THE WEIGHTED LEBESGUE CONSTANT OF LAGRANGE INTERPOLATION FOR EXPONENTIAL WEIGHTS ON $[-1,1]$

S. B. Damelin (Johannesburg) *

Dedicated to Prof. Mario Rosario Occorsio on his 65th birthday

Abstract

We establish uniform estimates for the weighted Lebesgue constant of Lagrange interpolation for a large class of exponential weights on $[-1, 1]$.

We deduce theorems on uniform convergence of weighted Lagrange interpolation together with rates of convergence.

1. Introduction

In this paper, we investigate the Lebesgue function and Lebesgue constant of Lagrange Interpolation for weights $w := \exp(-Q)$ where $Q : (-1,1) \longrightarrow \mathbb{R}$ is even, convex in $(-1,1)$ and grows sufficiently rapidly and smoothly near ± 1. Classical examples of these weights are:

$$w_{0,\alpha}(x) := \exp\left(-(1-x^2)^{-\alpha}\right),\ \alpha > 0 \tag{1.1}$$

$$w_{k,\alpha}(x) := \exp\left(-\exp_k(1-x^2)^{-\alpha}\right),\ \alpha > 0, k \geq 1. \tag{1.2}$$

Here $\exp_k := \exp(\exp(\exp(...)))$ denotes the kth iterated exponential.

We note that for $\alpha \geq \frac{1}{2}$, $w_{0,\alpha}$ violates Szegő's condition,

$$\int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}} dx > -\infty.$$

We are interested in approximating continuous functions $f : (-1,1) \longrightarrow \mathbb{R}$ by weighted polynomials $P_n w$ of degree at most n, $n \geq 1$ in the uniform norm

*The research of this paper was completed while the author was visiting the University of South Florida during the Fall Semester, 1996.

and to this end it will be necessary to impose a decay condition on the given function f. More precisely, we will suppose henceforth that our given f satisfies

$$\lim_{|x|\longrightarrow 1} |fw|(x) = 0. \tag{1.3}$$

The following important notation will be used in the sequel. Let $\mathcal{P}_n$ denote the class of algebraic polynomials of degree at most n and set:

$$E_n[f]_{w,\infty} := \inf_{P\in\mathcal{P}_n} \|(f-P)(x)w(x)\|_{L_\infty[-1,1]}. \tag{1.4}$$

This quantity is the error of best weighted polynomial approximation to f from $\mathcal{P}_n, n\geq 1$ and it is well known [8] that

$$E_n[f]_{w,\infty} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

For our approximation we will use weighted Lagrange interpolation operators and to this end we let

$$\chi_n := \{\xi_{1,n}, \xi_{2,n}, ..., \xi_{n,n}\},\ n\geq 1$$

be an arbitrary set of nodes in $[-1,1]$. The Lagrange interpolation polynomial to f with respect to χ_n is denoted by $L_n[f,\chi_n]$. Thus, if $l_{j,n}(\chi_n) \in \mathcal{P}_{n-1}$, $1\leq j\leq n$ are the fundamental polynomials of Lagrange interpolation at the $\xi_j,\ 1\leq j\leq n$ satisfying for $1\leq j\leq n$ then it is well known that

$$L_n[f,\chi_n](x) := \sum_{j=1}^{n} f(\xi_{j,n}) l_{j,n}(\chi_n)(x) \in \mathcal{P}_{n-1}. \tag{1.5}$$

It is customary to write

$$\begin{aligned} &\big\|w\big(f-L_n[f,\chi_n]\big)\big\|_{L_\infty[-1,1]} \\ &\leq E_{n-1}[f]_{w,\infty}\left(1+\left\|w(x)\sum_{j=1}^{n}\big|l_{j,n}(\chi_n)(x)\big|w^{-1}(\xi_{j,n})\right\|_{L_\infty[-1,1]}\right) \\ &= E_{n-1}[f]_{w,\infty}\Big(1+\big\|\Lambda_n(w,\chi_n)\big\|_{L_\infty[-1,1]}\Big). \end{aligned} \tag{1.6}$$

Here, $\|\Lambda_n(w,\chi_n)\|_{L_\infty[-1,1]}$ is called the Lebesgue constant with respect to the weight w and the set of nodes χ_n and $\lambda_n(w,\chi_n,x)$ is the corresponding Lebesgue function.

Using (1.6), it is well known and easy to see that estimates of the size of the Lebesgue constant and the error $E_{n-1}[f]_{w,\infty}$ yield theorems on uniform convergence of Lagrange interpolation.

Recently, Szabados in his paper [12] investigated the order of the weighted Lebesgue constant for Freud weights on $\mathbb{R}$. His methods were further explored by the author in [1] for Erdős weights on $\mathbb{R}$. These papers laid the ground for our current investigations. We mention that this paper also complements recent work of Mastroianni and Russo in [10] and Vértesi in [14]. The author thanks the above authors for showing him their preprints.

Our class of weights will be assumed to be admissible in the sense of the following definition.

