Marcinkiewicz-Zygmund inequalities and the Numerical approximation of singular integrals for exponential weights: Methods, Results and Open Problems, Some new, some old.

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Abstract

In this short article, we explore some methods, results and open problems dealing with weighted Marcinkiewicz-Zygmund inequalities as well as the numerical approximation of integrals for exponential weights on the real line and on finite intervals of the line. The problems posed are based primarily on ongoing work of the author and his collaborators and were presented at the November 2001 Oberwolfach workshop: 'Numerical integration and Complexity.'

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1 Introduction

This short article deals with the subject of weighted Marcinkiewicz-Zygmund inequalities as well as the numerical approximation of integrals with decaying or singular kernels for exponential weights on the real line or finite intervals of the line. The problems and methods posed, are based primarily, on ongoing and recent work of the author and his collaborators and were presented at the November 2001 Oberwolfach workshop: 'Numerical integration and Complexity.' We hope that this article, together with other comprehensive although less recent surveys on this subject, (see [29], [30], [42], [46] and the references cited therein), will help to advertise this area of research which continues to attract much interest.

To set the scene for our investigations, let $I := (c, d)$ with $-\infty \leq c < 0 < d \leq \infty$ and

$$\chi_n := \{x_{1,n},\, x_{2,n}, \ldots, x_{n,n}\},\ n \geq 1$$

a triangular array of points on I. By a *weight* w on I, we will mean a positive function on I with $x^n w(x) \in L^1(I)$, $n = 0, 1, ...$

It is folklore that the *Gauss quadrature* formula for w takes the form

$$\sum_{j=1}^{n} \lambda_{j,n} P(x_{j,n}) = \int_I (Pw)(x)dx \tag{1.1}$$

for $P \in \Pi_{2n-1}$, the linear space of algebraic polynomials of degree $\leq 2n-1$, $n \geq 1$. Here $\lambda_{j,n}$, $1 \leq j \leq n$ are the Cotes numbers associated with w, see [19].

More precisely, in this paper we will be interested in a class of *admissible* exponential weights w on I for which the following are archetypal examples:

- symmetric exponential weights on the line of polynomial decay:

$$w_\alpha(x) := \exp\left(-|x|^\alpha\right), \ \alpha > 1, \qquad x \in (-\infty, \infty);$$

- non-symmetric exponential weights on the line with varying rates of polynomial and faster than polynomial decay:

$$w_{k,l,\alpha,\beta}(x) := \exp(-Q_{k,l,\alpha,\beta}(x))$$

  with

$$Q_{k,l,\alpha,\beta}(x) := \begin{cases} \exp_l\left(x^\alpha\right) - \exp_l(0), & x \in [0, \infty), \\ \exp_k\left(|x|^\beta\right) - \exp_k(0), & x \in (-\infty, 0) \end{cases}$$

  where $l, k \geq 1$ and $\alpha, \beta > 1$;

- non-symmetric exponential weights on $(-1, 1)$ with varying rates of decay near ± 1:

$$w^{k,l,\alpha,\beta}(x) := \exp(-Q^{k,l,\alpha,\beta}(x))$$

  with

$$Q^{k,l,\alpha,\beta}(x) := \begin{cases} \exp_l(1-x^2)^{-\alpha} - \exp_l(1), & x \in [0, 1), \\ \exp_k(1-x^2)^{-\beta} - \exp_k(1), & x \in (-1, 0) \end{cases}$$

  where $l, k \geq 1$ and $\alpha, \beta > 1$.

Here and throughout, $\exp_{\rm k}$ and $\log_{\rm k}$ denote kth iterated exponentials and logarithms respectively.

The weights w_α are called even Freud weights (the Hermite weight is just w_2) in the literature and $w_{k,k,\alpha,\beta}$ and $w^{k,k,\alpha,\beta}$ are called even Erdös and even generalized Pollaczek weights respectively. See [3], [11], [37], [25] and the references cited therein. Throughout Π will denote the linear space of algebraic polynomials and C will be an absolute positive constant independent of t, x, n, f and P which will take on different values at different times.

The connection between convergence of Lagrange interpolation and convergence of Gauss quadrature is well known, see [19]. Let $f : \mathbb{R} \to \mathbb{R}$ be in the

$L_2(w)$ closure of Π. Let $L_n[f,\chi_n] := L_n[f]$ be the Lagrange interpolation polynomial of degree $n-1$ to f with respect to χ_n. Then, if χ_n consists of the n simple zeroes of the unique orthonormal polynomials $p_n(w^2)$ with respect to the weight w^2, we have Shohat's extension to infinite intervals, of Erdös and Turan's classical result on L_2 convergence of Lagrange interpolation, see [19], [14] and the references cited therein.

