Willpower and the Optimal Control of Visceral Urges*

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Abstract

Common intuition and experimental psychology suggest that the ability to self-regulate ("willpower") is a depletable resource. We investigate the behavior of an agent with limited willpower who optimally consumes over time an endowment of a tempting and storable consumption good or "cake." We assume that restraining consumption below the most tempting feasible rate requires willpower. Any willpower not used to regulate consumption may be valuable in controlling other urges. Willpower thus links otherwise unrelated behaviors requiring self-control. An agent with limited willpower will display apparent domain-specific time preference. Such an agent will almost never perfectly smooth his consumption, even when it is feasible to do so. Whether the agent relaxes control of his consumption over time as experimental psychologists predict or tightens it as most behavioral theories predict depends in our model on the net effect of two analytically distinct but opposing forces.

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1 Introduction

Common patterns of intertemporal choice that are inconsistent with standard models have inspired a literature on the economics of self-control. Tendencies to seek self-commitment, to procrastinate, and to seize small immediate rewards despite their important future costs, have motivated studies of quasi-hyperbolic time discounting (e.g., Laibson 1997, O'Donoghue and Rabin, 1999), temptation costs (e.g., Gul and Pesendorfer 2001, 2004), and conflicts between selves or systems (e.g., Thaler and Shefrin, 1988; Bernheim and Rangel, 2004; Fudenberg and Levine, 2006). These models are consistent with a great deal of experimental evidence, and have been fruitfully applied to economic problems ranging from portfolio choice to labor supply to health investment.

Our paper is focused on an aspect of self control that has received less attention in the economics literature: the *variation* in an individual's exercise of self restraint across time and circumstances. Specifically, we are concerned with anecdotal and laboratory evidence that the exercise of self control at one time, or in one domain, leaves a person less able or willing to exert self-restraint at another time, or in another domain. This evidence suggests that the exercise of self control draws on a limited and fungible cognitive resource – a resource that is often called *willpower*.

Common behaviors indicate that the ability to self-regulate may be a limited resource. Many who resist unhealthy food and fruitless websurfing all day, and who might prefer to go to bed early after a light dinner, find themselves up late, watching T.V. and gorging on junk food. Dieters can often maintain their discipline for short periods but find such self-restraint unsustainable over the long term.[1] Households can often keep to a budget for a while, but then fall back into old spending patterns and well short of their saving goals.

Experimental psychology (Baumeister *et al.,* 1994; Baumeister and Vohs, 2003) has gone beyond such anecdotes and demonstrated that individuals depleted by prior acts of self-restraint tend to behave later as if they have less capacity for self-control. The typical experiment has two phases. Every subject participates in the second phase but only a randomly chosen subset participates in the first, with the remainder serving as a control group. In the first phase, subjects are asked to perform a task that is meant to deplete their willpower; in the second phase, their endurance in an unrelated activity requiring self control is measured.[2] Subjects who participate in the first phase display substantially less endurance in the second phase. This apparent link between the exercise

[1] Kessler (2009) offers especially vivid accounts of this pattern.

[2] For example, in the first phase subjects have been asked not to eat tempting foods, not to drink when thirsty, and to inhibit automated/habitual behaviors such as reading the subtitles of film or "reading" the color of the ink that a word is written in, rather than the word itself. Typically, the self-control tasks differ in the two phases of these experiments. However, Vohs and Heatherton (2000) have obtained similar results in experiments where restraining consumption of tempting foods was used in both phases of the experiment, though not identical foods.

of self-control in one activity and later self-discipline in another has been observed repeatedly, with many different manipulations and measures of self-regulation (Baumeister and Vohs, 2003, Vohs and Faber, 2004). Moreover, subjects exhibit foresight, performing less well on the initial self-control task if informed at the outset that they will be asked to complete an additional task requiring self-control at the end (Muraven et al., 2006, and the references therein). Two experiments (Vohs and Faber, 2004 and Dewitte et al., 2005) show that willpower depletion and prior cognitive loads affect subsequent *economic* behavior.[3] Individual experiments have weaknesses, but we view the collection of these experimental findings as reinforcing the intuitive notion of willpower as a depletable cognitive resource.

Self-control may be exercised for a variety of reasons. In this paper we investigate the use of willpower to limit immediate consumption of a good so that more of that good can be enjoyed later. To do so, we add a willpower constraint to an especially simple model of intertemporal choice: the canonical cake-eating problem. In our formulation, moderating consumption of tempting goods, including leisure, requires willpower; the greater restraint the consumer exercises, the faster his willpower erodes.[4] We also take account of other activities besides intertemporal consumption that require willpower: cramming for and taking exams, training for performances, maintaining a diet, or sustaining mental focus over long periods.

While our model of limited willpower is simple, it generates a rich set of implications. We find that a willpower-constrained consumer would behave in ways that seem inconsistent with having a single rate of time discount. He might, for example, exhibit considerable patience when it comes to work effort and, at the same time, appear myopic when it comes to the long-term consequences of food and drink. His time preference would thus appear *domain-specific*.

Limited willpower also has important consequences for preferences over the timing of consumption. The hallmark of the canonical model, intertemporal smoothing, virtually disappears when a willpower constraint is introduced. Even when the agent has enough willpower to smooth consumption perfectly over the entire time horizon, he is almost always better off foregoing this option and

[3]In Vohs and Faber (2004), subjects who were willpower-depleted purchased a wider assortment of merchandise and spent a larger portion of their experimental earnings than the control group. The experiments of Dewitte et al. should be distinguished from research that showed the effects of cognitive loads on *contemporaneous* choices. In these latter studies, respondents were more likely to choose cake over fruit (Shiv and Fedorikhin, 1999) or a smaller earlier reward over a larger later one (Hinson, et al. 2003) when *simultaneously* asked to perform a memory task. In contrast, Dewitte et al. (2005) found that even when such memory tasks were performed prior to the consumption decision, they affected the choice of how much candy to eat.

[4]Experiments indicate that willpower may also be drained by other activities such as inhibiting automatic responses or bearing cognitive loads. Our model accommodates these causes of depletion either as anticipated uses of willpower that come after consumption choices, or as a shock to the initial willpower stock.

using more willpower to regulate alternative activities.[5] In addition, the Baumeister experiments suggest that a person with limited willpower will relax his self control over time and therefore consume food or leisure at an increasing rate – a pattern consistent with decisions to postpone consumption or get the hard part of a project out of the way at the outset. The thrust of previous economic models of self-control is quite different: agents are predicted to reduce their consumption over time. Either outcome is possible in our model and we can distinguish analytically the opposing forces which determine whether consumption rises or falls.

In an extension to our basic model, we consider the case where, consistent with some psychology experiments (Muraven *et al.,* 1999 and Muraven and Baumeister, 2000), the ability to self-regulate is like a muscle; avoiding temptations depletes willpower over the short term, but the regular exercise of such restraint can eventually build willpower. This extended model shows how the benefits of building willpower through its exercise generates an incentive to apply more self-restraint up front. We find that muscle-building causes optimal consumption to rise with time even when basic willpower concerns alone would imply that consumption would be constant.

Both our basic model and the extension differ from the bulk of the self-control literature by accommodating an effect of previous acts of self-restraint on current behavior that requires self-control. Models of hyperbolic discounting (Laibson 1997, O'Donoghue and Rabin, 1999) or temptation costs (Gul and Pesendorfer 2001), for example, would predict that, so long as the agent faces the same choice set (cake and feasible rates of consumption), the history of his previous choices would not matter. Models of addiction, habituation, or cues, (Becker and Murphy, 1986, Laibson, 2001, Gruber and Kőszegi, 2001, Gul and Pesendorfer, 2007, Bernheim and Rangel, 2004) allow the history of choices to affect behavior, but in these models the consumption of a good today affects later tastes for *the same good* whereas in our model, the willpower constraint links (induced) preferences across all forms of tempting consumption. The model in Benabou and Tirole (2004) also links past behavior to future choices as agents with hyperbolic discounting attempt to develop self-reputations for being patient. However, in their model the successful exercise of self-control in one domain invariably increases the likelihood of later self-control, even over the short term. We defer further discussion of the relation of our model to other particularly relevant contributions of the self-control literature (Fudenberg and Levine (2006), and Loewenstein and O'Donoghue (2005))

[5] Allowing willpower to have alternative uses has other implications. For example, using the typical two-stage protocol, Muraven (1998) finds that a given first-stage depletion activity has less of an effect on second stage self-control when subjects are paid more for exerting that latter control. This finding is consistent with a model where willpower has alternative uses. If we fix the marginal value of willpower remaining after the experiment, and increase the incentive for exerting self-control in the second phase activity, then the agent will optimally reallocate willpower to that second phase self-control activity and leave less in reserve for self-regulation after the experiment.

until the implications of our model have been presented.

The rest of the paper proceeds as follows. In section 2 we present the primary elements of our model. In the first part of that section, we focus on a particularly simple case in order to highlight the novel linkages in economic behavior induced by a willpower constraint. Because the willpower constraint does not induce time preference in this case, we can show many consequences of limited willpower using elementary calculus. In the remainder of the section, we discuss the patterns of time preference that the willpower constraint can induce in the more general problem. In section 3 we discuss further how our theory relates to other models of self control. In section 4 we consider an extension of our analysis to the case where the exercise of self control depletes willpower over the short term but builds it over the longer term. In section 5 we conclude.

