Technical Appendix for Willpower and the Optimal Control of Visceral Urges*

Emre Ozdenoren† Stephen Salant† Dan Silverman†

This Version: November, 2007

Abstract

This document contains proofs of some results from the main paper. These proofs were omitted from the main text for the sake of brevity.

A The Equivalence of Solving Problems P1 and P2

Here we show that by solving the problem in which the date when consumption ceases is a choice variable (P2), we in fact solve our original problem (P1). In problem (P1), the optimal consumption path either finishes the cake at time $t' < T$, or it does not. If the cake is exhausted at time $t' < T$, then the law of motion governing the depletion of willpower jumps to zero, and for any time $t \in (t', T]$, $\dot{W}(t) = c(t) = 0$. These same paths of consumption and willpower could be achieved in problem (P2) by choosing $s = t'$, and would generate the same payoff. Similarly, if in the original problem (P1) the cake is not exhausted before time T $(R(t) > 0$ for $t < T)$, these paths of consumption and willpower could also be achieved in the related problem by choosing $s = T$, and would generate the same payoff. Since the two problems share objective functions and laws of motion up to time $s = t'$, any program that is feasible in the original problem is also feasible in the related problem and will generate the same payoff.

*We thank Roy Baumeister, Gérard Gaudet and Miles Kimball for their important contributions to this paper. We also thank Roland Benabou, Doug Bernheim, Drew Fudenberg, Kai-Uwe Kuhn, Yoram Halevy, David Laibson, David Levine, Robert Mendelsohn, Muriel Niederle, Andrew Postlewaite, Matthew Rabin, Lones Smith, Joel Sobel, Kathleen Vohs, Itzhak Zilcha, and participants in seminars at Harvard, IAS/Princeton, SITE, Calgary, Central Florida, Michigan, Milano-Bicocca, Montreal, Penn and Pittsburgh for their helpful comments on a previous draft. Silverman gratefully acknowledges the support of the Institute for Advanced Study.

†Department of Economics, University of Michigan, 611 Tappan St., Ann Arbor, MI 48109-1220.

Given that any consumption path that is feasible in problem (P1) is also feasible in problem (P2), if the optimal consumption path in problem (P2) is feasible in problem (P1) then it is also optimal in problem (P1).[1] A sufficient condition for any consumption path in problem (P2) to be feasible in problem (P1) is that the consumption path depletes the cake by time s ($R(s) = 0$). After all, that consumption path generates the same willpower path up to time s in both problems and therefore $W(t) \geq 0$ up to time s. After time s in problem (P1), willpower remains constant (at $W(s)$) since the cake is depleted. Hence any such consumption path is feasible in problem (P1) as well.

To see that the optimal consumption path in (P2) exhausts the cake at time s, suppose the contrary—that the "optimal" program leaves $R(s) > 0$ cake uneaten at s. To dominate this program, consider a different program that duplicates the "optimal" consumption path up to time $s - \Delta$, for any $\Delta > 0$, and consumes $\frac{R(s)}{\Delta}$ more during the remaining interval of length Δ. This exhausts the cake, draws willpower down to the same level at $s - \Delta$ and depletes less willpower during $(s - \Delta, s)$. Indeed, one can always choose Δ small enough that $\frac{R(s)}{\Delta} \geq \bar{c}$. In this case, depletion of willpower during $(s - \Delta, s)$ ceases altogether. Not only would utility from the alternative activity be weakly larger than on the "optimal" path but utility from intertemporal consumption would be strictly larger. This follows since the alternative consumption path is uniformly higher throughout and strictly higher from time $s - \Delta$ to s. This contradicts the claim that any feasible path with $R(s) > 0$ can be optimal. Hence, without loss of generality, we can confine our attention to problem (P2).

B Self-control Models and Increasing Paths of Consumption

This section of the appendix discusses how increasing consumption sequences are inconsistent with prior models of self control. The prior literature can be grouped into three strands. The first focuses on the present-biased choices which result from (quasi-) hyperbolic time discounting; the second models self-control problems which arise from temptation costs; the third models self-control problems which derive from internal conflicts between multiple "selves," one a long-run self which may, at a direct utility cost, exert control over the choices of a sequence of short-term selves. The basic models with hyperbolic discounting (Laibson, 1997; O'Donoghue and Rabin, 1999) do not explain strictly increasing consumption since the impatience posited would always lead to

[1] To see this, suppose that the optimal consumption path in problem (P2) is feasible but not optimal in problem (P1). Then there is a strictly preferred consumption path that is feasible in problem (P1). But by our earlier argument this path is also feasible in problem (P2) and would dominate the program we claimed was optimal in problem (P1). Hence we have a contradiction.

larger immediate consumption and smaller consumption later.[2] Demonstrating that dual self-models (Thaler and Shefrin, 1988; Fudenberg and Levine, 2006) and models with temptation costs (Gul and Pesendorfer, 2001, 2004a,b; Benhabib and Bisin, 2004; Dekel, Lipman and Rustichini, 2005) do not explain the preference for increasing consumption sequences requires a more detailed investigation.

