# Weighted polynomial approximation and Hilbert transforms: Their connections to the numerical solution of singular integral equations

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**Abstract**

In this article, we explore the fascinating connection between two seemingly unrelated subjects: Weighted polynomial approximation and weighted Hilbert transforms. We show how together they allow for the rigorous study of stability and good numerical approximation of a large class of singular integral equations defined on the real line. The methods and material presented here are primarily ongoing work of the authors and and their collaborators and were presented by the first author at the Fourth international conference on dynamic systems and applications, Atlanta, 21–24 May 2003.

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## 1 Introduction

This article deals with the fascinating interplay between two seemingly unrelated subjects: Weighted polynomial approximation and Hilbert transforms. Our primary aim is to show how these two subjects have allowed the authors and their collaborators to prove stability theorems and obtain good numerical approximations for a large class of singular integral equations on the line. The exact form of the equations studied is motivated, in part, by concrete applications (see [1, 2, 12, 13] and the references cited therein), and so is of current interest and importance.

Indeed, in what follows, we will investigate the numerical approximation and stability of singular integral equations on the line of the form

$$\mu w^2 f - K[f] = g w^{2+\delta} \tag{1.1}$$

where $w : \mathbb{R} \to (0, \infty)$ is an even, exponential weight of smooth polynomial decay at infinity, $K[\cdot] := H[\cdot w^2]/\pi$ is a weighted Hilbert transform, $g$ is a fixed, real valued function in a weighted locally Lipschitz space of order $0 < \lambda < 1$ and $\delta$ and $\mu$ are finite positive numbers. Our main aim will be to show that for a large class of weights $w$ and given $g$ and $\delta$, there exist finite, positive numbers $\mu$ depending on $w$ and $\lambda$ such that solutions of (1.1) exist globally, are in the same weighted Lipschitz space as $g$ and may be well approximated in the sense that the numerical approximation technique we use is stable and we are able to obtain error bounds.

The problems and methods developed to address this problem were presented at the Fourth international conference on Dynamic Systems and Applications, Atlanta, 21–24 May 2003, by the first author. Our reference list is selective to the work posed in this article; indeed we emphasize that this is not a complete survey article, but merely an attempt to advertise this area of research which continues to attract much interest. Our results here have been made possible because of recent investigations of the authors and their collaborators, dealing with approximations of singular integrals, uniform bounds for weighted Hilbert transforms and pointwise convergence of weighted Lagrange interpolation polynomials and their derivatives.

We refer the interested reader to the references cited and the many cited therein for a detailed account of this exciting topic. In what follows, we outline the structure of this paper. In Section 2, we introduce our class of weights and our function class and discuss weighted Hilbert transforms. In Section 3, we consider the problem of weighted polynomial approximation and show how it can be used to study approximations of singular integrals. Finally, in Section 4, we state our main result on the stability and numerical approximation of singular integral equations.

In what follows, $C$ will denote an absolute positive constant which may take on different values from time to time and will be independent of $x$, $y$, $f$ and $n$. $\Pi_n$ will always denote the class of polynomials of degree at most $n \geq 1$.

# 2 Weighted Hilbert Transforms

In this section, we introduce the idea of a weighted Hilbert transform and show that it is a bounded operator from a space of locally Lipschitz functions of fractional order to itself.

## 2.1 Class of weights

We begin with the definition of a suitable class of weights which is contained in:

**Definition 1** Let $w := \exp(-Q)$ where $Q : \mathbb{R} \to \mathbb{R}$ is continuous and even. We shall call such a weight **admissible**, if it satisfies the following additional conditions:

(a) $Q'$ is continuous in $(0, \infty)$, $Q(0) = 0$ and

$$\lim_{|x| \to \infty} Q(x) = \infty.$$

(b) $Q''$ exists and is positive in $(0, \infty)$.

(c) The function

$$T(x) := \frac{xQ'(x)}{Q(x)},\ x \neq 0$$

is quasi-increasing in $(0, \infty)$ ($T(x) \leq CT(y)$, $x \leq y$) with

$$\beta \leq T(x) \leq C,\ x \in (0, \infty)$$

for some $\beta > 1$.