Definition 1.1. Let $w := \exp(-Q)$, where

$$Q : (-1, 1) \longrightarrow \mathbb{R}$$

is even and is twice continuously differentiable in $(-1, 1)$. Assume moreover that

$$Q^{'}(x) \geq 0 \text{ and } Q^{''}(x) \geq 0, \; x \in (0, 1). \tag{1.7}$$

$$\lim_{t \to 1^-} Q(t) = \infty. \tag{1.8}$$

The function,

$$T(t) := 1 + \frac{tQ^{''}(t)}{Q'(t)}, t \in [0, 1)$$

is increasing in $[0, 1)$ with

$$T(0+) > 1 \tag{1.9}$$

and there exist constants $C_j > 0, j = 1, 2$ of t such that

$$C_1 \leq \frac{T(t)}{\frac{Q'(t)}{Q(t)}} \leq C_2, t \to 1.$$

Suppose that for some $A \geq 2$,

$$T(t) \geq \frac{A}{1 - t^2}, t \to 1. \tag{1.10}$$

Then we write $w \in \mathcal{A}$

Remark 1.2. (a) The definition above appeared first in [7].

(b) The principle examples of $w \in \mathcal{A}$ are $w_{0,\alpha}$ and $w_{k,\alpha}$ defined by (1.1) and (1.2) respectively.

(c) The function T measures the rate of decay of the weight w at ± 1. It plays much the same role as its "cousin" for Erdős weights [3] and [5].

We need some notation that will be used repeatedly for a given $w \in \mathcal{A}$.

For real sequences (A_n) and (B_n) with $|B_n| \neq 0$, $n \in \mathcal{N}$, we adopt the following convention throughout:

$$A_n = O(B_n),\ A_n \sim B_n \text{ and } A_n = o\,(B_n)$$

will mean respectively that there exist constants $C_j > 0$, $j = 1, 2, 3$ independent of n such that

$$\frac{A_n}{B_n} \leq C_1, C_2 \leq \frac{A_n}{B_n} \leq C_3 \text{ and } \lim_{n\to\infty} \left|\frac{A_n}{B_n}\right| = 0.$$

Similar notation will be used for functions and sequences of functions. C, C_1, $C_2 ... > 0$ will always denote constants independent of n, x and $P \in \mathcal{P}_n$. The same symbol does not necessarily denote the same constant in different occurrences. We write $C \neq C(L)$ to indicate that C is independent of L.

As is traditional in weighted approximation for fixed weights, we define $a_u = a_u(w)$, $u \geq 1$ as the positive root of the equation

$$u = \frac{2}{\pi}\int_0^1 \frac{a_u t Q'(a_u t)dt}{\sqrt{1-t^2}},\ u > 0. \tag{1.11}$$

This number, which as a real valued function of u is uniquely defined and is strictly increasing in $(0, \infty)$ [6,15], is often called the Mhaskar-Rakhmanov-Saff number. Amongst its important uses is the infinite-finite range inequality

$$\|Pw\|_{L_\infty[-1,1]} = \|Pw\|_{L_\infty[-a_n,a_n]},\ P \in \mathcal{P}_n. \tag{1.12}$$

For the context of this paper, it is important that a_n, $n \geq 1$ depends only on the degree of the polynomial P and not on P itself.

For classical ultraspherical weights, $1 - a_n \sim n^{-2}$ uniformly for $n \geq 1$. In contrast, we have for $w_{0,\alpha}$ and $w_{k,\alpha}$ given by (1.1) and (1.2),

$$1 - a_n \sim n^{-\frac{1}{\alpha}+\frac{1}{2}} \text{ and } 1 - a_n \sim (\log_k n)^{\frac{1}{\alpha}} \text{ where } \log_k := \log(\log(\log(...)))$$

denotes the kth iterated logarithm.

Our interpolation points will be chosen as follows.

Define the orthonormal polynomials

$$p_n(x) := p_n(w^2, x) = \gamma_n x^n + ...,\ \text{with } \gamma_n := \gamma_n(w^2) > 0$$

satisfying

$$\int_{-1}^1 p_n(w^2,x)p_m(w^2,x)w^2(x)dx = \begin{cases} 0 & m \neq n \\ 1 & m = n. \end{cases}$$

It is well known [14] that such polynomials exist and they have n simple zeros. We denote these zeros by $x_{j,n}, 1 \leq j \leq n$ and order them decreasing from left to right as follows:

$$-1 < x_{n,n} < x_{n-1,n} < ... < x_{2,n} < x_{1,n} < 1.$$

We set

$$U_n := \{x_{j,n} : 1 \leq j \leq n\},\ n \geq 1. \tag{1.13}$$

Now fix $y_0 \in [-a_n, a_n]$ so that

$$|p_n w(y_0)| = \|p_n w\|_{L_\infty[-1,1]}$$

and set

$$V_{n+2} := U_n \cup \{-y_0, y_0\}. \tag{1.14}$$

As w is even, we may assume that $y_0 \geq 0$. Moreover, we will show [cf.,(3.18)], that y_0 is very close to the largest zero of p_n which in turn is very "close" to a_n. We will assume henceforth that y_0 is always fixed for the given $w \in \mathcal{A}$.

In order to describe the spacing of our interpolation points, we need to define some special sequences which appeared first in [7].