Consider a finite-horizon, undiscounted version of Fudenberg-Levine's saving model and assume no growth. Although they restrict attention to a specific functional form of the utility function and the cost function, we consider the more general case where utility is any strictly increasing and strictly concave function and cost is any strictly increasing and weakly convex function, each twice differentiable. Then in the Fudenberg-Levine model the single-agent chooses $(c_1, ..., c_t)$ to maximize

$$\sum_{t=1}^{T} \{u(c_t) - g(u(R_t) - u(c_t))\} \text{ subject to } R_{t+1} = R_t - c_t \text{ and } R_1 = \bar{R}, \tag{B1}$$

where $c_t \geq 0$, $R_t \geq 0$ and $\bar{R} > 0$ and we assume that the utility function $u(\cdot)$ is strictly increasing and strictly concave and the cost function $g(\cdot)$ is strictly increasing and weakly convex. As before $\bar{R}$ is the size of the cake, and R_t is the remaining cake in period t. To show that $c_{t+1} \geq c_t$ can never occur in their model, we verify that it would always violate the first-order condition for this problem. Assuming differentiability, the following condition must hold at an optimum:

$$u'(c_t)\cdot[1+g'(u(R_t)-u(c_t))]-u'(c_{t+1})\cdot[1+g'(u(R_{t+1})-u(c_{t+1}))]+u'(R_{t+1})[g'(u(R_{t+1})-u(c_{t+1}))] = 0 \tag{B2}$$

The final term of (B2) is strictly positive since it is the product of two strictly positive factors. As for the other two terms, they would combine to something weakly positive, violating condition (B2) whenever $c_{t+1} \geq c_t$.[3] Weakly increasing consumption sequences are, therefore, inconsistent with Fudenberg-Levine's model.

For similar reasons, weakly increasing consumption sequences are inconsistent with Gul-Pesendorfer's model. In the finite horizon, undiscounted version of their problem the agent maximizes:

$$\sum_{t=1}^{T} \{[u(c_t) + v(c_t)] - v(R_t)\} \text{ subject to } R_{t+1} = R_t - c_t \text{ and } R_1 = \bar{R}, \tag{B3}$$

[2] Benabou and Tirole (2004) have a model with hyperbolic discounting and imperfect recall of both preferences and actions that can also explain increasing paths of consumption.

[3] To see this, note that each term is the product of two factors. If consumption is weakly increasing the first factor of the first term $(u'(c_t))$ is weakly larger than its counterpart $(u'(c_{t+1}))$. Moreover, the argument of g' in the first term is also strictly larger than its counterpart in the second term: $u(R_t) - u(c_t) > u(R_{t+1}) - u(c_{t+1})$ since the cake remaining will be strictly smaller and consumption at $t+1$ will be weakly larger. Since g is assumed weakly convex, the first product of two factors must weakly exceed the second product of two factors and hence these terms make the left-hand side of the first-order condition even more positive.

where $c_t \geq 0$ and $R_t \geq 0$. Both the u and v functions are assumed to be strictly increasing. To insure concavity of the objective function, the u is assumed to be strictly concave, v is convex and $u+v$ is strictly concave. It is straightforward to verify that this formulation insures strictly decreasing consumption.[4] Intuitively, an agent smoothing consumption between adjacent periods can always do better by marginally redistributing consumption to the earlier of the two periods. This perturbation in consumption has no first-order effects on the sum of utilities of consumption in the two periods while strictly reducing the uneaten cake and hence the temptation cost. Thus, their model also predicts strictly decreasing consumption. In contrast, increasing consumption sequences can arise in a model with willpower depletion because the same amount of self-control is less costly in terms of willpower expended if it is exercised when the willpower stock is larger. Hence, there is an incentive to exercise more self control in the earlier of two periods ("use it or lose it").