(d)

$$\frac{Q^{(2)}(x)}{|Q'(x)|} \leq C\frac{|Q'(x)|}{Q(x)},\ x \in \mathbb{R}.$$

Definition 1 defines a general class of weights of smooth polynomial decay at infinity. A typical example of such a weight is given by

$$w(x) := w_\beta(x) := \exp\left(-|x|^\beta\right),\ \beta > 1,\ x \in \mathbb{R}$$

of which the Hermite weight ($\beta = 2$) is a special case. Here $T = \beta$ identically. Such weights are often called *Freud weights* in the literature. The conditions (a-b) are weak smoothness assumptions, whereas conditions (c-d) are regularity conditions. Indeed, it is easy to check that condition (c) forces $Q$ to grow as a polynomial at $\pm\infty$ and (d) ensures that the derivatives of $Q$ do not grow too much faster than $Q$ itself.

We mention that it is not essential for the reader to absorb Definition 1 for the remainder of this paper. Indeed, it is enough to consider the canonical example $w_\beta$ for many of the applications described below.

We refer the interested reader to the sections below and to the book [16], the surveys [3], [9] and the references cited therein for further perspectives and applications of this weight class.

## 2.2 Function Class

Given an admissible weight $w$ and a fixed number $0 < \lambda < 1$, we are now able to define our function class. This is contained in

**Definition 2** Define

$$X = C_\lambda^w := \{f : \mathbb{R} \to \mathbb{R}, \lim_{|x|\to\infty} |fw|(x) = 0, \ fw \text{ locally Lipschitz of order } \lambda\}.$$

Since we will need to approximate in this space, we need a suitable notion of distance and thus we find it convenient to metrize $X$ with a natural norm given by

$$\|f\|_X := \|fw\|_{L_\infty(\mathbb{R})} + L_\lambda^w(f), \ f \in X.$$

Here and throughout, $\|\cdot\|_{L_p(I)}$ denotes the $L_p(1 < p \leq \infty)$ norm on a finite or infinite interval $I$ when well defined, and $L_\lambda^w(f)$, is the smallest constant $D > 0$ (depending on $f$, $\lambda$ and $w$), such that

$$|f(x)w(x) - f(y)w(y)| \leq D|x-y|^\lambda$$

for all $x$ and $y$ sufficiently close in $\mathbb{R}$.

We have:

**Lemma 1** $X$ is a Banach space with respect to the norm $\|\cdot\|_X$.

Henceforth, when we refer to the space $X$, we mean that $X$ is defined with respect to the norm $\|\cdot\|_X$ above, and depends on a fixed and given admissible weight $w$ and constant $0 < \lambda < 1$.

## 2.3 Definition of the Weighted Hilbert Transform

Formally, for measurable $f : \mathbb{R} \to \mathbb{R}$, we define

$$K[f] := \frac{1}{\pi} H[fw^2](x) := \frac{1}{\pi} \lim_{\varepsilon\to 0^+} \int_{|x-t|\geq\varepsilon} \frac{fw^2(t)}{x-t} dt, \ x \in \mathbb{R}$$

to be the *weighted Hilbert transform* of $f$, where the integral above is understood as a Cauchy principal value integral.

It is straightword to check that this integral exists if, for example,

- $fw^2$ is locally Lipschitz of some fractional order and
- $|fw^2|(x) = O\left(|x|^{-\delta}\right)$ as $|x| \to \infty$ for some $\delta > 0$.

## 2.4 Analogue of Privalov's Theorem

We may now state the following result which, for compact intervals and with $w \equiv 1$, is an analogue of Privalov's classic theorem, see [11, Section 14.1].

**Theorem 1** Let $0 < \lambda < 1$ and $w$ be admissible. Then

$$K[\cdot] : X \to X.$$

Moreover, $K[\cdot]$ is a bounded operator.

Theorem 1 and its proof is due to Damelin and Diethelm and is contained in [8]. Important tools in the proof are the following uniform boundedness results, which follow from results of Damelin in [4].