Let $n \geq 1$ and $L > 0$ be fixed but large enough. We set

$$\delta_n := \big(nT(a_n)\big)^{\frac{-2}{3}},\ n \geq 1. \tag{1.15}$$

(1.16)

$$\Phi_n(x) := \begin{cases} \max\left\{\sqrt{1 - \frac{|x|}{a_n} + 2L\delta_n};\ \frac{1}{T(a_n)\sqrt{1-\frac{|x|}{a_n}+2L\delta_n}}\right\} & , |x| \leq a_n(1 + L\delta_n) \\ \Phi\big(a_n(1 + L\delta_n)\big) & , a_n(1 + L\delta_n) \leq |x| \leq 1. \end{cases}$$

$$\Psi_n(x) := \min\left\{\left(1 - \frac{|x|}{a_n} + 2L\delta_n\right)^{-1};T(a_n)\right\},\ x \in [-1, 1]. \tag{1.17}$$

To see how these sequences relate to our examples consider the weight $w_{0,\alpha}$ as in (1.1). There, [cf.,[7]]

$$\delta_n \sim n^{\frac{2}{3}(-\frac{2\alpha+3}{2\alpha+1})},\ n \to \infty$$

and

$$T(a_n) \sim (1-a_n)^{-1} \sim n^{\frac{1}{\alpha+\frac{1}{2}}},\ n \to \infty.$$

In particular as $\alpha \longrightarrow 0^+, w_{0,\alpha}$ approaches the classical Legendre weight and "roughly"

$$T(a_n) \longrightarrow n^2 \text{ and } \delta_n \longrightarrow n^{-2},\ n \longrightarrow \infty.$$

In general (see (3.8)), for some $\varepsilon > 0$,

$$T(a_u) = O(u^{2-\varepsilon}) \text{ and } \delta_u = O\big(T(a_u)\big)^{\frac{-2}{3}(\frac{3-\varepsilon}{2-\varepsilon})},\ u \to \infty.$$

We finish this section, with a note on the structure of this paper.

This paper is organized as follows:

2. Statement of results

2.1. Bounds for Lebesgue constants and uniform convergence of Lagrange Interpolation for $U_n, n \geq 1$.

We begin our investigation with the sequence of nodes, U_n, $n \geq 1$ defined by (1.13) and prove:

Theorem 2.1. *Let $w \in \mathcal{A}$ and $r \geq 1$. Then, uniformly for $n \geq 1$,*

$$\|\Lambda_n(U_n)\|_{L_\infty[-1,1]} \sim n^{\frac{1}{6}} T(a_n)^{\frac{1}{6}}. \tag{2.1}$$

In particular, for some $\varepsilon > 0$, there exists $C > 0$ independent of n such that

$$\|\Lambda_n(U_n)\|_{L_\infty[-1,1]} \leq C n^{\frac{1}{2}-\frac{\varepsilon}{6}}. \tag{2.2}$$

Remark 2.2. (a) Under the hypotheses of Theorem 2.1, (2.1) together with (1.6), (3.8), [9,Theorem 1.2] and [8,Corollary 1.6] immediately imply the following result on uniform convergence of Lagrange interpolation.

Let $n \geq N_0$. There exist $C_j > 0$ $j = 1, 2$ independent of f and n so that,

$$\begin{aligned} \big\|\big(f - L_n[f,U_n]\big)w\big\|_{L_\infty[-1,1]} &\leq C_1 E_{n-1}[f]_{w,\infty} n^{\frac{1}{6}} T(a_n)^{\frac{1}{6}} \\ &\leq C_2 \omega_{r,\infty}\left(f, w, \frac{1}{n-1}\right) n^{\frac{1}{6}} T(a_n)^{\frac{1}{6}}. \end{aligned} \tag{2.3}$$

Here, [cf.,(8)]

$$\omega_{r,\infty}(f,w,t) : = \left[\sup_{0<h\leq t} \left\| w\Delta^r_{hH_t(x)}\big(f,x,(-1,1)\big) \right\|_{L_\infty(|x|\leq a_{\frac{1}{2t}})} + \inf_{P\in\mathcal{P}_{r-1}} \|(f-P)w\|_{L_\infty(a_{\frac{1}{4t}}\leq |x|\leq 1)} \right] \tag{2.4}$$

is the weighted modulus of continuity of f,

$$H_t(x) := \sqrt{\left|1-\frac{|x|}{a_{\frac{1}{t}}}\right| + T(a_{\frac{1}{t}})^{-\frac{1}{2}}},\ x\in(-1,1) \tag{2.5}$$

and for an interval J and $h>0$,

$$\Delta^r_h(f,x,J) := \begin{cases} \sum_{i=0}^r \binom{r}{i}(-1)^i f(x+\frac{rh}{2}-ih) & ,x\pm\frac{rh}{2}\in J \\ 0 & ,\text{otherwise.}\end{cases}$$

(b) Moreover, if f satisfies $f^{(r)}w\in L_\infty[-1,1]$ then for some $\varepsilon>0$ there exist $C_j>0, j=3,4$ independent of n such that

$$\big\|\big(f-L_n[f,U_n]\big)w\big\|_{L_\infty[-1,1]} \leq C_3 n^{\frac{1}{6}-r} T(a_n)^{\frac{1}{6}} \leq C_4 n^{\frac{1}{2}-\frac{\varepsilon}{6}-r}. \tag{2.6}$$

See (3.8)

Thus, we can ensure uniform convergence for every $r\geq 1$.

We remark that it is possible to replace $n-1$ in (2.3) by n [cf.,(4)] but this is non-trivial as the modulus is not necessarily monotone increasing in t. Moreover, the Jackson estimate in (2.3) can be shown to hold for $n\geq r$. See [4].