2 A Model of Limited Willpower

To investigate the consequences of limited willpower in an especially simple setting, we consider the canonical cake-eating problem where a consumer chooses his consumption path $c(t)$ to maximize his discounted utility $U(c(t))$ over a finite horizon. We assume $U(0) = 0$ and, for $c \in (0, \bar{c})$, $U(c)$ is strictly concave, strictly increasing, and twice differentiable. We interpret the consumption rate $\bar{c}$ as reflecting either (1) a physical limit on the rate of consumption,[6] or (2) the rate of consumption resulting in the highest utility flow ($U(\bar{c}) \geq U(c)$ for $c > \bar{c}$), in which case consuming at a faster rate than $\bar{c}$ is never optimal.[7] The analysis that follows permits either interpretation. Our focus is on non-addictive goods like those captured in experiments by Baumeister and others, so we assume that U is independent of prior consumption. We denote the stock of cake remaining at time t as $R(t)$ and assume that $R(0)$ is given. There is no return to saving in the model; thus at time t the cake declines at rate $c(t)$.

2.1 Willpower Depletion

We depart from the canonical model by assuming that the consumption good is tempting and that the agent, initially endowed with willpower stock $W(0)$, may deplete his willpower when he limits his rate of consumption. More precisely, we assume the rate of willpower depletion at t depends on the stock of remaining willpower, $W(t)$, and the current rate of consumption $c(t)$. We assume that depletion function, denoted as $f(W, c)$, is nonnegative and $f(W, c) = 0$ for $c \geq \bar{c}$. We

[6] This interpretation is especially appropriate if the consumption good is leisure time. If, for example, time is measured in hours, leisure cannot be consumed faster than one hour per hour: $c \in [0, 1]$.

[7] If consuming at a rate faster than $\bar{c}$ is feasible, it would never be chosen since it saves no willpower and by shortening the time to exhaustion without increasing the utility flow it reduces overall utility.

assume f is decreasing in W, strictly decreasing and strictly convex for $c \in [0, \bar{c}]$, and $f_{cW} \geq 0$. When no cake remains, the rate of willpower depletion is zero.

These assumptions follow, in large part, from a natural view of the determinants of temptation costs, a view with foundations established in the literature on self-control and dual selves (Gul and Pesendorfer, 2001, and Fudenberg and Levine, 2006).[8] In this view, the temptation (willpower) costs associated with consuming at rate c are increasing in the utility forgone by choosing c instead of the most tempting feasible rate. Thus, in our setting, no willpower is expended if the consumer chooses rate $\bar{c}$, and no willpower is expended if no cake remains to be consumed. Similarly, if the agent is committed to consume only at rate c (i.e., no other rate is feasible) then, again, no willpower is depleted. More generally, given that utility is strictly increasing in consumption for $c \in [0, \bar{c}]$, whenever cake remains the rate of willpower depletion will, as we have assumed, strictly decline with c. Assuming, in addition, that f is strictly convex in c, captures the natural idea that increases in the rate of consumption save less willpower when the rate is closer to $\bar{c}$.

Adopting this view also implies, given our continuous-time formulation, that the utility forgone by restraining the rate of consumption does not depend on the size of the remaining cake. This follows because, in continuous time, as long as some cake remains ($R(t) > 0$), it is generally feasible to consume at any rate below $\bar{c}$. Hence, if cake remains and there are no external restrictions on the menu of consumption rates, the rate of willpower depletion at time t depends only on the rate of consumption and not the size of the cake.[9]

We clarify later in this section both the motivation for, and implications of, our other substantive behavioral assumption, that the rate of willpower depletion is weakly decreasing in W.

[8] In the appendix, we formalize this argument.

[9] In a discrete-time formulation of our model, the utility forgone by restraining consumption, and thus the rate of willpower depletion, *would* depend on the size of the cake. So long as the remaining cake were less than $\bar{c}$, the most tempting choice at any time would be to consume the entire cake. This is also why in Gul and Pesendorfer's (2004) discrete time model the most tempting choice in any period is consumption of all remaining wealth. If Gul and Pesendorfer's model were, like ours, formulated in continuous time, the most tempting choice at any time would also be the most tempting feasible rate of consumption (a flow), not the entirety of remaining wealth (a stock). Even in continuous time, though, there is a sense in which richer people with their greater consumption opportunities (more cake) endure greater temptation. If, for example, a rich person duplicates the *entire* consumption path of a poor person who at some point exhausts his (smaller) cake, the rich person will deplete more of his willpower. After he has finished his cake, the poor person requires no willpower to consume nothing. The rich, however, must use his willpower to resist temptation and set his consumption to zero.

2.2 Alternative Uses of Willpower

We allow for the possibility that any willpower that remains after moderating intertemporal consumption may be used to restrain urges in other activities. We think of this willpower "bequest" not as generating utility directly, but as altering the feasible set of choices in other activities and therefore *indirectly* determining the utility generated from those activities.[10] We denote this indirect utility function as $m(\cdot)$. To illustrate, suppose the intertemporal consumption problem is how a student allocates time between having fun and studying, a predetermined 12 hours, for an exam to be given in 2 days. His time horizon is 48 hours, during which time he can consume 36 hours of leisure (his cake); and studying involves resisting the urge to consume that tempting leisure. In this case, the bequest function may capture the value of willpower that remains after studying and is used to maintain concentration and avoid stupid mistakes during the exam. We also allow $m(\cdot)$ to be a constant; i.e., willpower's value might derive solely from its use in moderating intertemporal consumption. More generally, we view the bequest function as a stand-in for the value of willpower applied to subsequent activities in which the agent might benefit from self-control.

2.3 Cake Eating Problem

The agent cannot choose a consumption path which results in negative willpower. From the set of consumption paths which maintain nonnegative stocks of both willpower and cake, the consumer chooses the one which maximizes the sum of his discounted utility of consumption and the value of the willpower bequeathed to the regulation of his alternative activities.

If a consumer with initial willpower $\overline{W}$ and initial cake $\overline{R} < \bar{c}T$ chooses his consumption path $c(t) \in [0, \bar{c}]$ optimally, he maximizes:

$$
\begin{aligned}
& V(0) = \int_0^T e^{-\rho t} U[c(t)]\, dt + e^{-\rho s} m(W(s)) \\
\text{subject to} \quad & \dot{R}(t) = -c(t) \qquad\qquad (P1) \\
& \dot{W}(t) = \begin{cases} -f(W(t), c(t)) \ \text{ if } R(t) > 0 \\ 0, \text{ otherwise} \end{cases} \\
& R(T) \geq 0, \ W(T) \geq 0 \\
& R(0) = \overline{R} \in (0, \bar{c}T) \\
& W(0) = \overline{W} \geq 0,
\end{aligned}
$$

[10] None of our results depends on the units in which willpower is measured. The reason is simple. For any particular way of measuring willpower, we identify the utility-maximizing path among those the agent has sufficient self-control to implement. Since using a different willpower metric would identify the same set of decision paths and since utility depends only on these paths and never on the willpower number ascribed to them, our invariance result follows.

where ρ is the subjective rate of time discount and $s = \sup \{t \in [0, T] : R(t) > 0\}$.[11] For the bulk of our analysis, we interpret the relevant time horizon as short (a day, several days, or a week), and will therefore set $\rho = 0$. Variations in the rate of consumption are thus attributable to the effects of limited willpower. Note that the function describing changes in the stock of willpower $\left(\dot{W}(t)\right)$ jumps to zero when the cake is exhausted.

As discussed below, the inclusion of a willpower constraint affects consumer behavior in two ways: it induces linkages between behaviors in different domains and it can induce time preference in consumption. To study these effects separately, we investigate first a particularly simple case where willpower induces linkages but not time preference and then move on to the more general case where willpower concerns can also induce time preference.[12]

2.4 Induced Linkages

We begin our analysis by studying optimal choice in the case where the rate of willpower depletion depends only on the level of consumption, and not on the willpower stock. Formally, we assume that $f(W(t), c(t)) = g(c(t))$. Moreover, we assume that additional willpower devoted to alternative activities has a constant marginal value ($m'(W(t)) \equiv m$). Again, we note that m might equal zero, and thus a willpower bequest might have no value.

Under these assumptions, it is optimal (1) to exhaust the cake and (2) to consume it at a constant rate as long as cake remains. To see why this is true, suppose the contrary. If it is optimal to leave some cake unconsumed, the agent must consume at a rate slower than $\bar{c}$ over some interval. This follows because we have assumed the cake is too small to consume at the maximal rate over the entire time horizon ($\bar{R} < \bar{c}T$). But then an increase in the consumption rate over part of that interval would increase overall utility and would at the same time save willpower. Hence, the original program is not optimal and the optimal program must exhaust the cake. Now suppose it were optimal for the agent to vary his consumption rate before the cake is exhausted. Because utility is strictly concave in tempting consumption, marginally reducing larger consumption while marginally increasing the smaller consumption by an offsetting amount would strictly increase utility from consumption without violating the cake constraint. Whether this perturbation strictly increases overall utility depends on its effect on willpower. Recall, however, we are assuming

[11] We have assumed, for convenience, that the value of the willpower bequest to alternative activities is realized either when the cake is exhausted (s) or at the end of the horizon (T), whichever comes first. This assumption is not material to our results.

[12] Because the willpower depletion technology is discontinuous, problem ($P1$) is somewhat nonstandard. Therefore, in each investigation, we will solve problem ($P1$) by analyzing a simpler, conventional problem with the same solution. Details about the equivalent problem are provided in the appendix.

that willpower depletion is independent of the willpower stock and is a decreasing function of consumption. It follows that the proposed perturbation will actually make more willpower available for use elsewhere: increasing the smaller consumption releases more willpower than is depleted by reducing the larger consumption by an offsetting amount, and no changes would occur in willpower depletion at other times since consumption would be unchanged then. So if the original program was feasible, the perturbed program will also be feasible but will yield higher utility. Hence, in the optimal program consumption must be constant as long as cake remains.[13]

The preceding argument shows that consumption is constant when it is positive. In principle, however, cake may be exhausted prior to T, after which consumption must be zero. A perturbation that further smooths consumption in this situation has a decidedly different effect than the one described above. In this situation, marginally reducing consumption in the phase when it is positive and marginally increasing it in the phase when it is zero depletes willpower in *both phases*. This occurs because when no cake is available no willpower is depleted, but once cake becomes available in the second phase, consuming a small amount requires additional willpower. As the analysis below clarifies, even when feasible it is not always optimal to stretch the phase of constant consumption to the end of the horizon. Intuitively, when the return to alternative uses of willpower is high, it will be better to complete consumption before time T and save more willpower for other purposes. We will call the situation when the agent chooses to stretch consumption to time T the "perfect-smoothing regime," and call the alternative the "no-smoothing regime."