**Theorem 2: Uniform Boundeness I** Let $w$ be admissible, $x \in \mathbb{R}$ and let $\varepsilon$ be any small positive number. Then

$$\begin{aligned}|wK(f)(x)| &\leq \|fw\|_{L_\infty(\mathbb{R})} \\ &\quad + \int_0^\varepsilon \frac{|f(x+y/2)w^2(x+y/2) - f(x-y/2)w^2(x-y/2)|}{y} dy\end{aligned}$$

for all measurable $f : \mathbb{R} \to \mathbb{R}$ for which the left hand side exists and the right hand side is finite. In particular, if $f \in X$ then

$$\|wK[f]\|_{L_\infty(\mathbb{R})} \leq C\|f\|_X.$$

**Theorem 3: Uniform Boundedness II** Let $g : \mathbb{R} \to \mathbb{R}$ be measurable and supported in $[-A, A]$ for some $A > 0$. Let $x \in [-A, A]$ and let $\varepsilon > 0$ be any small and positive number. Set $u_b(x) := (1 + |x|)^b$, $b \in \mathbb{R}$. Then:

$$\left|H[gu_{-1/4}]u_{1/4}\right|(x) \quad \leq \quad A^{1/4}\|g\|_\infty + u_{1/4}(x)\left[\int_{x-\varepsilon}^{x+\varepsilon}\left|\frac{(gu_{-1/4})(t)}{t-x}dt\right|\right].$$

## 2.5 Some History

Indeed, the subject of bounds on Hilbert transforms and singular integrals has a rich history and has been studied by many people. In particular, and in connection with our work, we mention the work of Riesz, Nevai, Xu, Shi, Lubinsky, Rabinowitz, Jha, König, Nielsen, Askey, Waigner, Muckenhoupt, Mikhlin, Prössdorf, Makovoz, De Bonis, Della Vecchia, Criscuolo, Mastroianni, Damelin, Diethelm, Jung and Kwon. See [3, 5, 6, 7, 8, 9, 14, 17] and the many references cited therein. In particular, as natural analogues of Theorems 2 and 3, we mention the following classical results of Riesz and Muckenhoupt.

**Theorem 4** Let $1 < p < \infty$ and $u : \mathbb{R} \to [0, \infty)$ be measurable. Then

$$\int_{\mathbb{R}} |H[f](x)|^p u(x)dx \leq C\int_{\mathbb{R}} |f(x)|^p u(x)dx$$

for all measurable $f : \mathbb{R} \to \mathbb{R}$ for which the left hand side exists and the right hand side is finite if and only if there exists $C_1 > 0$ such that for every interval $I$,

$$\left[\frac{1}{|I|}\int_I u(x)dx\right]\left[\frac{1}{|I|}\int_I u(x)^{\frac{1}{1-p}}dx\right]^{p-1} \leq C_1.$$

**Theorem 5** Let $b < 1 - 1/p$, $B > -1/p$, $b \leq B$ and $1 < p < \infty$. Then we have

$$\|H[f](1 + |x|)^b\|_{L_p(\mathbb{R})} \leq C\|f(1 + |x|)^B\|_{L_p(\mathbb{R})}$$

for all measurable $f : \mathbb{R} \to \mathbb{R}$ for which the left hand side exists and the right hand side is finite.

# 3 Weighted Polynomial Approximation and Approximation of Singular Integrals

## 3.1 The Possibility of Weighted Polynomial Approximation

An important step in the proof of Theorem 1 in Section 2 is to establish good approximations for the singular integrals defined by the weighted Hilbert transform. These were investigated in detail by Damelin and Diethelm in their papers [6, 7]. Apart from the problem of the singularity of the integrand, a natural problem arises in that the continuous functions $f : \mathbb{R} \to \mathbb{R}$ studied are defined over the whole real line and indeed may be allowed to become unbounded at $\pm\infty$. If we wish to approximate by polynomials, which themselves are unbounded at $\pm\infty$, this problem leads naturally to the idea of weighted polynomial approximation: Let $w$ be an admissible weight. Then if

$$\|Pw\|_{L_\infty(\mathbb{R})} < \infty$$

for every polynomial $P$, we have that

$$\|x^n w\|_{L_\infty(\mathbb{R})} < \infty,\ n = 0, 1, 2, \ldots$$

which forces the condition

$$\lim_{|x|\longrightarrow\infty} x^n w(x) = 0, \; n = 0, 1, 2, \ldots.$$

Since we are approximating, it is thus natural to assume that our continuous function $f$ satisfies

$$|fw|(x) \to 0, \; |x| \to \infty.$$

## 3.2 The Possibility of Weighted Approximation on the Real Line. Bernstein's Problem and its Solution.

The following is folklore and may be viewed as an analogue of the classical Weierstrass's density theorem for continuous functions on compact intervals, see [9, 18] and the references cited therein.