Remark 2.3. It is possible to show that the rate in (2.3) cannot be improved in the following sense:

Under the hypotheses of Theorem 2.1, there exists a sequence of continuous functions G_n and a constant $C_5>0$ independent of n such that

$$\lim_{|x|\longrightarrow 1} G_n(x)w(x) = 0$$

and satisfying

$$\big\|\big(G_n-L_n[G_n,U_n]\big)w\big\|_{L_\infty[-1,1]} \geq C_5\omega_{r,\infty}\left(G_n,w,\frac{1}{n-1}\right) n^{\frac{1}{6}} T(a_n)^{\frac{1}{6}}. \tag{2.7}$$

We are not able to remove the dependence on n in (2.7) and pose this as an open problem.

2.2. A better behaving Lebesgue function

We observe that although (2.6) yields uniform convergence for every $r \geq 1$, we can substantially improve our results, by choosing our interpolation points more carefully. The idea goes back to D. L. Berman, J. Egerváry, G. Freud, J. Sántha, P. Szász and P. Turán [13] although exploited for the first time for Freud weights by Szabados [12].

In order to introduce our results, we recall the definition of V_{n+2} defined as in (1.14).

We prove:

Theorem 2.4. *Let $w \in \mathcal{A}$. Then uniformly for $n \geq 1$,*

$$\|\Lambda_{n+2}(V_{n+2})\|_{L_\infty[-1,1]} \sim \log n. \tag{2.8}$$

Remark 2.5. It has recently been shown in [14], that the lower bound in (2.8) holds for an arbitrary set of nodes χ_n.

Remark 2.6.: Uniform convergence of Lagrange interpolation with respect to $V_{n+2}, n \geq 1$.

(a) Under the hypotheses of Theorem 2.1, (3.8), (1.6) [9,Theorem 1.2] and [8,Corollary 1.6] immediately imply the following result on uniform convergence of Lagrange interpolation.

Let $n \geq N_0$. Then there exist $C_j > 0$ $j = 1, 2$ independent of f and n so that,

$$\begin{aligned} \left\| (f - L_{n+2}[f, V_{n+2}])w \right\|_{L_\infty[-1,1]} &\leq C_1 E_{n+1}[f]_{w,\infty} \log n \\ &\leq C_2 \omega_{r,\infty}\left(f, w, \frac{1}{n+1}\right) \log n. \end{aligned} \tag{2.9}$$

(b) Moreover,if f satisfies $f^{(r)}w \in L_\infty \in [-1, 1]$ there exists $C_3 > 0$ independent of n such that

$$\left\| (f - L_{n+2}[f, V_{n+2}])w \right\|_{L_\infty[-1,1]} \leq C_3 \frac{\log n}{n^r}. \tag{2.10}$$

Remark 2.7. It is possible to show that the rate in (2.9) cannot be improved as in the same sense as Remark 2.3.

3. Preliminary lemmas

Throughout this section, we assume that $w \in \mathcal{A}$.

3.1. Orthogonal polynomials on $[-1,1]$, some essential estimates

Our first three lemmas, are a collection of results concerning the behavior of a_n, p_n, the spacing of the zeros of p_n and bounds on $l_{j,n}(w,;)$. Recall L was fixed in (1.16).

Lemma 3.1. *Set*

$$x_{0,n} := x_{1,n} + \delta_n \text{ and } x_{n,n+1} := -x_{0,n}. \tag{3.1}$$

(a) There exists an $A > 0$ independent of n and L such that for $n \geq N_0$,

$$\left| \frac{x_{1,n}}{a_n} - 1 \right| \leq A\delta_n. \tag{3.2}$$

(b) Uniformly for $n \geq N_0$ and $0 \leq j \leq n-1$,

$$x_{j,n} - x_{j+1,n} \sim \frac{1}{n}\Phi_n(x_{j,n}). \tag{3.3}$$

(c) Uniformly for $n \geq N_0$ and $0 < j \leq n-1$,

$$1 - \frac{|x_{j,n}|}{a_n} + L\delta_n \sim 1 - \frac{|x_{j+1,n}|}{a_n} + L\delta_n. \tag{3.4}$$

and

$$\Phi_n(x_{j,n}) \sim \Phi_n(x_{j+1,n}). \tag{3.5}$$

(d) For $n \geq 1$,

$$\sup_{x \in [-1,1]} |p_n w|(x) \left| 1 - \frac{|x|}{a_n} \right|^{\frac{1}{4}} \sim 1 \tag{3.6}$$

and

$$\sup_{x \in [-1,1]} |p_n w|(x) \sim n^{\frac{1}{6}} T(a_n)^{\frac{1}{6}}. \tag{3.7}$$

Proof. (3.2), (3.6) and (3.7) are respectively (1.33), (1.38) and (1.39) in [7]. (3.3) for $1 \leq j \leq n-1$ is (1.35) in [7] and follows for $0 \leq j \leq n$ using (1.15), (1.16) and (3.1). (3.4) follows using (3.3) much as in [7] an (3.5) follows from (3.4). $\square$

Lemma 3.2. *(a) For some $\varepsilon > 0$,*

$$T(a_u) = O(u^{2-\varepsilon}) \text{ and } \delta_u = O\big(T(a_u)\big)^{\frac{-2}{3}\left(\frac{3-\varepsilon}{2-\varepsilon}\right)},\ u \to \infty. \tag{3.8}$$

(b) *Given $0<\alpha<\beta$, we have uniformly for $u\geq 1$,*

$$T(a_{\alpha u})\sim T(a_{\beta u}). \tag{3.9}$$

(c) *Given fixed $r>1$,*

$$\frac{a_{ru}}{a_u}-1\sim T(a_u)^{-1},\ u\in[1,\infty). \tag{3.10}$$

Proof. (3.9) and (3.10) are (3.7) and (3.10) of [7]. Moreover, (3.8) follows from (3.8) of [7] and (1.15). □

The final result in this subsection is a lemma on the fundamental polynomials of Lagrange Interpolation. Its proof can be found in [7].