2.4.1 Simple Comparative-Statics

In the rest of this section, we assume the agent has enough cake or willpower to implement perfect smoothing if he chooses.[14] Then the payoff from consuming at constant rate $c \in [0, \bar{c}]$ and exhausting the cake at time $s \in [s_{min}, T]$, where $s_{min} = \bar{R}/\bar{c}$, is $sU(c) + m[\overline{W} - sg(c)]$, where $s = \frac{\overline{R}}{c}$. Provisionally substituting $\frac{\overline{R}}{s}$ for c we obtain a continuous, strictly concave objective function of one variable, s. The agent's problem is equivalent to choosing the time to exhaust the cake, s, to solve:

$$\max_{s \in [s_{min}, T]} sU(\frac{\overline{R}}{s}) + m[\overline{W} - sg(\frac{\overline{R}}{s})].$$

It is straightforward to verify that an optimum exists and, since the maximand is strictly concave, any solution to the first-order condition is optimal. The optimum may occur at $s < T$ or at $s = T$.

[13] See footnote 24 for a more formal proof of these arguments.

[14] That is, $\overline{W} - Tg(\frac{\overline{R}}{T}) > 0$. The case where he lacks enough cake or willpower to implement perfect smoothing is equally tractable but less interesting.

No-smoothing Regime When no smoothing is optimal, the first-order condition[15] implies:

$$U(\frac{\overline{R}}{s}) - U'(\frac{\overline{R}}{s})\frac{\overline{R}}{s} - mg(\frac{\overline{R}}{s}) + \frac{\overline{R}}{s}mg'(\frac{\overline{R}}{s}) = 0. \tag{1}$$

The objective function is strictly concave, so the left-hand side of (1) is strictly decreasing in s. If it did not decline to zero for $s < T$, then perfect smoothing ($s = T$) would be optimal.

To determine the rate of consumption in the no-smoothing regime, we rewrite the first-order condition in terms of the constant consumption rate c:

$$U(c) - cU'(c) - mg(c) + cmg'(c) = 0. \tag{2}$$

Notice that equation (2) implicitly determines the rate of constant consumption (c) as a function of the exogenous marginal value (m) of bequeathed willpower; in the no-smoothing regime, therefore, the rate of consumption does not depend on either of the other exogenous variables ($\bar{W}$ and $\bar{R}$). If an agent in this regime begins with larger initial willpower, he optimally consumes at the same rate and, with a cake of unchanged size, exhausts it at an unchanged time (s), bequeathing all of the additional willpower to the alternative activities.

Turning next to changes in initial cake sizes, suppose someone in the no-smoothing regime began with a marginally larger initial cake. Then his rate of consumption would not change (since m has not changed); but since he would take longer to exhaust his larger cake later (s increases) he would have less willpower left for the alternative activities.[16]

Now consider changes in the value (m) of willpower applied to alternative activities. Suppose someone in the no-smoothing regime began with a marginally higher m. It is straightforward to verify that the rate of consumption is a strictly increasing function of m.[17] Hence, the optimal rate of consumption would increase in response and, given that the cake size is unchanged, it would be exhausted sooner leaving more willpower for the subsequent activity.

Perfect-smoothing Regime In contrast, when perfect-smoothing is optimal, no behavior changes in response to an exogenous increase in the value of willpower in the alternative activities (m). For,

[15] It may also be optimal to consume at rate $\bar{c}$ until the cake is exhausted at s_{min}. This occurs if and only if the left-hand side of (1) is weakly negative at $s = s_{min}$. Similarly, it may be optimal, when m is sufficiently small, not to smooth perfectly and yet carry no willpower over into alternative activities. Each of these cases is less interesting and, given space constraints, we do not discuss them.

[16] Indeed, as noted in the previous footnote, it is possible, if m is sufficiently small, that the consumer will choose to carry no willpower over into alternative activities.

[17] Totally differentiating the first-order condition determining c in the no-smoothing regime, we conclude that $dc/dm = (g(c) - cg'(c)) \,/\, (c[mg''(c) - U''(c)]) > 0$, where the sign follows since $g(\cdot)$ is positive, strictly decreasing, and strictly convex while U is strictly concave.

	c	s	$W(s)$
m	$+$	$-$	$+$
$\bar{R}$	0	$+$	$-$
$\overline{W}$	0	0	$+$

No Smoothing

	c	s	$W(s)$
m	0	0	0
$\bar{R}$	$+$	0	$+$
$\overline{W}$	0	0	$+$

Perfect Smoothing

Table 1: **Comparative-Static Results in Each Region.** Sign of changes in optimal consumption (c), time to exhaust the cake (s), and willpower bequest ($W(s)$), resulting from changes in the return to willpower applied to alternative activites (m), changes in initial cake size ($\overline{R}$), or changes in willpower ($\overline{W}$). Results apply on the interior of each region.

perfect smoothing requires that the cake still be exhausted at T and, given a cake of unchanged size, the consumer would devour it at the same rate and would therefore have the same amount of willpower to bequeath. If someone who would perfectly smooth received marginally more willpower, he would not alter his consumption path or date of exhaustion but would simply use the additional willpower in the alternative activities. If such a person instead began with a larger cake, then he would consume it at a faster rate, exhausting it at an unchanged date ($s = T$), and he would have more willpower left over for the alternative activities.[18] We summarize the comparative-static results for each regime in Table 1 and then turn to the behavioral implications of these results.

2.4.2 Smoothing Consumption May Not Be Optimal

Consumption smoothing is the hallmark of the standard model with stationary utility and no discounting. However, it does not survive the addition of willpower concerns when willpower is needed not only to resist tempting consumption but also to regulate other urges. These urges may include the temptation to slack off at work or school, the urge to participate in risky financial or sexual behaviors, or the urge to express anger or jealousy to co-workers, friends or family. Suppose the return to the alternative activities is high enough, so that the agent is in the no-smoothing regime.[19] If this agent had more willpower, he would use no more to restrain tempting

[18] Agents with very large cakes may consume at a rate very close to $\bar{c}$ for the entire horizon, and thus have nearly all of their initial willpower stock leftover to devote to self-control in alternative activities. The model thus predicts that the very rich will appear especially disciplined in non-consumption activities requiring self-control (maintaining exercise regimes, preparing for performances or tests, etc.) since their greater wealth means that resisting tempting consumption is often unnecessary.

[19] To see which regime is optimal for given parameters m and $\overline{R}$, set $s = T$ in the left hand side of (1). If it is nonnegative, then perfect smoothing is optimal; otherwise no-smoothing is optimal. Notice that, for any cake size, no smoothing is optimal for any m above a critical level.

consumption; instead he would use the additional willpower to restrain urges in other activities. (See the bottom rows in Table 1.) Therefore, if m is high enough, the agent would continue to avoid perfect smoothing of consumption no matter how large is his initial stock of willpower.

2.4.3 Domain-Specific Time Preference

This simple model also shows how, for an agent with limited willpower, time preference may differ sharply by decision domain. The individual may appear willing to postpone gratification in one set of activities and, at the same time, profoundly myopic about choices in another, even though he discounts time at a single rate. Consider the following concrete example. Suppose willpower is used both to regulate consumption and to exert concentrated effort on dull but professionally important tasks at work – tasks which provide only longer-term rewards. If the return (m) to more concentrated effort at work is high enough so that the agent is in the no smoothing regime, an increase in m will result in the agent consuming his tempting cake faster, exhausting it more quickly, and therefore having more willpower for work related activities. (See Figure 1 and the top rows in Table 1.) Hence, the agent would appear at the same time more myopic when it comes to consumption choices, but more forward-looking when it comes to work.

Short-sighted consumption behavior and diligence at work might also occur if the individual were instead truly myopic but simply enjoyed work. This explanation can be distinguished from the effects of willpower. In our model as the return (m) to additional willpower allocated to work increases, the individual would appear increasingly impatient in his consumption and increasingly disciplined in his regulation of alternative activities. In contrast, an increase in returns to effort at work would not affect short-run time preference for consumption in a model that attributed such domain-specific time preference to tastes for work or to domain-specific rates of time discount. Thus, the limited-willpower model makes distinctive predictions about changes in self-restraint exercised in one activity as a function of the return to exerting self-restraint in another.[20]

2.5 Induced Time Preference

Besides inducing linkages between otherwise unrelated behaviors, the introduction of a willpower constraint may also induce time preference in consumption of the tempting good. To investigate this aspect of behavior, we now permit the rate of willpower depletion to depend on the stock of willpower remaining ($f_W < 0$). Analysis of this case requires a more complete examination of the

[20] In addition, even if a good is not tempting (e.g. deodorants, household cleaners, car tires), expenditures on it can still be affected by willpower considerations if a budget constraint links the non-tempting goods to the tempting ones. For an illustration see Ozdenoren, Salant, and Silverman (2008).

consumer's dynamic optimization problem ($P1$) since consumption need not be constant.[21]

2.5.1 The Time Path of Consumption

To begin, consider the intuitive tradeoffs revealed by the first-order conditions characterizing optimal choice. When consumption is interior ($c(t) \in (0, \bar{c})$), the first-order condition (A1) is:

$$\underbrace{U'(c(t))}_{\text{direct marginal benefit}} + \underbrace{(-f_c)\lambda(t)}_{\text{indirect marginal benefit}} = \underbrace{\alpha(t)}_{\text{marginal cost}} . \tag{3}$$

As equation (3) reflects, consuming at a slightly faster rate at time t generates two marginal benefits and one marginal cost. The *direct* marginal benefit, $U'(c(t))$, is the usual one: the increase in utility at time t from consuming more at t. The marginal cost of consuming at a faster rate at t, $\alpha(t)$, is also standard: the utility foregone because the additional cake consumed at t cannot be consumed at another time.