Fact

$$\lim_{n\to\infty}\int_I (f - L_n[f,\chi_n])^2(x)w(x)dx = 0$$

iff for every $P \in \Pi$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\lambda_{j,n}(f-P)^2(x_{j,n}) = \int_I (f-P)^2(x)w(x)dx.$$

In studying weighted mean convergence of Lagrange interpolation for $1 < p < \infty$, $p \neq 2$ and other arrays χ_n on I, one needs extensions of Shohat's result and forward (and converse) quadrature sum inequalities called *Marcinkiewicz-Zygmund* inequalities which are weaker than (1.1). These take the form:

$$\int_I (|Pw|(x))^p dx \leq (\geq) C \tag{1.2}$$
$$\sum_{j=1}^{n}\lambda_{j,n}w^{-2}(x_{j,n})|(Pw)(x_{j,n})|^p, \quad P \in \Pi_{n-1}.$$

1.1 Forward Quadrature

Suppose that we have a *forward quadrature* estimate of the form:

$$\sum_{j=1}^{n}\lambda_{j,n}w^{-2}(x_{j,n})V(x_{j,n})|(Pw)(x_{j,n})|^p \leq C\int_I (|Pw|(x))^p V(x)dx \tag{1.3}$$

for some smooth decaying function V near c and d and an array χ_n. For example suppose that $I = \mathbb{R}$,

$$V(x) = (1+x^2)^{-1/p},\ x \in \mathbb{R}$$

and let us assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies that for every $\varepsilon > 0$ there exists $x_0 \in \mathbb{R}$ so that

$$|fwV^{-1}|(x) \leq \varepsilon,\ |x| > x_0. \tag{1.4}$$

For w_α and $w_{k,k,\alpha,\beta}$, (1.4) is enough to guarantee that f is in the $L_\infty(wV^{-1})$ closure of Π, see [10]. Now let $q := \frac{p}{p-1}$ and observe by duality that we have

$$||L_n[f]w||_p = \sup_{\{g:\ ||gw||_q = 1\}}\int_{\mathbb{R}} L_n[f](x)g(x)w^2(x)dx. \tag{1.5}$$

For such g, let

$$S_n[g] = \sum_{j=0}^{n-1} c_j p_j(w^2)(x),\ c_j = \int_{\mathbb{R}} (gp_j(w^2)w^2)(x)dx$$

be the $(n-1)$th partial sum of the orthonormal expansion of g. Then, as $g - S_n[g]$ is orthogonal to Π_{n-1}, we may apply (1.1), (1.2) (with $p = 1$), the mean boundedness of S_n from L_p to L_p with suitable weights and a general method as outlined in [14], [10],[39], [40], [41] and the references cited therein, to show that given any $\varepsilon > 0$,

$$\left| \int_{\mathbb{R}} (L_n[f]gw^2)(x)dx \right| \leq C\varepsilon ||V||_p \leq \varepsilon. \tag{1.6}$$

Since f is in the $L_\infty(wV^{-1})$ closure of Π and $L_n[P] = P$ for all $P \in \Pi_{n-1}$, (1.5) and (1.6), show that given any $\varepsilon > 0$, we have for every continuous $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.4),

$$\text{limsup}_{n\to\infty} ||(L_n[f] - f)w||_p \leq \varepsilon. \tag{1.7}$$

Letting $\varepsilon \to 0+$, gives weighted mean convergence of $L_n[f]$ to f.

1.2 Converse Quadrature

Suppose that we have a *converse quadrature* estimate of the form:

$$||Pw||_p \leq C \left\{ \sum_{j=1}^{n} \lambda_{j,n} w^{-2}(x_{j,n}) |Pw|^p(x_{j,n}) \right\}^{1/p} \tag{1.8}$$

for $P \in \Pi_{n-1}$ and an array χ_n. Then with I, V and f as above and using (1.8) and (1.4), one may deduce, see [20], [9], [15], [26], [27] and the references cited therein, that for every $\varepsilon > 0$

$$||L_n[f]w||_p \leq C \left\{ \sum_{j=1}^{n} \lambda_{j,n} w^{-2}(x_{j,n}) |fw|^p(x_{j,n}) \right\}^{1/p} \tag{1.9}$$

$$\leq \varepsilon \left\{ \sum_{j=1}^{n} \lambda_{j,n} w^{-2}(x_{j,n}) V^p(x_{j,n}) \right\}^{1/p}.$$

Suppose now that $\frac{\lambda_{j,n} w^{-2}}{|x_{j,n} - x_{j-1,n}|}$ is bounded uniformly from above and below by positive constants uniformly in j and n. This is true for w_α, $w_{k,l,\alpha,\beta}$ and $w^{k,l,\alpha,\beta}$, see [25] and [15]. Then as $\frac{V(t)}{V(x_{j,n})}$ is bounded uniformly for $t \in [x_{j-1,n}, x_{j,n}]$ and for $0 \leq j \leq n$, the last term in (1.9) is bounded by $\varepsilon \left(\int_{\mathbb{R}} V^p(x)dx \right)^{1/p}$ for all n. Again we may deduce weighted mean convergence. We mention that in practice, the choice of V is not arbitrary and depends heavily on the rate of decrease of w, see for example [10], [14] and [28].