Lemma 3.3. *(a)* *Uniformly for $n\geq 1$, $1\leq j\leq n$ and $x\in(-1,1)$*

$$|l_{j,n}(U_n)(x)|\sim \phi(x_{j,n})\frac{1}{n}w(x_{j,n})\left(1-\frac{|x_{j,n}|}{a_n}+L\delta_n\right)^{\frac{1}{4}}\left|\frac{p_n(x)}{x-x_{j,n}}\right|. \tag{3.11}$$

(b) *There exists $C>0$ such that uniformly for $n\geq 1$, $1\leq j\leq n$ and $x\in(-1,1)$,*

$$l_{j,n}(U_n)(x)w(x)w(x_{j,n})\leq C. \tag{3.12}$$

(c) *Uniformly for $n\geq 1$ and $1\leq j\leq n$,*

$$\left[n\min\left\{\left(1-\frac{|x_{j,n}|}{a_n}+L\delta_n\right)^{-1}, T(a_n)\right\}\right]^{-1}|p_n^{'}w|(x_{j,n}) \tag{3.13}$$

$$\sim |p_{n-1}w|(x_{j,n})\sim\left(1-\frac{|x_{j,n}|}{a_n}+L\delta_n\right)^{\frac{1}{4}}.$$

(d) *There exists $C>0$ such that uniformly for $n\geq 1$, $1\leq j\leq n$ and for*

$$|x-x_{j,n}|\leq C\frac{\Phi(x_{j,n})}{n},\ x,\ x_{j,n}\in[-a_n,a_n], \tag{3.14}$$

we have

$$|p_n(x)|w(x)\sim n|x-x_{j,n}|\Phi_n^{-1}(x_{j,n})\left(1-\frac{|x_{j,n}|}{a_n}+L\delta_n\right)^{\frac{-1}{4}}. \tag{3.15}$$

3.2. Infinite-finite range inequality and some important spacing results

In this subsection, we present three lemmas which appeared for Freud weights in [12] and Erdős weights in [1]. As the proofs are similar, we omit them and refer the reader to the cited references.

We begin with an infinite-finite range inequality.

Lemma 3.4. *Given $m \in \mathbb{N}$ and $n \geq 1$, we have for every $\{P_k\}_{k=1}^m \in \mathcal{P}_n$*

$$\left\| w \sum_{k=1}^{m} |P_k| \right\|_{L_\infty[-1,1]} = \left\| w \sum_{k=1}^{m} |P_k| \right\|_{L_\infty[-a_n, a_n]} . \tag{3.16}$$

Next we present an important spacing lemma.

Lemma 3.5. *(a) For $n \geq N_0$, we have uniformly for $1 \leq j \leq n$,*

$$|y_0 - |x_{j,n}|| \sim a_n \left(\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + L\delta_n \right). \tag{3.17}$$

(b) Let y_0 be as in (1.14). Then, we have for $n \geq N_0$,

$$a_n(1 - B\delta_n) \leq y_0 \leq a_n \tag{3.18}$$

for some $B > 0$ independent of n and L.

Our final lemma is a bound for the fundamental polynomials of Lagrange Interpolation.

Lemma 3.6. *Let $l_{n+1,n+2}(V_{n+2})$ and $l_{n+2,n+2}(V_{n+2})$ be respectively the fundamental polynomials of degree $\leq n+1$ at the points y_0 and -y_0. Then there exists $C > 0$ such for all $x \in [-1, 1]$,*

$$|l_{n+1,n+2}(V_{n+2})|(x)w(x)w^{-1}(y_0) \leq C \tag{3.19}$$

and

$$|l_{n+2,n+2}(V_{n+2})|(x)w(x)w^{-1}(-y_0) \leq C. \tag{3.20}$$

3.3. Fundamental polynomials revisited

In this subsection, we set

$$\Delta x_{j,n} := x_{j,n} - x_{j+1,n}, \ 1 \leq j \leq n \tag{3.21}$$

and prove

Lemma 3.7. *Let $n \geq N_0, r > 1$ and $|x| \leq a_{rn}$. Then there exist $C_j > 0$, $j = 1, 2$, such that for $1 \leq j \leq n$,*

$$\begin{aligned} &w(x)l_{j,n}(U_n)(x)w^{-1}(x_{j,n}) \\ &\quad \leq C_1 \left(\left| 1 - \frac{|x|}{a_n} \right| + L\delta_n \right)^{-\frac{1}{4}} \left(\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + L\delta_n \right)^{\frac{1}{4}} \frac{\Delta x_{j,n}}{|x - x_{j,n}|}, \end{aligned} \tag{3.22}$$

and

$$(3.23)\ w(x)l_{j,n+2}(V_{n+2})(x)w^{-1}(x_{j,n})$$
$$\leq C_2\left(\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n\right)^{\frac{-3}{4}}\left(\left|1-\frac{|x|}{a_n}\right|+L\delta_n\right)^{\frac{3}{4}}\frac{\Delta x_{j,n}}{|x-x_{j,n}|}.$$