What is distinctive in (3) is that increasing consumption at t also has an *indirect* marginal benefit ($-f_c\lambda(t) > 0$). This indirect benefit arises from two factors. First, in raising the rate of consumption at t, the agent depletes willpower more slowly at t ($-f_c(W(t), c(t))$). Second, because $f_W < 0$, expanding consumption at t launches a sequence of subsequent willpower savings which cumulatively *magnify* the initial savings by $\lambda(t)$ utils per unit of willpower saved at t.[22] We refer to the first of these factors as "the immediate indirect benefit" and the second as the "magnifier."

We depict the determinants of optimal consumption at t in Figure 2. The marginal benefit schedule (the *sum* of the direct and indirect marginal benefits) is downward sloping since $U(c)$ and $-f(W, c)$ are, by assumption, strictly concave in consumption at time t. The marginal cost curve, $\alpha(t)$, is horizontal since the marginal value of additional cake at an instant is unaffected by the rate of consumption at that instant. Optimal consumption occurs where the two curves intersect.[23]

[21] See the appendix for details of this analysis.

[22] Using the differential equation (A5) describing the evolution of $\lambda(t)$ and the endpoint condition (A8), the indirect marginal benefit in equation (3) can be re-written as the product of two factors:

$$(-f_c)\lambda(t) = -f_c(W, c) \cdot \left[m'(W(s))\, e^{\int_{x=t}^{\min(s,T)} -f_W(W(x), c(x))dx} \right],$$

where for expositional clarity we have chosen as the endpoint the case where some willpower is carried over to regulate the alternative activity ($W(s) > 0$). If none is carried over, $m'(W(s))$ would be replaced by a different constant, but this would not affect our discussion.

[23] The figure can also be used to illustrate the two corner cases. If $c = 0$ is optimal, the marginal cost curve must weakly exceed the marginal benefit curve at $c = 0$: if $c = \bar{c}$ is optimal, the marginal benefit curve must weakly exceed the marginal cost curve at $c = \bar{c}$.

As time elapses, the marginal cost curve remains fixed (see equation A4) because additional cake is equally valuable whenever it arrives; its mere availability, before it is consumed, provides no services and its future arrival can be anticipated by consuming more in advance. Hence, optimal consumption changes over time if and only if the marginal benefit curve shifts. If the marginal benefit curve shifts outward (inward) over time, it is optimal to increase (decrease) consumption. Since the direct component of the marginal benefit schedule ($U'(c)$) does not change with time, shifts in the indirect marginal benefit determine whether it is optimal to increase or decrease consumption over time.

If the stock of remaining reserves has no effect on the rate of willpower depletion ($f_W = 0$), neither factor varies over time and it is optimal to consume at a constant rate while cake remains.[24] We provisionally adopted this assumption to simplify the exposition in subsection 2.4.

The Possibility of Negative Time Preference In the case where regulating consumption depletes willpower more quickly when willpower reserves are smaller ($f_W < 0$), each factor determining the indirect marginal benefit changes over time. The *immediate* indirect benefit of consuming at a faster rate increases as time elapses because, with time, willpower reserves are depleted by use. At lower levels of willpower, consuming marginally faster releases more willpower. However, this effect is magnified less since magnification occurs over a shorter interval of time.

Whether consumption rises or falls over time depends on which these two factors dominates. It is instructive first to identify the boundary case where consumption is constant even though both factors vary over time. For this to occur, the magnifier must decline in percentage terms by exactly as much as the immediate indirect benefit increases in percentage terms. The magnifier declines in percentage terms by $-\dot{\lambda}/\lambda = -f_W(W,c) > 0$. When consumption is constant, the immediate indirect benefit increases in percentage terms by $\frac{f_{cW}\dot{W}}{f_c} = -\frac{f_{cW}f}{f_c} > 0$. When consumption is locally constant these two percentage changes must be equal. When consumption strictly increases (strictly decreases) locally, the percentage increase in the immediate benefit must be strictly larger (strictly smaller) than the percentage reduction in the magnifier.

To summarize, the following proposition characterizes the determinants of the (local) rate of change in consumption.

Proposition 1 *Suppose $c(t) \in (0, \bar{c})$ for some $t \leq s$. Then $\dot{c} \gtreqless 0$ as $\lambda \frac{d}{dW}\frac{d}{dc}\ln f(W,c) \gtreqless 0$.*

Proof. Since $\alpha(t)$ is constant over time, differentiating equation (3) with respect to t implies

[24] To see that $c(t)$ must be constant for $t \leq s$, note again that (A4) implies that $\alpha(t)$ is constant. If $f_W \equiv 0$ then $f_c(W(t), c(t))$ does not vary with $W(t)$ and, given (A5), $\lambda(t)$ is constant. Since we are assuming that $\rho = 0$, the solution to (A1) will be the same for all $t \leq s$.

$[U''(c) - \lambda f_{cc}]\dot{c} + \{\dot{\lambda}(-f_c) + \lambda f_{cW}(-\dot{W})\} = 0$. Because $[U''(c) - \lambda f_{cc}] < 0$, $\dot{c} \gtreqless 0$ if and only if $-f_c\lambda\{\frac{f_{cW}}{-f_c}(-\dot{W}) + \frac{\dot{\lambda}}{\lambda}\} \gtreqless 0$. Then, using (A5) and (A2) to replace $\frac{\dot{\lambda}}{\lambda}$ and $-\dot{W}$, we obtain $\dot{c} \gtreqless 0$ if and only if
$\lambda\{ff_{cW} - f_c f_W\} \gtreqless 0$. The result then follows because $\lambda\{ff_{cW} - f_c f_W\} \gtreqless 0$ as $\lambda \frac{d}{dW}\frac{d}{dc}\ln f(W,c) \gtreqless 0$.
■

This proposition implies that, when interior, consumption is constant in only three circumstances: (1) when willpower is not scarce ($\lambda = 0$), (2) when the rate of willpower depletion is independent of the stock of willpower ($f_W = 0$), or (3) when, despite neither of the preceding circumstances, the indirect marginal benefit of a marginal increase in consumption is constant.

Proposition 1 can also be seen to say that when willpower is scarce, consumption strictly increases whenever the logarithm of the depletion function has a strictly positive cross partial derivative.[25] Knowing this, it is straightforward to construct examples where the consumption path will rise or fall monotonically before jumping to zero. For instance, consider the following function of willpower depletion:

$$f(W,c) = v(c)A + K(W)\Gamma(v(c)),$$

where $v(c) \equiv U(\bar{c}) - U(c)$.[26] We assume $K(\cdot) > 0$, $K'(\cdot) < 0$, $\Gamma'(\cdot) > 0$, $\Gamma''(\cdot) > 0$ and $K(\bar{W})\Gamma'(0) + A > 0$. Note that the last condition is always satisfied when $A \geq 0$, and it will only be violated when A is too negative. These conditions together ensure that $f(W,c)$ satisfies all the assumptions imposed earlier (i.e., $f(W,\bar{c}) = 0$, $f_c < 0$, $f_W \leq 0$, $f_{cc} \geq 0$, $f_{cW} \geq 0$.) Differentiating we obtain

$$\frac{d}{dW}\frac{d}{dc}\ln f(W,c) = \frac{-AU'(c)K'(W)[v(c)\Gamma'(v(c)) - \Gamma(v(c))]}{[v(c)A + K(W)\Gamma(v(c))]^2}$$

Since $\Gamma'' > 0$, $v(c)\Gamma'(v(c)) - \Gamma(v(c)) > 0$. Therefore, the sign of the above expression is the same as the sign of A. Thus, by Proposition 1 whenever $\lambda > 0$ we have

$$\operatorname{sign}\dot{c} = \operatorname{sign}A.$$

This example shows how limited willpower may induce either positive or negative time preference.

Increasing paths of consumption have previously been attributed to the effects of habit or to utility from beliefs (anticipation).[27] Here we see how problems of self-control may also generate

[25] The reader may recognize this condition as logsupermodularity of the depletion function.

[26] In terms of the underlying framework developed in the appendix, the willpower depletion function $h(W,y)$ that leads to this $f(W,c)$ is $h(W,y) = yA + K(W)\Gamma(y)$, where $y = \sup_{x\in B(t)} U(x) - U(c)$, $B(t)$ is the set of available consumption rates and $c \in B(t)$ is the consumption rate chosen at time t.

[27] See Lowenstein and Prelec (1993) for a review.

incentives to "save the best for last" or "get the hard part out of the way first."

While our model is flexible enough to explain both positive and negative time preference revealed in consumption choices, it can be rejected. The problem of predicting the effects of willpower on consumption profiles is analogous to predicting the effects of price on demand or labor supply; in the absence of additional information, income and substitution effects make the net effect ambiguous; but with sufficient information about preferences, unambiguous predictions can be derived. Here, as Proposition 1 shows, with sufficient information about the willpower depletion technology the model makes strong predictions about the qualitative features of the consumption path. Importantly, choice experiments can provide the requisite information. It is straightforward to design, for example, choice experiments that would identify whether the rate of willpower depletion depends on the stock of remaining willpower.[28]

2.5.2 *Almost* Never Smooth Even When Feasible

In section 2.4, we showed that perfect smoothing, the signature prediction of the standard model, may not occur when $f_W = 0$. In this subsection we derive necessary and sufficient conditions for perfect smoothing to occur under more general assumptions ($f_W \leq 0$ and $m(W)$ either zero or strictly increasing and weakly concave). We find that if an agent lacks an alternative use of willpower, a necessary and sufficient condition for him to smooth perfectly is simply that he have enough willpower initially for such smoothing to be feasible. If, instead, willpower has alternative uses, in *almost* all circumstances he will refrain from perfect smoothing no matter how large his initial stock of willpower.