**Bernstein's Problem (1910/1911)**

Let

$$W := \exp(-Q)$$

where

$$Q : \mathbb{R} \longrightarrow \mathbb{R}$$

is even and $Q(e^x)$ is convex in $(0, \infty)$. Is it true that for every continuous $f : \mathbb{R} \longrightarrow \mathbb{R}$ with

$$\lim_{|x|\to\infty} |fW|(x) = 0$$

and for all $\varepsilon > 0$, there exists a polynomial $P$ such that

$$\|(f - P)\, w\|_{L_\infty(\mathbb{R})} < \varepsilon?$$

The answer to this question depends on $Q$. Indeed it is precisely known for which choices of $Q$ we may expect a positive answer. We have the following result which in its present form should be attributed to Carleman, Dzhrbashyan, Akhiezer and Babenko.

**Theorem 6:** Let

$$W := \exp(-Q)$$

where

$$Q : \mathbb{R} \longrightarrow \mathbb{R}$$

is even and $Q(e^x)$ is convex in $(0, \infty)$. Then the following are equivalent:

(a) For every continuous $f : \mathbb{R} \longrightarrow \mathbb{R}$ with

$$\lim_{|x|\to\infty} |fW|(x) = 0$$

and for every $\varepsilon > 0$, there exists a polynomial $P$ such that

$$\|(f - P)W\|_{L_\infty(\mathbb{R})} < \varepsilon.$$

(b)

$$\int_{\mathbb{R}} \frac{Q(x)}{1 + x^2} dx = \infty.$$

### 3.3 Error Bounds for Numerical Quadratures

Notice that the condition (b) above forces $Q$ to grow as fast as a polynomial at infinity. In particular, density holds for admissible weights $w$ with the weight $w_\beta$, a typical example. Armed with the above, Damelin and Diethelm began a program in [6, 7, 8] to discretize the equations

$$\mu w^2(x)f(x) - K[f](x) = g(x)w^{2+\delta}(x), \quad x \in \mathbb{R}, \tag{3.1}$$

given in (1.1). To do this, we define a sequence of approximating functions $(f_n)_{n\geq 1}$, $f_n : \mathbb{R} \to \mathbb{R}$ by

$$\mu w^2(x)f_n(x) - K[L_n(f)](x) = g(x)w^{2+\delta}(x),\ x \in \mathbb{R} \tag{3.2}$$

where $L_n[f]$ is a suitable interpolation polynomial which interpolates $f$ at any interpolatory array of $n \geq 1$ points. For convenience, we shall set $K_n[\cdot] := H[L_n[\cdot]w^2]$, $n \geq 1$ and denote by $L(X, X)$ the space of all bounded operators from $X$ to $X$.

We recall that Theorem 1 says that $K \in L(X, X)$.

We now numerically approximate the solutions of (3.1) by proving a stability result and error bounds. For this, we will need to interpolate.

**Interpolation array choice**: Define orthonormal polynomials $p_n(w^2)$ of full degree $n$ by

$$\int_{\mathbb{R}} p_k(w^2)p_n(w^2)w^2 = \delta_{kn}.$$

and order their $n$ simple zeroes by

$$x_{n,n}(w^2) < x_{n-1,n}(w^2) < \cdots < x_{2,n}(w^2) < x_{1,n}(w^2).$$

We also define $x_{0,n}$ to be a point with the property that

$$|p_n(x_{0,n})w(x_{0,n})| = \|p_n w\|_{L_\infty(\mathbb{R})}$$

and set $x_{n+1,n} = -x_{0,n}$. Let us set $V_{n+2}$ to be the $n$ zeroes above together with the two additional points $\pm x_{0,n}$, $n \geq 1$ and write $L_{n+2}[f] := L_{n+2}(f, V_{n+2})$ as the unique *Lagrange* interpolating polynomial in $\Pi_{n+1}$ for the function $f$ with nodes $x_{j,n}$, $0 \leq j \leq n+1$, i.e.

$$L_{n+2}[f](x_{j,n}) := L_{n+2}(f, V_{n+2})(x_{j,n}) = f(x_{j,n}),\ 0 \leq j \leq n+1.$$

Finally let

$$E_n[f]_{w,\infty} := \inf_{P \in \mathcal{P}_n} \|(f-P)\,w\|_{L_\infty(\mathbb{R})}$$

denote the error of best weighted polynomial approximation to $f$ from the space $\Pi_n$ of polynomials of degree at most $n$.