Proof. We begin with the proof of (3.22). By (3.6), (3.7), (3.11) and (3.13)

$$(3.24)\ w(x)l_{j,n}(U_n)(x)w^{-1}(x_{j,n})$$
$$\leq C\left(\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n\right)^{\frac{1}{4}}\left(\left|1-\frac{|x|}{a_n}\right|+L\delta_n\right)^{\frac{-1}{4}}\frac{\Phi_n(x_{j,n})}{n|x-x_{j,n}|}.$$

Then using (3.3), (3.24) becomes

$$w(x)l_{j,n}(U_n)(x)w^{-1}(x_{j,n})$$
$$\leq C_1\left(\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n\right)^{\frac{1}{4}}\left(\left|1-\frac{|x|}{a_n}\right|+L\delta_n\right)^{\frac{-1}{4}}\frac{\Delta x_{j,n}}{|x-x_{j,n}|}$$

as required and so (3.22) is proved. We now proceed with (3.23).

First observe that for $1 \leq j \leq n$,

$$l_{j,n+2}(V_{n+2})(x)=\left(\frac{y_0^2-x^2}{y_0^2-x_{j,n}^2}\right)l_{j,n}(U_n)(x). \tag{3.25}$$

Next, we claim that

$$|y_0-x|\leq C_2a_n\left(\left|1-\frac{|x|}{a_n}\right|+L\delta_n\right). \tag{3.26}$$

We consider two cases:

Case 1: $|x|\leq a_n$. Here,

$$\big|\,y_0-|x|\,\big|\leq Ba_n\delta_n+a_n\left(1-\frac{|x|}{a_n}\right)\leq C_3a_n\left(\left|1-\frac{x|}{a_n}\right|+L\delta_n\right)$$

if L is large enough.

Case 2: $a_n<|x|\leq a_{rn}$. Here, using (3.10),

$$|x|-a_n\leq a_{rn}-a_n\leq C_4a_nT(a_n)^{-1}\leq C_5a_n\left(\left|1-\frac{|x|}{a_n}\right|\right).$$

Thus,

$$\big|\, y_0 - |x| \,\big| \leq |a_n - y_0| + \big|\, a_n - |x| \,\big| \leq C_6 a_n \left(\left| 1 - \frac{|x|}{a_n} \right| + L\delta_n \right)$$

and so (3.26) is established. Then (3.17), (3.22) and (3.25) easily yield (3.23). □

4. The proofs of our upper bounds

In this section we establish our upper bounds for (2.1) and (2.8). Throughout we assume that $w \in \mathcal{A}$, $x \in [-1,1]$ is fixed and $x_{k(x),n}$ is that zero of p_n closest to x.

We need a lemma which appeared in [1] and in a slightly different form in [2] and [12].

Lemma 4.1. *Uniformly for $1 \leq j \leq n$ and $n \geq N_0$,*

$$\sum_{\substack{j=1 \\ j \notin [k(x)+2,k(x)-2]}}^{n} \frac{\Delta x_{j,n}}{|x - x_{j,n}|^{\alpha}} = \left\{ \begin{array}{ll} O(1) & ,0<\alpha<1 \\ O(\log n) & ,\alpha = 1 \\ O\left(\frac{n}{\Phi_n(x)}\right)^{\alpha-1} & ,\alpha>1. \end{array} \right\} \tag{4.1}$$

Proof. This follows much as in [1].

We are now ready to proceed with the proofs of our upper bounds. We begin with

The proof of the upper bound in (2.1).

From (3.22) we have for $1 \leq j \leq n$,

$$w(x) l_{j,n}(U_n)(x) w^{-1}(x_{j,n}) \leq C_1 \left(\frac{\left|1 - \frac{|x_{j,n}|}{a_n}\right| + L\delta_n}{\left|1 - \frac{|x|}{a_n}\right| + L\delta_n} \right)^{\frac{1}{4}} \frac{\Delta x_{j,n}}{|x - x_{j,n}|}. \tag{4.2}$$

Now using (4.2) and the definition of $\lambda_n(U_n, x)$ (see (1.6)), we have,

$$\begin{aligned} \lambda_n(U_n,x) &= \sum_{j=1}^{n} w(x) \left| l_{j,n}(U_n)(x) \right| w^{-1}(x_{j,n}) \\ &\leq \sum_{j \in [k(x)+2,k(x)-2]} w(x) \left| l_{j,n}(U_n)(x) \right| w^{-1}(x_{j,n}) \\ &\quad + C_1 \sum_{j \notin [k(x)+2,k(x)-2]} \left(\frac{\left|1 - \frac{|x_{j,n}|}{a_n}\right| + L\delta_n}{\left|1 - \frac{|x|}{a_n}\right| + L\delta_n} \right)^{\frac{1}{4}} \frac{\Delta x_{j,n}}{|x - x_{j,n}|}. \end{aligned} \tag{4.3}$$