C When Willpower May be Built Through Its Exercise: Details

Again we consider a related but more tractable problem and argue that, by solving it, we solve problem (P3). In the related problem, the agent chooses both an optimal consumption path $c(t)$ and the date $s \leq T$ after which consumption ceases, where $\dot{W}(t) = \dot{M}(t) = c(t) = 0$ for all $t \in (s, T]$, to maximize:

$$V(0) = \int_0^s e^{-\rho t} U[c(t)]\, dt \tag{P4}$$

subject to the constraints of problem (P3) except constraints (11)-(13) which are replaced by:

$$\begin{aligned}
\dot{W}(t) &= \gamma M(t) - f(W(t), c(t)) \\
\dot{M}(t) &= f(W(t), c(t)) - \sigma M(t) \\
R(t) &\geq 0,\ W(t) \geq 0,\ M(t) \geq 0 \text{ for } t \in [0, s].
\end{aligned}$$

As in the problem without muscle, to show that this tractable problem has the same consumption path for $t \in [0, s]$ as the solution to the actual problem (P3), it suffices to show that in problem $(P4)$ the entire cake is consumed by $t = s$ in the optimal solution. To see this note that if $R(s) > 0$ in the optimal program then $M(s) > 0$ and $W(s) \geq 0$. But then we could choose Δ small enough that $\frac{R(s)}{\Delta} \geq \bar{c}$. We could then duplicate the proposed optimal path until $s - \Delta$ and augment it by $\frac{R(s)}{\Delta}$ in this final interval. The payoff would be strictly higher and the program would be feasible

[4] The first-order condition is $[u'(c_t) + v'(c_t)] - [u'(c_{t+1}) + v'(c_{t+1}) - v'(R_{t+1})] = 0$. Given that $u+v$ is strictly concave and v is strictly increasing, the first-order condition can only be solved by a strictly decreasing consumption sequence.

since the willpower left at $s-\Delta$ is the same in the two programs ($W(s-\Delta) \geq 0$) and no willpower is depleted in the final interval.

Having established that the solution of our related problem ($P4$) solves the actual problem (P3), we make two observations which simplify the analysis. First, since the muscle is initially positive and decays exponentially even if it is never augmented, muscle will be strictly positive and will at no time violate the nonnegativity constraint. Second, since the stock of cake can only decline, requiring that it is nonnegative at s insures that it will be nonnegative previously.

Given that for $t \in [0, s)$ these two state variables must be nonnegative, we can simplify our formulation by replacing $R(t) \geq 0$ and $W(t) \geq 0$ by $R(s) \geq 0$ and $W(s) \geq 0$. However, since $W(s) \geq 0$ no longer implies that $W(t) \geq 0$ for $t < s$, the conditions which must necessarily hold at an optimum whenever $W(t) > 0$ will differ from those that hold while $W(t) = 0$. Given our focus, we consider only the former situation in detail.[5] The Hamiltonian for this problem is:

$$
\begin{aligned}
&H\left(c\left(t\right), R\left(t\right), W\left(t\right), t, \alpha\left(t\right), \lambda\left(t\right), \pi\left(t\right)\right) \\
&= e^{-\rho t} U\left(c\left(t\right)\right) - \alpha\left(t\right) c\left(t\right) + \lambda\left(t\right)\left(\gamma M\left(t\right) - f\left(W\left(t\right), c\left(t\right)\right)\right) + \\
&\pi\left(t\right)\left(f\left(W\left(t\right), c\left(t\right)\right) - \sigma M\left(t\right)\right).
\end{aligned}
$$

[5] To derive conditions which must hold across both cases, Seierstad and Sydsœter (1987), and also Léonard and Long (1992)) begin by forming the Lagrangean $H + \Theta(t)W(t)$, where $\Theta(t)$ is a Lagrange multiplier. In such problems the multiplier (λ) on the state variable (willpower) may jump discontinuously as the nonnegativity constraint is just reached or as it becomes slack. If that multiplier does jump, then consumption would jump as well at such dates.

