \( \gamma, \gamma_0 \) is not a geodesic, both of which are contradictions. It remains to show that the “only if” holds here. This appears to be true when looking at representative examples, but no proof is currently known.

This conjecture leads us to our second, related conjecture:

**Conjecture 3.0.3.** If \( P \) is a right-angled polygon with \( n \) sides, a geometric code \((\gamma_i)\) necessitates crossing a a geodesic \( \gamma_0 \) within the 1-skeleton of the tiling polygons twice if and only if \((\gamma_i)\) contains the subsequence \((k, k \pm 1, k \mp 1, \ldots, k)\) for some \( 1 \leq k \leq n \) (where addition is mod \( n \)).

Again, we can prove the “if” statement. If a geodesic crosses from a polygon \( P \) to polygon \( P' \) in the tiling through the \( k \)-th side, then \((k+1)\)-th and \((k-1)\)-th sides of \( P \) and \( P' \) come from the same geodesic in the 1-skeleton of the tiling because the polygons are right-angled and the tiling is constructed by reflections along sides. Thus, crossing repeatedly through \((k+1)\)-th and \((k-1)\)-th sides of polygons implies that the \( k \)-th side of every polygon is part of the same geodesic.

Thus, it is clear that the subsequence \((k, k \pm 1, k \mp 1, \ldots, k)\) requires crossing the geodesic marked by \( k \) twice. Recall that we do not need to consider the case where any element of the sequence is repeated because this is impossible by construction. From this observation and looking at many examples, it appears that this conjecture describes the only problem that occurs. But, as before, the proof of the “only if” statement is currently not known.

Combining these two conjectures gives us the following:

**Conjecture 3.0.3 (“Golden Rule”).** If \( P \) is a right-angled polygon with \( n \) sides, a geometric code \((\gamma_i)\) is admissible if and only if \((\gamma_i)\) contains the subsequence \((k, k \pm 1, k \mp 1, \ldots, k)\) for some \( 1 \leq k \leq n \) (where addition is mod \( n \)).

### 4 Hyperbolic Buildings (Bourdon’s Buildings)

To give an idea of the structure of a Building, let’s start with a simple example. Take \( \mathcal{B} := \{(x, y, z)|x \in \mathbb{Z} \text{ or } y \in \mathbb{Z} \text{ or } z \in \mathbb{Z}\} \). This is a 2-dimensional Euclidean Building (which should align with our intuition of what a building is). This Building consists of simplices of dimension 0, 1, and 2. The maximal dimension simplices are called chambers; in this case, our chambers are (open) unit squares, 2-simplices.

Note that this building is a union of infinitely many planes - for example, we can take \( \mathcal{B} = \bigcup_{n \in \mathbb{Z}} A_n, B_n, C_n; \) where \( A_n := \{(x, y, z)|x = n\} \) and so forth. Each of these planes is called an Apartment. These apartments are sufficient to produce the building, however these are not the only apartments. The set \( S \cup T \) where \( S := \{(x, y, z)|x = 0 \text{ and } y \leq 0\}; T := \{(x, y, z)|y = 0 \text{ and } x \geq 0\} \) is an apartment. Simplicially, this apartment is equivalent to the other apartments. Specifically, apartments are isometric to one another. Another important property of buildings to note here is that given any two chambers, we can find an apartment that contains them both - in fact in this case, there are infinitely many. (This should not be too hard to convince yourself is true.)

What is truly interesting about buildings is that they are a union of the same metric space infinitely many times, and certainly we can extend metric to the building in a very natural way. (For any two points \( p, q \in \mathcal{B} \), there are some chambers \( P, Q \subset \mathcal{B} \) such that \( p \in P \) and \( q \in Q \); define \( dist(p, q) := \inf\{dist_A(p, q)\}|P, Q \subset A\} \). We can define the Isometry Group of this metric space and that is the primary focus of our study - we’ll come back to that.

#### 4.1 Tessellations

Take any regular right (90-degree) polygon := \( P \) in the Poincaré Disc \( \mathcal{U} \) with \( n \) sides \( (n \geq 4) \).

**Fact.** \( P \) is a fundamental domain for the Fuchsian group generated by inversion (reflection) over the geodesics which contain the sides of \( P \). This relies on the fact that we have a right polygon; if we draw the geodesics which contain each side, then reflect these geodesics over one another, then reflect those new geodesics over
one another, we end up with a Hyperbolic Plane (disc) which is covered by geodesics which intersect one another at right angles, with a polygon isometric to \( P \) appearing infinitely many times on a grid. This is called a tesselation of the hyperbolic plane by the polygon \( P \).

In fact, what we’ve defined here is the simplest case of a Hyperbolic Building. There is but a single apartment (the whole plane) and the chambers are isometric copies of \( P \). This isn’t a particularly exciting building, but its construction gives intuition for how more complex hyperbolic buildings are constructed.

### 4.2 Bourdon’s Buildings

Editing in Progress

### References