Using (3.16), we may suppose without loss of generality that $|x| \leq a_n$. Then as in [1], using (1.15), (3.12), (4.1) and (4.3), we have

$$
\begin{aligned}
(4.4)\quad \lambda_n(U_n, x) \quad &\leq \quad C_2 \sum_{j \in [k(x)+2, k(x)-2]} 1 \\
&\quad + O\left(\sum_{j \notin [k(x)+2, k(x)-2]} \frac{n^{\frac{1}{6}} T(a_n)^{\frac{1}{6}} \Delta x_{j,n}}{a_n^{\frac{1}{4}} |x - x_{j,n}|^{\frac{3}{4}}} \right) \\
&\quad + O\left(\sum_{j \notin [k(x)+2, k(x)-2]} \frac{\Delta x_{j,n}}{|x - x_{j,n}|} \right) \\
&= \quad O(1) + O(\log n) + O\left(n^{\frac{1}{6}} T(a_n)^{\frac{1}{6}}\right) = O\left(n^{\frac{1}{6}} T(a_n)^{\frac{1}{6}}\right).
\end{aligned}
$$

whence,

$$(4.5) \qquad \|\Lambda_n(U_n)\|_{L_\infty[-1,1]} = O\left(n^{\frac{1}{6}} T(a_n)^{\frac{1}{6}}\right)$$

as required. □

We now present

The proof of our upper bound in (2.8).

We follow the ideas of [12]. Firstly, from (3.23) we have for $1 \leq j \leq n$,

$$(4.6)\; w(x) l_{j,n+2}(V_{n+2})(x) w^{-1}(x_{j,n}) \leq C_1 \left(\frac{\left|1 - \frac{|x_{j,n}|}{a_n}\right| + L\delta_n}{\left|1 - \frac{|x|}{a_n}\right| + L\delta_n} \right)^{\frac{-3}{4}} \frac{\Delta x_{j,n}}{|x - x_{j,n}|}.$$

Using (3.16), we may assume without loss of generality that $|x| \leq a_{n+2}$. Then as in [1], (1.6), (3.2), (3.3), (3.4), (3.12), (3.17), (3.19), (3.20) and (4.6) give

$$(4.7) \qquad \lambda_{n+2}(V_{n+2}, x) = \sum(x) + O(1),$$

where

$$(4.8) \qquad \sum(x) := C \sum_{\substack{j=1 \\ j \notin [k(x)+2, k(x)-2]}}^{n} \left(\frac{\left|1 - \frac{|x_{j,n}|}{a_n}\right| + L\delta_n}{\left|1 - \frac{|x|}{a_n}\right| + L\delta_n} \right)^{\frac{-3}{4}} \frac{\Delta x_{j,n}}{|x - x_{j,n}|}.$$

We now turn to the delicate estimation of $\sum(x)$.

Observe that for $1 \leq j \leq n$,

$$\left(\frac{\left|1-\frac{|x|}{a_n}\right|+L\delta_n}{\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n}\right)^{\frac{3}{4}} \leq 1+\frac{|x-x_{j,n}|^{\frac{3}{4}}}{a_n^{\frac{3}{4}}\left(\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n\right)^{\frac{3}{4}}}. \tag{4.9}$$

Then, using (4.9), we may write

$$\sum(x)=O\left(\sum_{j\in S}\frac{\Delta x_{j,n}}{|x-x_{j,n}|}+\sum_{j\in S}\frac{\Delta x_{j,n}}{a_n^{\frac{3}{4}}|x-x_{j,n}|^{\frac{1}{4}}\left(\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n\right)^{\frac{3}{4}}}\right)$$

where,

$$S=\Big\{j:1\leq j\leq n, j\notin[k(x)+2,k(x)-2]\Big\}.$$

We continue our estimate as

$$=O(\log n)+O\left(\sum_{j\in S}\frac{\Delta x_{j,n}}{|x-x_{j,n}|^{\frac{1}{4}}\left(\left|\,a_n-|x_{j,n}|\,\right|+a_nL\delta_n\right)^{\frac{3}{4}}}\right)$$

whence by (4.1)

$$\begin{aligned} &=O(\log n)+O\left(\sum_{\substack{j\in S\\ |x_{j,n}|\leq a_n(1-\delta_n)}}\frac{\Delta x_{j,n}}{|x-x_{j,n}|^{\frac{1}{4}}\left(a_n-|x_{j,n}|\right)^{\frac{3}{4}}}\right) \\ &\quad +O\left(\sum_{\substack{j\in S\\ |x_{j,n}|>a_n(1-\delta_n)}}\frac{\Delta x_{j,n}n^{\frac{1}{2}}T(a_n)^{\frac{1}{2}}}{|x-x_{j,n}|^{\frac{1}{4}}a_n^{\frac{3}{4}}}\right). \end{aligned} \tag{4.10}$$

Next, using the geometric and arithmetic mean inequality and (4.1) again, we may continue (4.10) as

$$\begin{aligned} \sum(x) \quad = \quad &O(\log n)+O\left(\sum_{\substack{j\in S\\ |x_{j,n}|\leq a_n(1-\delta_n)}}\frac{\Delta x_{j,n}}{|x-x_{j,n}|}\right) \\ &+O\left(\sum_{\substack{j\in S\\ |x_{j,n}|\leq a_n(1-\delta_n)}}\frac{\Delta x_{j,n}}{a_n-|x_{j,n}|}\right) \end{aligned} \tag{4.11}$$

$$+O\left(\frac{n^{\frac{1}{2}}T(a_n)^{\frac{1}{2}}}{a_n^{\frac{3}{4}}}\sum_{\substack{j\in S\\|x_{j,n}|>a_n(1-\delta_n)}}\frac{\Delta x_{j,n}}{|x-x_{j,n}|^{\frac{1}{4}}}\right)$$

$$= O(\log n)+O\left(\sum_{\substack{j\in S\\|x_{j,n}|>a_n(1-\delta_n)}}1\right)$$

where in the last line we used (1.15), (1.16), (3.2) and (3.3).