The first order conditions are given by,

$$c(t) \geq 0,\ e^{-\rho t}U'(c(t)) - \alpha(t) - (\lambda(t) - \pi(t)) f_c \leq 0 \text{ and c.s.} \tag{C4}$$

$$\dot{W}(t) = \gamma M(t) - f \tag{C5}$$

$$\dot{M}(t) = f - \sigma M(t) \tag{C6}$$

$$\dot{\alpha}(t) = 0 \tag{C7}$$

$$\dot{\lambda}(t) = f_W(\lambda(t) - \pi(t)) \tag{C8}$$

$$\dot{\pi}(t) = \pi(t)\sigma - \lambda(t)\gamma = -\gamma\left(\lambda(t) - \frac{\sigma}{\gamma}\pi(t)\right) \tag{C9}$$

$$T - s \geq 0,\ H(c(s), R(s), W(s), s, \alpha(s), \lambda(s), \pi(s)) \geq 0 \text{ and c.s.} \tag{C10}$$

$$W(t) > 0 \text{ and c.s.} \tag{C11}$$

$$R(s) \geq 0,\ \alpha(s) \geq 0 \text{ and c.s.} \tag{C12}$$

$$W(s) \geq 0,\ \lambda(s) \geq 0 \text{ and c.s.} \tag{C13}$$

$$M(s) \geq 0,\ \pi(s) \geq 0 \text{ and c.s.} \tag{C14}$$

To analyze the dynamics of consumption, it will be useful to sign $\lambda(t) - \pi(t)$ and $\lambda(t)\left(\frac{\gamma}{\sigma}\right) - \pi(t)$. First assume that $\gamma/\sigma \leq 1$. In this case, we show that each of the preceding terms is weakly positive while $\dot{\lambda}$ and $\dot{\pi}$ are weakly negative. These results can be most readily understood using the phase diagram depicted in Figure 1. Provisionally assume that $f_W < 0$ and $\gamma > 0$. Then we can plot the locus of (λ, π) pairs such that $\dot{\lambda} = 0$. By the $\dot{\lambda}$ equation $(C8)$, these points lie on the 45^0 line $\pi = \lambda$. Horizontal motion above this locus is to the right and below it is to the left. Similarly, we can plot the locus of (λ, π) pairs such that $\dot{\pi} = 0$. By the $\dot{\pi}$ equation $(C9)$, these points lie on a flatter ray provided $\frac{\gamma}{\sigma} < 1$. In the extreme case where $\frac{\gamma}{\sigma} = 1$, the two rays coincide. Vertical motion above the $\dot{\pi} = 0$ locus is upward and below this locus it is downward. As long as muscle exists at any time in the program, some will remain at the end ($M(T) > 0$), because it at most decays exponentially and therefore never reaches zero. It follows from condition $(C14)$ that the endpoint condition $\pi(T) = 0$ is satisfied. As long as willpower considerations matter $(\lambda(0), \pi(0) \neq 0)$, the endpoint condition and dynamics preclude initial multipliers set at or above the lower of the two rays since then $\dot{\pi} \geq 0$ implying $\pi(T) > 0$. Thus $\pi(T) = 0$ requires that the initial multipliers be set below the lower of the two rays. But this in turn implies that $\lambda - \pi > 0, \frac{\gamma}{\sigma}\lambda - \pi > 0, \dot{\lambda} < 0$, and $\dot{\pi} < 0$.

Now consider the case where $\gamma/\sigma > 1$, depicted in Figure 2. The endpoint condition $\pi(T) = 0$, together with these dynamics imply that there will be a final phase in which the multipliers will lie strictly below the $\dot{\lambda} = 0$ locus, and thus, again, $\lambda - \pi > 0, \frac{\gamma}{\sigma}\lambda - \pi > 0, \dot{\lambda} < 0$, and $\dot{\pi} < 0$.

To see that the optimal consumption path in the muscle model involves consuming the entire

cake, note that having assumed $U'(\cdot) > 0$ and $f_c < 0$, and shown $(\lambda - \pi) > 0$, we can satisfy (C4) only if $\alpha > 0$; and then (C12) requires that the cake be exhausted by time s. To see that, if we start with an initial level of willpower sufficient for perfect smoothing, decreases in that stock eventually lead to a willpower level $\left(W_{\tilde{H}}\right)$ where any further reduction in the initial stock of willpower will make the perfectly smooth path infeasible, note that since $\lambda(0) \geq 0$, we know that utility is increasing in the initial stock of willpower. Because perfect smoothing is the optimal path in the absence of willpower concerns path, it follows that such a path is infeasible for any $W(0) < W_{\tilde{H}}$.