No Alternative Uses of Willpower We consider first the case where willpower has no alternative uses and thus willpower bequeathed to the future has no marginal value. Since our model coincides with the canonical model when the willpower constraint is slack ($\lambda(t) = 0$), perfect smoothing is optimal whenever feasible.

Proposition 2 *Let W_H be the minimum level of initial willpower such that setting $c(t) = \frac{\overline{R}}{T}$ for $t \in [0, T]$ is feasible. Denote the optimal consumption path as $c^*(t)$. Let $\rho = m(W) = 0$. Then if*

[28] Consider a variation on Baumeister's two-phase experiments. In the first phase, subjects would perform a quantifiable willpower-depleting activity (A). In the second phase, the same subjects would perform another, quantifiable willpower-depleting activity (B). The treatment group would perform A before B, and the control B before A. All subjects would be informed ahead of time about the two activities and their order. If the level of willpower stock does not affect its rate of depletion the two groups should, on average, perform as well on a given activity regardless of whether it precedes or follows the other activity. If, instead, depletion is anticipated to be more rapid when reserves are lower, subjects should restrain themselves more on a given activity when it occurs first.

$\overline{W} \geq W_H$ the cake is exhausted, and $c^*(t) = \frac{\overline{R}}{T}$ for $t \in [0,T]$. If instead $\overline{W} < W_H$ then both the cake and willpower are exhausted $(R(s) = W(s) = 0)$, and $\lambda(t) > 0$ for all $t \in [0,s]$.

Proof. See appendix. ■

Corollary 1 *When $\rho = m(W) = 0$, $\overline{W} < W_H$, and $c(t) \in (0, \bar{c})$, consumption is strictly increasing (resp. constant, strictly decreasing) if and only if $\frac{d}{dW}\frac{d}{dc}\ln f(W,c)$ is strictly positive (resp. zero, strictly negative).*

Proof. This follows as a consequence of Propositions 2 and 1. ■

Remark 1 *(Procrastination) Corollary 1 describes the path of consumption when it is interior $(c(t) \in (0, \bar{c}))$. As we have seen, however, this description does not preclude intervals of zero consumption. Suppose that the agent must spend a fixed amount of time to complete a project, and the cake represents the remaining hours of leisure. Then he may work hard on the project at the beginning, slack off as time goes by, but then cram for some interval just before the deadline.*

When Willpower Has Alternative Uses Proposition 2 shows that if there is no other use of willpower, perfect smoothing is always optimal when feasible; and thus the standard model is nested in ours. We now move on to investigate when perfect smoothing is optimal if willpower has alternative uses. We assume throughout this subsection that devoting more willpower to controlling alternative urges is always useful and find that perfect smoothing is *almost* never optimal even when the cake is exhausted at T.

To see why the optimality of smoothing depends on the alternative uses of willpower, note that if it is optimal to use some willpower to regulate consumption and the remainder to regulate other urges, then it must be that willpower cannot be advantageously re-allocated between the two uses ($\lambda(s) = m'(W(s))$ by condition (A8)). However, we have assumed that additional willpower devoted to alternative activities is valuable ($m'(\cdot) > 0$). Therefore $\lambda(s) > 0$. Since this shadow value weakly declines over time (condition (A5)) and is strictly positive when consumption ceases, it must also be strictly positive previously. But then our previous result implies that perfect smoothing occurs only if $\frac{d}{dW}\frac{d}{dc}\ln f(W,c) = 0$.

We know from the analysis in subsection 2.4 that even if consumption is flat when positive, perfect smoothing still may not occur. It may be optimal instead to consume at a constant but rapid rate in order to exhaust the cake before T and save more willpower for the alternative activities. Intuitively, this would occur if additional willpower devoted to regulating other urges was sufficiently valuable.

How valuable must willpower be in alternative uses for perfect smoothing to fail even though optimal consumption is flat whenever positive? For perfect smoothing to be optimal it must be that, when $c(t) = \frac{\overline{R}}{T}$ and $s = T$, the marginal return from allocating a bit more willpower to alternative activities, $m'(W(T))$, is weakly less than the marginal value of additional willpower allocated to consumption under perfect smoothing. We show below that this latter marginal value can be bounded above by:

$$m_H(W,c) = \left(U(c) - U'(c)\,c\right) / \left(f(W,c) - f_c(W,c)\,c\right).^{29}$$

Thus, if we denote by $\hat{W}$ the willpower available at T if perfect smoothing has been implemented, and the rate of consumption under perfect smoothing by c_H,then if $m'(\hat{W}) > m_H(\hat{W}, c_H)$, perfect smoothing never occurs.[30]

The following proposition summarizes the necessary conditions derived above and, in addition, provides sufficient conditions for perfect smoothing to be optimal.

Proposition 3 *If additional willpower devoted to alternative activities is valuable ($m'(\cdot) > 0$), and if the agent perfectly smooths consumption then $\frac{d}{dW}\frac{d}{dc}\ln f(W,c) = 0$, for $c = c_H$ and $W \in \left[\hat{W}, \bar{W}\right]$, and $m'(\hat{W}) \leq m_H(\hat{W}, c_H)$. Moreover, if $\frac{d}{dW}\frac{d}{dc}\ln f(W,c) = 0$,* for all *$c \geq c_H > 0$ and $W \in \left[\hat{W}, \bar{W}\right]$ and $m'(\hat{W}) \leq m_H(\hat{W}, c_H)$ then perfect smoothing must occur.*[31]

Proof. We have already proved, in the text, that when $m'(\cdot) > 0$ perfect smoothing requires $\frac{d}{dW}\frac{d}{dc}\ln f(W,c) = 0$, for $c = c_H$ and $W \in \left[\hat{W}, \bar{W}\right]$. Thus, to prove the first statement, it remains only to derive that $m'(\hat{W}) \leq m_H(\hat{W}, c_H)$. If any willpower is used to restrain consumption, optimality of this decision requires that $m'(W(s)) \leq \lambda(s)$. Next we find an upper bound on $\lambda(s)$ using the first-order conditions. Condition (A1) implies that $U'(c(s)) - \lambda(s) f_c \leq \alpha$. Multiplying both sides by $c(s)$ we get,

$$\left[U'(c(s)) - \lambda(s) f_c\right] c(s) \leq \alpha c(s).$$

Condition (A6) requires

$$H(s) = U(c(s)) - \alpha c(s) - \lambda(s) f(W(s), c(s)) \geq 0.$$

Combining these two inequalities we see that

$$\lambda(s) \leq \left(U(c) - U'(c)\,c\right) / \left(f(W,c) - f_c(W,c)\,c\right) = m_H(W,c).$$

---

[29] If no smoothing occurs, then $m_H(W,c) = m'(W(s))$. This generalizes equation (2).

[30] Clearly $\hat{W}$ depends on the initial levels of willpower and cake but we suppress this dependence for simplicity.

[31] Recalling the case studied in Section 2.4, where $f(W,c) = g(c)$ and $m'(\cdot) = m$, this condition reduces to $m \leq (U(c_H) - c_H U'(c_H)) / (g(c_H) - c_H g'(c_h))$.

To prove that the specified conditions are sufficient for perfect smoothing to be optimal, assume the contrary. Since the conditions imply that consumption must be constant when positive, perfect smoothing could only fail because $s < T$. If consumption terminates at $s < T$, then $c = \frac{\bar{R}}{s} > c_H$. Since less restraint will be exercised over a shorter interval $W(s) > \hat{W} > 0$. This in turn has two implications. Since more willpower will be bequeathed to the alternative activities, $m'(\hat{W}) \geq m'(W(s))$. In addition, since $W(s) > 0$, (A8) implies that $\lambda(s) = m'(W(s)) > 0$. By hypothesis, $s < T$. Then, as shown above, the first-order conditions imply $\lambda(s) = m_H(W(s), c)$. Since $m_H(\cdot, \cdot)$ is weakly increasing in the first argument and strictly increasing in the second argument, $m_H(W(s), c) > m_H(\hat{W}, c_H)$. Given the hypothesis that $s < T$ we therefore conclude that $m'(\hat{W}) \geq m'(W(s)) = \lambda(s) = m_H(W(s), c) > m_H(\hat{W}, c_H)$. But $m'(W(s)) > m_H(\hat{W}, c_H)$ violating $m'(\hat{W}) \leq m_H(\hat{W}, c_H)$. ■

Thus Proposition 3 shows that when willpower has alternative uses, the necessary conditions for perfect smoothing are restrictive, but not impossible to meet. Limited willpower *largely* undoes the signature prediction of the standard theory of intertemporal choice. If willpower is depleted by restricting consumption in the right way, and if the marginal value of willpower allocated to other activities is not too high, then the consumer may still smooth perfectly.

3 Comparison with Alternative Models of Self-Control

The preceding sections were largely focused on how the predictions of a willpower model differ from those of a standard model of intertemporal choice. In this section we compare our willpower model with existing models of self-control.