Remark 1

Historically, forward and converse quadrature sum estimates were first considered by Marcinkiewicz and Zygmund in [51] for convergence of trigonometric polynomials interpolating a trigonometric 2π periodic function at equidistant nodes in $[0, 2\pi)$. Thereafter they were applied by Askey, Nevai (and his students), Xu and Mastrioanni for Jacobi, generalized Jacobi weights on $(-1, 1)$ and the Hermite weight on $\mathbb{R}$ and by Mastrioanni, Russo, Totik and Erdélyi for doubling weights on finite intervals, see [1], [18], [35], [36], [39], [40], [41], [48], [49] and the references cited therein. We also mention related work of Mhaskar, Narkowich and Ward [38], on spheres in $\mathbb{R}^{d+1}$, $d \geq 1$, work of Lubinsky for general arrays on finite intervals, see [31] and that of Damelin and Rakhmanov for weighted polynomials of the form $P_n w^n$, see [16]. Recently, see [3], [4], [10], [9], [11], [12], [13], [14], [15], [26], [27], [28], [32], [33] and the references cited therein, Lubinsky, Damelin and their collaborators have concentrated on proving (1.3), (1.8) for the even weights w_α, $w_{k,k,\alpha,\beta}$ and $w^{k,k,\alpha,\beta}$ with applications to mean and uniform convergence of Lagrange and Hermite-Fejér of higher order. The main motivation in studying these classes of weights, in these contexts, is so that in the infinite case one can approximate functions that may become unbounded at $\pm\infty$ or in the finite case, functions which may have exponential singularities at ± 1.

Remark 2

We remark that the use of Marcinkiewicz-Zygmund inequalities, as we have used them in $L_p (1 < p < \infty)$, do not yield rates of convergence in general for the weights w_α, $w_{k,k,\alpha,\beta}$ and $w^{k,k,\alpha,\beta}$. For $p = \infty$, one can estimate $||L_n[f]w||_p$ differently for certain arrays and prove estimates of the form

$$||L_n[f]w||_\infty \leq Cg(n)||fw||_\infty \tag{1.10}$$

for suitable $g : (0, \infty) \to \mathbb{R}$ and for all f satisfying (1.4) with a suitable damping factor V. This yields rates of convergence. For example if $w = w_\alpha$ and χ_n consists of the zeroes of $p_n(w^2)$ together with two additional points near the largest zeroes of $p_n(w^2)$, see for example [44] and [3], then $g(n)$ in (1.10) may be taken as $\log n$ and this estimate is essentially best possible for all such f. If we allow a smaller subclass of f than that of [7], it is possible to apply the method of (1.6) and recent results of [5] to remove the factor $\log n$ from (1.10) completely. See [8].

If χ_n consists of the zeroes of $p_n(w^2)$ where $w = w_\alpha$, then we know now from [17], that in order to obtain estimates such as (1.10) in L_p, one needs to approximate by $L_n[f^*]$ where f^* is f truncated near the largest zeroes of $p_n(w^2)$.

1.3 Open Problem 1

The tools required to establish forward and converse quadrature estimates for weights such as w_α, $w_{k,l,\alpha,\beta}$ and $w^{k,l,\alpha,\beta}$ rely on Markov-Bernstein inequalities, large Sieve methods and orthogonal expansions, (see [39],[40], [41], [14], [10], [33] and the references cited therein), duality and Konig's method and its improvements, (see [20], [15], [9], [26], [32] and the references cited therein), Caleson measures, (see [25], [50] and the references cited therein) and estimates for $p_n(w^2)$, its zeroes and $\lambda_{j,n}$,(see [25] and the references cited therein).