Finally, we turn to the time path of optimal consumption in the absence of discounting $(\rho = 0)$. Differentiating condition (C4), we obtain:

$$\begin{aligned} U''(c)\,\dot{c} &= \dot{\alpha} + \left(\dot{\lambda} - \dot{\pi}\right) f_c + (\lambda - \pi)\left(f_{cc}\dot{c} + f_{cW}\dot{W}\right) \\ &= \left(f_W(\lambda - \pi) - (\pi\sigma - \lambda\gamma)\right) f_c + (\lambda - \pi)\left(f_{cc}\dot{c} + f_{cW}\dot{W}\right) \\ &= (\lambda - \pi)\left(f_W f_c + f_{cc}\dot{c} + f_{cW}\dot{W}\right) - (\pi\sigma - \lambda\gamma) f_c \end{aligned}$$

which implies

$$\dot{c} = \overbrace{\frac{(\lambda - \pi)\left(f_W f_c - f_{cW} f\right)}{\Delta}}^{\text{direct willpower effect}} + \overbrace{\frac{(\lambda - \pi) f_{cW}\gamma M(t)}{\Delta}}^{\text{muscle service flow}} + \overbrace{\frac{\left(\lambda\left(\frac{\gamma}{\sigma}\right) - \pi\right)\sigma f_c}{\Delta}}^{\text{muscle building}} \tag{C15}$$

where $\Delta = \left[U''(c) - (\lambda - \pi) f_{cc}\right] < 0$. The inequality follows because c maximizes the Hamiltonian. If $\lambda(t) = \pi(t) = 0$ for $t \geq 0$, equation (C15) yields the classical result that consumption is constant as long as it is positive and equation (C10) requires that it be positive until T.

Referring to equation (C15), if $f_W = 0$ then both the direct willpower effect and the muscle service flow are absent. First consider the case where muscle decays at a faster rate than it contributes to willpower $[(\gamma/\sigma) \leq 1]$. In this case, the muscle building term of equation (C15) is always positive (see Figure 1) and the associated discussion). Thus when $(\gamma/\sigma) \leq 1$ and $f_W = 0$, consumption is always increasing.

When muscle decays more slowly $[(\gamma/\sigma) > 1]$, more complex consumption paths may emerge. If the multipliers λ and π are initial located in regions II or III of Figure 2, optimal behavior has the same qualitative features as when $(\gamma/\sigma) \leq 1$. Consumption is always increasing. If, however, the multipliers are initially located in region I of Figure 2, consumption will decrease with time until the multipliers pass into region II. That is, the consumption profile is $\cup$ - shaped.

Referring again to equation (**??**), if $(f_W f_c - f_{cW} f) = 0$ then only the direct willpower effect is inactive. Relative to the optimal path in the absence of muscle, the ability to build of willpower with exercise again induces time preference. Consider, for example, a situation where the initial

muscle stock is zero $(M(0) = 0)$,and thus, at the beginning of the program, the muscle service flow term is inactive. In the beginning of the program, optimal behavior in this case is like that in the case where $f_W = 0$. For example, when $(\gamma/\sigma) \leq 1$, consumption will increase in these early stages of the optimal path.

Figures

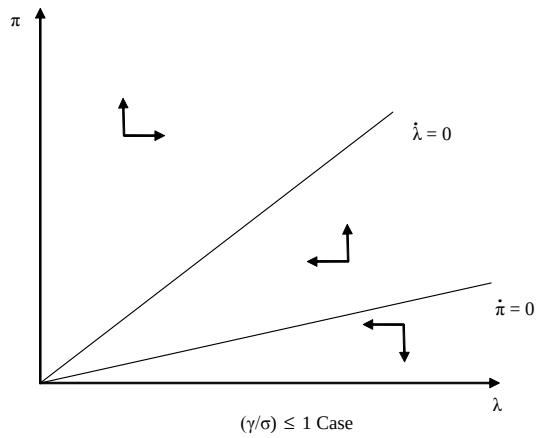

Figure 1: Phase Diagram of Costate Variables in Muscle Model Where $\frac{\gamma}{\sigma} \leq 1$.

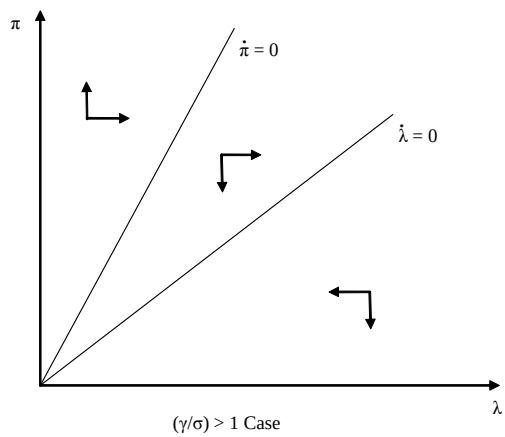

Figure 2: Phase Diagram of Costate Variables in Muscle Model Where $\frac{\gamma}{\sigma} > 1$.