We note first that limited willpower provides complementary explanations for the hallmark predictions of self-control models: a preference for commitment, profound procrastination, and apparent time-inconsistencies. Consider a taste for commitment. As in each of the leading models of self-control,[32] an agent in our model would strictly prefer to have his "cake" or paycheck doled out to him by a savings club. For if the entire amount were available, not spending it would deplete his valuable willpower. Next consider profound procrastination. Like other models of self-control, limited willpower also explains extreme forms of procrastination. Suppose, for example, an agent must spend a fixed amount of time on an assignment before a certain deadline, but can allocate the remaining leisure time optimally.[33] Then if the agent is willpower constrained, he may enjoy

[32] Benabou and Tirole (2004) is an exception, as in that model commitments may prevent the agent from obtaining valuable information about his willpower.

[33] This is the setup Fischer (2001) used to investigate procrastination but we have introduced a willpower depletion constraint. Without it, Fischer finds that explaining procrastination with a standard model requires an unrealistic

leisure early in the program, and later work non-stop until the deadline.

Depletable willpower is also consistent with apparent intertemporal preference reversals. When asked to choose between a smaller reward that will arrive immediately and a larger one that would arrive after some delay, an agent with limited willpower may choose the former. The willpower cost of resisting the immediate, smaller reward may outweigh the utility gain from receiving the later, larger reward. However, if asked to choose now one of these same two options set off in the temporal distance, the consumer may choose the later, larger prize. The choice now of a prize to be delivered in the future allows the consumer to commit irreversibly to an option. He then prefers the later, larger prize since willpower ceases to be required to resist the earlier, smaller prize. In this way a willpower-constrained agent may appear to behave in a time-inconsistent manner.

While it offers complementary explanations for a taste for commitment, preference reversals and profound procrastination, the willpower model also has implications that distinguish it from existing models of self-control.

A distinguishing feature of the limited-willpower model is the linkages that it generates between otherwise unrelated choices. We saw the implications of these linkages in the "no smoothing" results and the domain-specific time preference. Perhaps most basically, we also saw an implication of these links as consumption of even a non-addictive, but storable good in the past had influence on consumption of that good in the future. Thus, different from models of hyperbolic discounting (Laibson 1997, O'Donoghue and Rabin, 1999), and temptation costs (e.g., Gul and Pesendorfer 2001, 2004), keeping the current choice set (remaining cake and feasible rates of consumption) constant, in our model the history of past choices matters for behavior.

In addition, by linking otherwise unrelated, and non-addictive self-control activities, and by explicitly modeling a willpower reserve, our model can be distinguished from models of cues like Laibson (2001) and Bernheim and Rangel (2004). And even if attention is restricted to the effects of past self-restraint in one activity on future self-restraint in the same activity, there remains a difference. In the cue models, past efforts at self-restraint make it *less* likely that one will enter a hot state and thereby lose all self-control. In our model, by contrast, the past exercise of self-restraint will, at least over the short term, make future self-control less likely.

Benabou and Tirole (2004) offer a model of willpower where agents are quasi-hyperbolic discounters, but have limited knowledge of the degree of their present-bias (willpower for Benabou and Tirole). In the moment of greatest temptation agents know their willpower, but later can only infer

subjective rate of time discount. Moreover, with standard time discounting, such procrastination is not something to be avoided; even if they could, agents would not seek commitments to better smooth their work effort. Our model can generate procrastination (zero consumption of leisure for the final phase of the planning horizon) for any discount rate, and agents in our model would, if possible, seek commitments to avoid it.

their preferences from previous experience. To build self-reputation and show their future selves that they are not impatient, agents apply self discipline. The mechanism through which willpower works in their model is thus quite different from ours, and would in general lead to different predictions. One difference is in how various self-control activities would be linked. In their model, an agent who successfully uses self-restraint in one domain would become increasingly confident in his willpower and, thus, more likely to use self-restraint in another domain. In our model, the opposite is true, an agent who is depleted from using willpower in one situation will be less likely to use self-restraint in another seemingly unrelated situation.

Loewenstein and O'Donoghue (2005) and Fudenberg and Levine (2006) also have related models of costly self-control. In Loewenstein and O'Donoghue's framework the past exercise of self-restraint affects the direct cost of self-control in the future. Their model is quite general and they apply it both to intertemporal choice and to static choice under uncertainty. We view our paper as complementing theirs with a further examination of the implications of limited willpower for intertemporal choice and with greater emphasis on the allocation of willpower across competing demands for self-restraint. Fudenberg and Levine (2006) model a game between a long-run self and a sequence of short-run (myopic) selves. The short-run selves are both the "doer" selves (they "physically" make the choices) and the conduits of flow utility to the long-run self. At a cost, the long-run self may alter the preferences of the short-run self and thereby exercise self-control. By allowing cognitive loads to affect the cost of a given act of self-control, Fudenberg and Levine (2006) can accommodate the influence of such loads on the willingness to exercise self-control within a period. They also discuss how within period linkages between activities requiring self-control may emerge if self-control costs depend on multiple activities in a non-linear way. Our model is different in that it predicts longer-term effects, through the willpower constraint, of both shocks to willpower stocks and anticipated, alternative needs for self-restraint.[34]

In addition, the willpower model's accommodation of increasing paths of consumption distinguishes it from the bulk of existing models of self-control in which agents are tempted to overconsume.[35] Negative time preference is usually viewed as unrelated to self-control and is often attributed to utility from beliefs (anticipation). See, e.g., Loewenstein (1987). Models of anticipatory utility *can* accommodate increasing consumption paths, but cannot by themselves explain a taste for commitment, or profound procrastination, or links between seemingly unrelated acts

[34] There is also a conceptional difference between our approach and that taken by both Loewenstein and O'Donoghue (2005) and Fudenberg and Levine (2006) in that we model self-control costs as opportunity, rather than direct costs. See the end of this section for more on this issue.

[35] The exception is Benabou and Tirole (2004). A complete discussion of this issue is left to the online appendix at www.umich.edu/~dansilv/research.

of self-regulation. As we have shown, optimal willpower management can explain all of these phenomenon while, at the same time, inducing a preference for "saving the best for last."

Our analysis shows how willpower concerns may induce a preference for increasing consumption *in the absence of commitment.* Our willpower model predicts perfect smoothing of consumption if the consumer can commit to that path at any time prior to the moment consumption begins. The importance of commitment thus permits a simple way to distinguish the willpower motive from anticipatory utility in contributing to tastes for increasing consumption. To the extent that such tastes are revealed under commitment, it points to anticipatory utility from beliefs. To the extent that such tastes are *enhanced* by the absence of commitment, it points to willpower concerns.

Finally, we conclude this section by distinguishing our treatment of self-control as an endogenous shadow cost from the treatment in the previous literature as an exogenous direct cost.[36] Presumably there are circumstances in which the classical model predicts well. It is, therefore, an attractive feature of any new model supplanting a classical one that it identifies the circumstances in which the classical results arise endogenously. If willpower depletion is appended as a constraint, there will be circumstances in which perfect smoothing will be optimal and consumption behavior predicted by the willpower model will be observationally equivalent to the one predicted by the classical model.

4 Self-Control Builds Willpower Like a Muscle

Common intuition suggests that exercising self-control is like exercising one's muscles: after a short while fatigue sets in (willpower depletion) and one's self-control resolve weakens; but such exercise has a longer-run payoff since it builds up one's ability to resist temptation in the future (willpower renewal). Experiments in psychology seem to confirm this phenomenon. Moreover, the evidence suggests that the willpower built up in one domain can then be used to advantage in another domain (Muraven *et al.,* 1999 and Muraven and Baumeister, 2000).[37]

[36] When the consumer derives indirect utility from willpower remaining after regulating consumption (the bequest function is not constant), our model is formally analogous to one with direct utilty costs – as resisting temptation involves (indirect) disutility. (We thank an anonymous referee for pointing this out.) Note, however, the bequest function is not necessary for willpower to have value in our model; resisting temptation may be costly even if there is no indirect utility from a bequest (see Proposition 2 where $\lambda(t) > 0$). As important, we interpret the bequest function as giving the value of willpower applied to all subsequent activities in which the agent might benefit from self-control. Thus, consistent with the Baumeiester experiments, we would predict that a consumer who exerted greater self-control in the consumption problem would exhibit less in subsequent activities.

[37] In one experiment (Muraven *et al.,* 1999), for example, subjects who participated in two-week self-control drills (regulating moods, improving posture, etc.) later showed significant increases in the length of time they would squeeze a handgrip relative to those who did not participate in the drills.

In this section, we return to the special case of section 2.4 and ask whether it would still be optimal to maintain an unchanging grip on one's consumption of a tempting storable good if the exercise of self-control today augmented one's ability to resist temptation in the future. To incorporate the effects of renewable willpower, we introduce a third state variable, the stock of "muscle," denoted by $M(t)$. In investigating behavior from some date (labelled $t = 0$) onward, we assume that the initial stocks of cake, willpower, and self-control muscle are specified.

We assume that the stock of muscle provides an inflow to the willpower stock which can be drawn on in the future. To capture this, we change the transition equation for willpower (equation A2) to: $\dot{W}(t) = \gamma M(t) - f(W(t), c(t))$. As before, restraining consumption depletes willpower but now the self-control muscle provides a flow of willpower in the opposite direction. Thus, given the same stock of willpower and the same resistance to tempting consumption, willpower depletes more slowly the larger is one's self-control muscle.

The size of one's self-control muscle can be built up by the prior exercise of restraint. To capture this, we assume that the rate at which muscle develops (or deteriorates) is given by $\dot{M} = f(W(t), c(t)) - \sigma M(t)$. Thus, muscle is built through the current exercise of willpower and the existing muscle stock decays exponentially at a rate σ. In this way, even if the agent never exercises any self-control, his muscle stock will decay exponentially but will remain strictly positive. Since we assume here that willpower and muscle are not used in any alternative activity, the shadow price reflecting the value of having additional muscle at time s is zero ($\pi(s) = 0$).