The following result should be attributed to both Szabados and Damelin and Diethelm, see [7, 19].

**Theorem 7** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous with

$$\lim_{|x|\to\infty} |fw|(x) = 0.$$

Then uniformly for $f$ and large enough $n$,

$$\|L_{n+2}[f]w\|_{L_\infty(\mathbb{R})} \leq C\|fw\|_{L_\infty(\mathbb{R})} \log n.$$

Now set

$$R_n[\cdot] := K[\cdot] - K_n[\cdot],\ n \geq 1.$$

Theorem 7 allowed Damelin and Diethelm to estimate $R_n[\cdot]$ precisely in [7]. For simplicity, we state their result for $w_\beta$, $\beta > 1$. This result is essentially sharp in error for all admissible weights $w$.

**Theorem 8** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable with

$$\lim_{|x|\to\infty} |fw_\beta|(x) = 0$$

and with $\|f'w_\beta\|_{L_\infty(\mathbb{R})} < \infty$. Then uniformly for $f$ and large enough $n$

$$\|R_n[f]\|_{L_\infty(\mathbb{R})} \leq C\|f'w_\beta\|_{L_\infty(\mathbb{R})} n^{\frac{1}{\beta}-1} \log n.$$

## 4 Integral Equation: Error Bounds and Stability

In this final section, we state our main result from [8].

Let us fix, $w$ admissible, $0 < \lambda < 1$ and $g \in X$. Let $\mu$ be a positive number which will be chosen later and consider the formal integral equation

$$\mu w^2(x)f(x) - K[f](x) = g(x)w^{2+\delta}(x), \quad x \in \mathbb{R}, \tag{4.1}$$

with some $\delta > 0$ where

$$K[\cdot] := \frac{1}{\pi}H[w^2\cdot].$$

Here and in the following, $I$ denotes the identity operator on $X$.

**Theorem 9a** We have $(\mu I - K)^{-1} \in L(X, X)$, the solution $f$ of (4.1) satisfies $f \in X$ and

$$w^2(x)f(x) = \frac{\mu}{\mu^2+1}g(x)w^{2+\delta}(x) + \frac{1}{\mu^2+1}\cdot\frac{1}{\pi}H[w^{2+\delta}g](x), \quad x \in \mathbb{R}. \tag{4.2}$$

We now define the approximation sequence of functions $\{f_n\}$ , $n \geq 1$ by:

$$\mu w^2(x)f_n(x) - K[L_n[f_n]](x) = g(x)w^{2+\delta}(x), \quad x \in \mathbb{R}, \tag{4.3}$$

where, as above, $L_n[h]$ interpolates the function $h$ at the array $\{x_{1,n}, \ldots, x_{n,n}\}$ of $n$ interpolation points specified above.

We have:

**Theorem 9b** Assume that $\beta$ is as in Definition 1 with $\beta > 12/5$ and $f'w \in L_\infty(\mathbb{R})$. Then we have $(\mu I - K_n)^{-1} \in L(X, X)$ for each fixed $n \geq 1$ provided $\mu$ is not an eigenvalue of $K_n$. Moreover, if $\mu$ is not an eigenvalue of $K_n$, for all sufficiently large and fixed $n$ then

$$fw^2 - f_n w^2 = (\mu I - K_n)^{-1}K[f - L_n[f]]w^2. \tag{4.4}$$

Moreover

$$\|(f - f_n)w^2\|_{L_\infty(\mathbb{R})} \leq C\|(\mu I - K_n)^{-1}\|_{X\to X} n^{1/6} E_{n-1}[f]_{w,\infty}. \tag{4.5}$$

**Conclusion** We conclude by mentioning that Theorem 9 is only the beginning of the story. Indeed, see [1, 2, 12, 13], many current applications demand an intensive investigation of integral equations involving other weights of different decays on the real line, on finite intervals and domains in the plane as well as discretizations with other interpolation and approximation operators and arrays. So there is work to be done!

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