Now it remains to observe that the spacing (3.3) and (1.16), imply that there exist at most a finite number of j such that $|x_{j,n}|>a_n(1-\delta_n)$. Then (4.11) yields,

$$\sum(x)=O(\log n)+O(1)=O(\log n). \tag{4.12}$$

Combining (4.12) with (4.7) and taking $\sup s$ over $[-1,1]$ yields

$$\|\Lambda_{n+2}(V_{n+2})\|_{L_\infty[-1,1]}=O(\log n) \tag{4.13}$$

as required. □

5. Proof of Theorems 2.1 and 2.4

In this section we present the proof of our lower bound in (2.1). The corresponding bound in (2.8) has been shown in [14] for any system of nodes χ_n. We deduce Theorems 2.1 and 2.4.

We begin with

The Proof of our lower bound in 2.1.

Write

$$\lambda_n(U_n,x)=w(x)\,|p_n(x)|\sum_{j=1}^{n}p_n'(x_{j,n})^{-1}w(x_{j,n})^{-1}\,|x-x_{j,n}|^{-1}\,. \tag{5.1}$$

In particular, (5.1) becomes using (3.10), (3.13) and (3.18)

$$\lambda_n(U_n,y_0)\geq C_1 n^{\frac{-5}{6}}T(a_n)^{\frac{1}{6}}\sum_{0\leq x_{j,n}\leq\frac{a_n}{2}}1. \tag{5.2}$$

Now it remains to observe that the spacing (3.3) and (1.16) imply that there exist $\geq C_3 n$ j such that $x_{j,n} \in [0, \frac{a_n}{2}]$. Then, (5.2) becomes

$$\lambda_n(U_n, y_0) \geq C_4 n^{\frac{1}{6}} T(a_n)^{\frac{1}{6}}$$

so that

$$\|\Lambda_n(w, U_n)\|_{L_\infty[-1,1]} \geq \lambda_n(U_n, y_0) \geq C_5 n^{\frac{1}{6}} T(a_n)^{\frac{1}{6}} \tag{5.3}$$

as required. □

The proofs of Theorem 2.1 and Theorem 2.4. These follow using (4.5), (5.3), (4.13) and the result of [14]. □

Acknowledgments. The author would like to thank G. Mastroianni, J. Szabados and P. Vertesi for many useful discussions and valuable advice with regard to the presentation of this paper.

References

[1] S.B. Damelin, The Lebesgue function and the Lebesgue constant of Lagrange interpolation for Erdős weights, *J. Approx. Theory*, (to appear).

[2] S.B. Damelin and D.S. Lubinsky, Necessary and sufficient conditions for mean convergence of Lagrange interpolation for Erdős Weights, *Canad. Math. J.*, **40** (1996), 710-736.

[3] S.B. Damelin, Converse and smoothness theorems for Erdős weights, *J. Approx. Theory*, (to appear).

[4] S.B. Damelin, Smoothness theorems for Erdős weights II, *J. Approx. Theory*, (to appear).

[5] S B. Damelin and D.S. Lubinsky, Jackson theorems for Erdős weights, *J. Approx. Theory*, (to appear).

[6] G. Freud, Lagrangesche interpolation über die nullstellen der hermiteschen orthogonalpolynome, *Studia Sci. Math. Hungar.*, **4** (1969), 179-190.

[7] A.L. Levin and D.S. Lubinsky, Christoffel functions and orthogonal polynomials for exponential weights on $(-1, 1)$, *Memoirs of the Amer. Math. Soc.*, No 535, **111** (1994).

[8] D.S. Lubinsky, Forward and converse theorems of polynomial approximation for exponential weights on $[-1, 1]$, *J. Approx. Theory*, **91** (1997), 1-47.

[9] D.S. Lubinsky, Forward and converse theorems of polynomial approximation for exponential weights on $[-1,1]$ II, *J. Approx. Theory*, **91** (1997), 48-93.

[10] G. Mastroianni and M.G. Russo, Lagrange interpolation in some weighted uniform spaces, (Manuscript).

[11] D.M. Matjila, Bounds for Lebesgue functions for Freud weights, *J. Approx. Theory*, **2** (1994), 385-401.

[12] J. Szabados, Weighted Lagrange interpolation and Hermite-Fejér interpolation on the real line, *J. of Inequ. and Appl.*, **1** (1997), 99-123.

[13] J. Szabados and P. Vértesi, *Interpolation of Functions*, World Scientific, Singapore, 1991.

[14] P. Vértesi, On the Lebesgue function of weighted Lagrange interpolation II, (Manuscript).

[15] E.B. Saff and V. Totik, *Logarthmic Potentials with External Fields*, Springer-Verlag, Heidelberg (1997).

Department of Mathematics,
University of the Witwatersrand,
PO Wits 2050, Johannesburg,
South Africa.

Email address: Steven@gauss.cam.wits.ac.za