In general, the optimal consumption path in this model shares some qualitative features with that in our earlier model without muscle. First, in both formulations, the cake is entirely consumed. Second, *if willpower has no alternative uses*, perfect smoothing is optimal whenever it is feasible. In particular, for every initial level of muscle there is a willpower level $W_{\tilde{H}}$ (possibly zero)[38] above which the optimal path entails perfect smoothing.

Recall the case investigated in section 2.4. There, the assumption that $f_W = 0$ implied that it was optimal to restrain consumption of the tempting good equally over time until the cake was exhausted. We now show that the opportunity to build up one's willpower muscle may lead instead to increasing paths of consumption. To show this formally, first we write the Hamiltonian for this problem:

$$
\begin{aligned}
H(c(t), R(t), W(t), t, \alpha(t), \lambda(t), \pi(t)) = e^{-\rho t} U(c(t)) - \alpha(t) c(t) + \lambda(t)(\gamma M(t) - f(W(t), c(t))) \\
+ \pi(t)(f(W(t), c(t)) - \sigma M(t)).
\end{aligned}
$$

[38] If the initial muscle level is large enough, the agent will be able to achieve perfect smoothing with $W_{\tilde{H}} = 0$.

Suppose consumption is interior and consider the first-order condition, analogous to equation (A1):

$$\underbrace{U'(c(t))}_{\text{direct marginal benefit}} + \underbrace{(-f_c)(\lambda(t) - \pi(t))}_{\text{indirect marginal benefit}} = \underbrace{\alpha(t)}_{\text{marginal cost}}. \tag{4}$$

As before, equation (4) shows that consuming at a slightly faster rate at time t generates the usual *direct* marginal benefit, $U'(c(t))$ and marginal cost, $\alpha(t)$. As before, both are stationary. Therefore, as before, whether optimal consumption is increasing or decreasing locally at time t depends on whether the indirect marginal benefit is increasing or decreasing over time.

By consuming at a faster rate at t, the agent depletes willpower more slowly at t $(-f_c(W(t), c(t)))$. Having additional willpower at t is worth $\lambda(t)$ utils per unit of additional willpower. If $f_W = 0$, then additional willpower is equally valuable whenever it is acquired. Hence, the value of willpower for future use, $\lambda(t)$, is constant over time (i.e., there is no magnifier effect.)

But while consuming at a faster rate at t has the beneficial indirect effect of freeing up valuable willpower, the net indirect benefit is smaller. For, in relaxing his self-control, the agent slows his muscle building. The loss of muscle at t has current and future consequences because a smaller muscle has a smaller willpower flow. The utility lost by relaxing self-discipline at t is given by $\pi(t)$ utils per unit of willpower released at t.

Unlike additional willpower, which is valuable whenever it is acquired, additional muscle is more valuable early in the program since it can then generate a willpower inflow over a longer time interval; additional muscle at time s is worthless. The *net* indirect marginal benefit of consuming at a marginally faster rate at time t is given by $(-f_c)(\lambda(t) - \pi(t))$ utils. If the value of additional muscle declines monotonically, the indirect benefit of consuming at a faster rate rises over time and hence consumption should rise over time. To see this formally note that:

$$\dot{\pi}(t) = -\gamma\lambda(t) + \sigma\pi(t).$$

Since $f_W = 0$, $\lambda(t)$ is constant. Let's assume that willpower is scarce and let $\lambda(t) = \lambda > 0$. Now we can solve for $\pi(t)$ to obtain:

$$\pi(t) = \lambda\frac{\gamma}{\sigma}\left(1 - e^{\sigma(t-s)}\right), \tag{5}$$

where we made use of the fact that muscle has no value time s $(\pi(s) = 0)$. Since $t \in [0, s]$, $\pi(t)$ is decreasing $(\dot{\pi(t)} < 0)$.

Finally, we show that, if $f_W = 0$, consumption must *increase* over time. To see this take the derivative of equation (4) to obtain:

$$\dot{c} = \frac{\dot{\pi}(-f_c)}{U'' - (\lambda(t) - \pi(t)) f_{cc}}.$$

The numerator is negative since $\dot{\pi} < 0$ and $-f_c > 0$. The denominator is negative since we are assuming $c(t) \in (0, \bar{c})$ and the second-order condition holds since $c(t)$ maximizes the Hamiltonian.

Therefore consumption must increase over time. Relative to the optimal path in the absence of muscle, the ability to build willpower through the exercise of self-control leads the agent to resist temptation strongly at the outset and to relax his restraint over time.[39]

5 Conclusion

This paper has explored the consequences of including in a conventional model of intertemporal choice a cognitive constraint consistent with a large body of experimental evidence. Specifically, we assumed that if an agent restricts his consumption of a tempting good, then exercising self-restraint depletes his finite stock of willpower – a resource useful for regulating urges of all kinds. This willpower constraint captures the common notion, consistent with laboratory experiments, that an individual has a limited, though positive, capacity to control his own impulsive behaviors.

To study how a willpower constraint affects choices over time, we introduced it in the simplest intertemporal model, where an agent decides how to consume a storable good (a cake, paycheck, or stock of leisure time) over a fixed time period. Any willpower left over after regulating intertemporal consumption is used to control urges in other activities.

The model is simple but generates a rich set of implications. A willpower constrained consumer regards seemingly unrelated activities as linked because he uses the same cognitive resource to exercise self control in different activities. As a consequence, prior acts of self-restraint may affect an agent's subsequent conduct because they reduce the stock of willpower he can use to regulate his behavior. The link between seemingly unrelated behaviors also implies that agents with limited willpower may appear to have domain-specific time preference – behaving as if completely myopic with regard to some choices and extremely forward-looking with regard to others. As important, time preference in one domain will change with the returns to self-control in another.

Limited willpower also has implications for the time-path of consumption. We saw that, because he draws on a single cognitive resource to regulate various urges, an agent may never smooth consumption, even if his initial willpower makes it feasible. In addition, an agent in our model may *increase* his consumption over time because exercising self control later, when his stock of willpower is reduced, requires more willpower than exercising the same self control earlier.

We extended the model to investigate what would happen if the current exercise of self control, while immediately depleting willpower, also builds willpower reserves in the future. We found that

[39] When $f_W < 0$, more intricate consumption patterns arise when there is opportunity to build muscle. A more detailed analysis of this case is available in an online appendix. See www.umich.edu/~dansilv/research.

renewable willpower induces a time preference even when consumption would have been constant if willpower had been nonrenewable.

That willpower is scarce and depletable accords with both introspection and experiment. To explore how this constraint affects intertemporal behavior, we appended it to the canonical model of intertemporal choice. To our surprise, we found that the augmented model accounts *both* for prominent anomalies of intertemporal choice that have been the focus of the self-control literature *and* for other anomalies that are often treated as altogether separate phenomena requiring separate models. Future research should clarify experimentally the form of the willpower depletion function. Given that taking account of a willpower constraint is so tractable, it should then be embedded in other economic models where self-control issues arise.

6 Appendix

6.1 Motivating the Functional Form of Willpower Depletion

Here we offer a simple model of willpower depletion to motivate the functional form of f that we assumed in the main text. Let willpower depletion at time t is given by the function $h(W(t), y(t))$, where $W(t)$ is the willpower stock and $y(t)$ is the utility foregone by not consuming at the most tempting feasible rate at time t. More precisely, $y(t) = \sup_{x \in B(t)} U(x) - U(c(t))$, where $B(t)$ denotes the set of feasible rates of consumption at time t and $c(t) \in B(t)$ denotes the actual rate of consumption at time t. We refer to $B(t)$ as the menu of available rates at time t, a menu that may reflect commitments to limit the rate of consumption at t. We make two primary assumptions about willpower depletion function.

Assumption 1 *Willpower depletion is a nonnegative, strictly increasing and convex function of the utility foregone by not consuming at the most tempting feasible rate. If the agent consumes at the most tempting feasible rate then there is no depletion of his willpower. That is,* $h(W(t), y(t))$ *is nonnegative, strictly increasing and convex in* $y(t)$. *Furthermore,* $h(W(t), 0) = 0$ *for all* $W(t) \geq 0$.

It follows from Assumption 1 that, given the menu of available consumption rates $B(t)$, the more the agent restrains his rate of consumption, the faster he depletes his willpower. It also follows from Assumption 1 that if an agent can choose his consumption rate either from a given menu or from a proper subset of that menu, then any consumption rate $c(t)$ available on both menus will involve weakly less depletion of willpower when it is chosen from the smaller one. To see this observe that if $B'(t) \subset B(t)$ then $y'(t) = \sup_{x \in B'(t)} U(x) - U(c(t)) \leq \sup_{x \in B(t)} U(x) - U(c(t)) = y(t)$ for all $c(t) \in B'(t)$ and thus $h(W(t), y'(t)) \leq h(W(t), y(t))$. In the case where these inequalities are strict, the agent will strictly prefer to choose a given consumption rate from the smaller menu whenever willpower has scarcity value. In the extreme, the agent prefers commitment to a menu containing just one rate over choosing that rate from a larger menu.

When no cake remains, Assumption 1 yields one more implication: since the only feasible (and thus the most tempting) rate of consumption is then zero, consuming nothing when no cake remains depletes no willpower.

We impose one other behavioral assumption, the implications of which are seen in section 2.5.

Assumption 2 *The same act of self-restraint results in (weakly) faster depletion of willpower if the agent's reserves of willpower are lower. That is,* $h(W(t), y(t))$ *is decreasing in* $W(t)$.

Besides the two substantive behavioral assumptions listed above, we also assume that $h_{yW} \leq 0$.

As noted in the main text, in a cake-eating problem formulated in continuous time, as long as some cake remains (i.e., $R(t) > 0$), it is feasible to consume at any finite rate. It follows that, so long as cake remains, the utility forgone by restraining the rate of consumption does not depend on the size of the cake. Hence, if cake remains and there are no external restrictions on the menu, willpower depletion at time t is simply $h(W(t), U(\bar{c}) - U(c(t)))$. Using this result we can simplify the depletion function. When cake remains, willpower depletion is a function of only the current rate of consumption and the willpower stock:

$$f(W, c) = h(W, U(\bar{c}) - U(c)).$$

Our assumptions on h and U imply that f is nonnegative and $f(W, c) = 0$ for $c \geq \bar{c}$. As for its derivatives, it is decreasing in W, strictly decreasing and strictly convex for $c \in [0, \bar{c}]$, and $f_{cW} \geq 0$. When no cake remains, Assumption 1 implies simply that willpower depletion is zero.

Note that as the cake shrinks to zero, willpower depletion is strictly positive for zero consumption, since $f(W, 0)$ is a strictly positive constant for all $R > 0$. However, when there is no cake remaining ($R = 0$), willpower depletion is zero for zero consumption.

6.2 Details on the Cake Eating Problem

As noted in the main text, problem ($P1$) is non-standard because of the discontinuity in the willpower depletion process. To circumvent this difficulty, we examine a related "free endpoint" problem ($P2$), which is standard in its formulation and which has the same solution as problem

$(P1)$. The consumer chooses $c(t) \in [0, \bar{c}]$ to maximize:

$$V(0) = \int_0^s e^{-\rho t} U[c(t)]\, dt + e^{-\rho s} m(W(s))$$

$$\begin{aligned} \text{subject to} \quad & \dot{R}(t) = -c(t) \\ & \dot{W}(t) = -f(W(t), c(t)) \\ & R(s) \geq 0,\ W(s) \geq 0 \\ & R(0) = \overline{R} \in (0, \bar{c}T) \\ & W(0) = \overline{W} \geq 0. \end{aligned} \tag{$P2$}$$

In problem $(P2)$, the agent chooses both an optimal consumption path and a date $s \leq T$ beyond which depletion of each stock ceases ($\dot{W}(t) = \dot{R}(t) = 0$ for all $t \in (s, T]$). Problem $(P2)$ therefore allows all of the paths that are feasible in $(P1)$ and evaluates each as $(P1)$ does. However, problem $(P2)$ also allows other feasible paths: those where willpower depletion ceases even though cake remains $(R(s) > 0)$. To establish that the solutions to the two problems are the same, we need to prove that these additional paths are never optimal in $(P2)$. Intuitively, this is obvious since one can always dominate such paths by consuming marginally more cake in the neighborhood of s. This additional consumption is feasible since (by hypothesis) more cake is available; moreover, consuming it would relax the willpower constraint.[40]

Working with problem $(P2)$, then, the Hamiltonian is given by

$$H(c(t), R(t), W(t), \alpha(t), \lambda(t), t) = e^{-\rho t} U(c(t)) - \alpha(t) c(t) - \lambda(t) f(W(t), c(t)) \ ;$$

we will refer to it as $H(t)$ when no confusion arises. The first-order conditions include:

$$c(t) \in \begin{cases} \{0\} \\ (0, \bar{c}) \\ \bar{c} \end{cases} \text{and} \ ,\ e^{-\rho t} U'(c(t)) - \alpha(t) - \lambda(t) f_c \begin{cases} \leq 0 \\ = 0 \\ \geq 0 \end{cases} \tag{A1}$$

$$\dot{W}(t) = -f \tag{A2}$$

$$\dot{R}(t) = -c \tag{A3}$$

$$\dot{\alpha}(t) = 0 \tag{A4}$$

$$\dot{\lambda}(t) = \lambda(t) f_W \tag{A5}$$

$$T - s \geq 0,\ H(s) - \rho e^{-\rho s} m(W(s)) \geq 0 \text{ and c.s.} \tag{A6}$$

$$R(s) \geq 0,\ \alpha(s) \geq 0 \text{ and c.s.} \tag{A7}$$

$$W(s) \geq 0,\ \lambda(s) - m'(W(s)) \geq 0 \text{ and c.s.} \tag{A8}$$

[40] For a proof of the equivalence of P1 and P2, see our online appendix at www.umich.edu/~dansilv/research.

To isolate the effects of the willpower constraint, we assume that there is no discounting ($\rho = 0$).

Proof of Proposition 2: From (A4) $\alpha(t)$ is a constant function, and with a slight abuse of notation we denote this constant as $\alpha \geq 0$. First assume $\overline{W} \geq W_H$. Consider the case where $\lambda(t) = 0$ for all $t \in [0, s]$. Since $U'(\cdot)$ is a strictly positive, stationary function, consumption will be constant. Condition (A1) admits three possibilities: (i) $c(t) = 0$ for all t and $\alpha > 0$; (ii) $c(t) = \bar{c}$ for all t and $\alpha \geq 0$; or (iii) $c(t) = c \in (0, \bar{c})$ and $\alpha > 0$ for all t. Other conditions necessary for an optimal program permit us to eliminate the first two possibilities. From (A6), either $c(s) > 0$ and $s = T$ or $c(s) = 0$ and $s \leq T$. We can rule out (i) since it would imply $R(s) = \bar{R} > 0$ and $\alpha > 0$, a violation of (A7). We can rule out (ii) since $\bar{R} < \bar{c}T$ and $c(t) = \bar{c}$ for $t \in [0, T]$ implies that $R(T) < 0$, another violation of (A7). Hence, possibility (iii) must hold. Since $\alpha > 0$, (A7) implies $R(T) = 0$ and $c(t) = c \in (0, \bar{c}) = \frac{R}{T}$. By definition of W_H, $W(T) \geq 0$. Thus (A8) is satisfied. This proves the first statement in the proposition.

Now assume $\overline{W} < W_H$. Then $\lambda(t) > 0$ for all $t \in [0, s]$, for suppose to the contrary that $\lambda(t) = 0$ for some $t \in [0, s]$. Equation (A5) implies that $\lambda(t)$ is weakly decreasing and can be written as $\lambda(t) = \lambda(0) e^{\int_{n=0}^{t} f_W(W(n), c(n)) dn}$. Since $e^{\int_{n=0}^{t} f_W(W(n), c(n)) dn} > 0$ for all t, $\lambda(t) = 0$ for some $t \in [0, s]$, implies that $\lambda(t) = 0$ for all $t \in [0, s]$. But as seen above the conditions above then imply that $c = \frac{\overline{R}}{T}$ which is infeasible when $\overline{W} < W_H$. Formally, it would result in $W(T) < 0$, violating (A8). So if $\overline{W} < W_H$ then $\lambda(t) > 0$ for all $t \in [0, s]$ and, by (A8), $W(s) = 0$. To satisfy (A1) with $U'(\cdot) > 0$ and $-\lambda f_c > 0$ requires either (i) $\alpha > 0$ or (ii) $\alpha = 0$ but $c(t) = \bar{c}$ for all $t \in [0, s]$. But (ii) cannot occur since no willpower would be used and yet this case requires that the willpower stock be drawn down from a strictly positive level ($\bar{W} > 0$) to $W(s) = 0$. Hence (i) must obtain and since $\alpha > 0$, (A7) requires that the cake is entirely consumed $(R(s) = 0)$. In summary, if $\overline{W} < W_H$, both stocks will be depleted and $\lambda(t) > 0$ for all $t \in [0, s]$, as was to be proved.

Figures

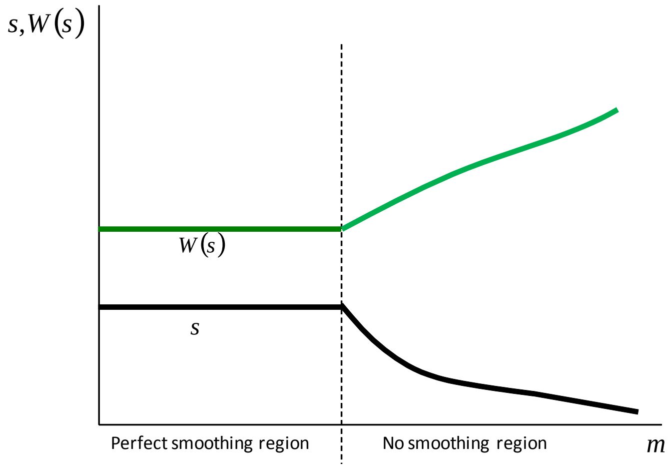

Figure 1: Optimal time (s) to exhaust the cake and optimal willpower bequest $(W(s))$ as a function of the return to willpower applied to alternative activities (m).

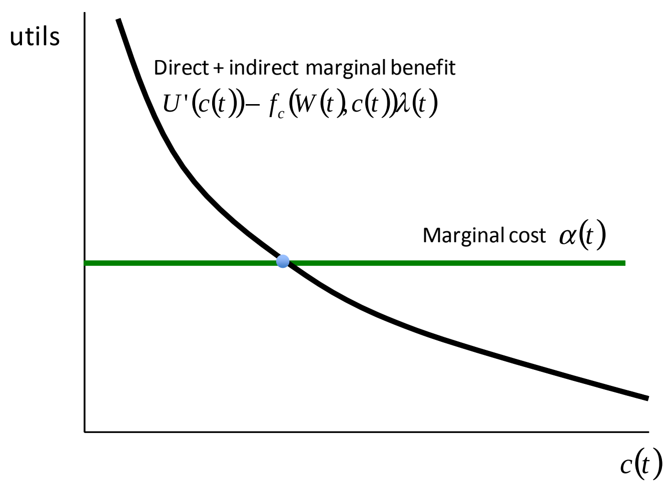

Figure 2: Determination of the optimal rate of consumption at time t

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