Consider the following class of *admissible weights* for which w_α, $w_{k,l,\alpha,\beta}$ and $w^{k,l,\alpha,\beta}$ are prime examples:

Let $w : I \to (0, \infty)$ satisfy the following conditions below:

- $Q := \log(1/w)$ is continuously differentiable and satisfies $Q(0) = 0$;
- Q' is nondecreasing in I with
$$\lim_{x\to c^+} Q(x) = \lim_{x\to d^-} Q(x) = \infty;$$
- The function
$$T(x) := \frac{xQ'(x)}{Q(x)},\ x \neq 0$$
is quasi-increasing in $(0, d)$ (i.e. $T(x) \leq CT(y)$, $0 < x \leq y < d$) and similarly quasi decreasing in $(c, 0)$ with
$$T(x) \geq \lambda > 1,\ x \in I\backslash 0;$$
- There exists $\varepsilon_0 \in (0, 1)$ such that for $y \in I\backslash\{0\}$
$$T(y) \sim T\left(y\left[1 - \frac{\varepsilon_0}{T(y)}\right]\right);$$
- For every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x \in I\backslash\{0\}$,
$$\int_{x-\frac{\delta x}{T(x)}}^{x+\frac{\delta x}{T(x)}} \frac{|Q'(s) - Q'(x)|}{|s - x|^{3/2}} ds \leq \varepsilon |Q'(x)| \sqrt{\frac{T(x)}{|x|}}.$$

For these later admissible weights, sharp bounds and asymptotics for $p_n(w^2)$ and its zeroes were worked out in [25]. Note that we allow w to have different rates of smooth decrease to the left and right of 0 and that Q'' need not exist, instead we require a a local Lipschitz condition on Q'. Admissible weights have since been applied for weighted Hilbert transforms under weaker hypotheses in [2].

In view of the remarks made above, it is now possible to raise the following open problem:

Formulate and prove analogues of (1.3) and (1.8) for admissible weights as well as density criteria such as those in (1.4) and use these tools to study mean and uniform convergence of Lagrange interpolation and orthogonal expansions. Also investigate methods to obtain rates of convergence. Some results related to this problem have been proved in [8] and [24].

1.4 Product integration rules and weighted Hilbert transforms

Let w be admissible and suppose we are given an unspecified smooth function $f : \mathbb{R} \to \mathbb{R}$, with fw decaying at some given smooth rate at c and d. We wish to numerically approximate

$$I[k, f] := \int_I f(t)k(t)dt \tag{1.11}$$

where k is a specified kernel which typically decays near c and d. For example, if $w = w_\alpha$, we may assume that fw decays at $\pm\infty$ as $(1+|x|)^{-\alpha}$ for some positive α and then k would satisfy for some $0 < p < \infty$ and real $\Delta = \Delta(p, \alpha)$

$$||(kw^{-1})(x)(1 + |x|)^{\Delta}||_p < \infty.$$

Alternatively, for the same f, we may set

$$I[k, f](x) := \int_I (fw)(t)k(t, x)dt,\ x \in I \tag{1.12}$$

where $k(t, x) := (x - t)^{-1}$ and the integral is understood as a Cauchy principal valued integral. Note that $I[k, f](x)$ is just the weighted Hilbert transform of f.

Now given an array χ_n, form the rules

$$I_n^*[k, f](x) := \sum_{j=1}^{n} w_{j,n}(x)f(x_{j,n}) \tag{1.13}$$

where the weights $w_{j,n}$ are chosen so that I_n reproduces polynomials $P \in \Pi_{n-1}$, $n \geq 1$. In [27], [6] and [7], conditions for pointwise convergence of $I_n^*[f, k](x)$ to $I[f, k](x)$ for even Freud weights such as w_α were established for arrays χ_n consisting of the zeroes of $p_n(w^2)$ and arrays consisting of the zeroes of $p_n(w^2)$ plus two additional points which dampen the growth of the Lebesgue constant of $L_n[.]$ near the largest zeroes of $p_n(w^2)$. Such arrays have also proved useful in studying mean and uniform convergence of Lagrange and higher Hermite Fejér interpolation, see [10], [3], [4], [26], [9], [11], [13], [28], [32], [44] and the references cited therein.

1.5 Open Problem 2

The tools required to establish error bounds and convergence for the above integration rules for admissible weights rely on Markov-Bernstein inequalities, bounds on weighted Hilbert transforms, bounds on functions of the second kind, Markov-Stieltjes inequalities, Peano kernel theory and estimates for $p_n(w^2)$, its zeroes and $\lambda_{j,n}$. See [6], [7], [2], [8], [27] and the references cited therein, for a detailed account of how these methods are applied for w_α, $w_{k,k,\alpha,\beta}$ and $w^{k,k,\alpha,\beta}$. In view of the discussion above, it is now possible to raise the following open problem: Formulate and prove necessary and sufficient conditions for mean and pointwise convergence of integration rules such as those given by (1.11) for admissible weights. Some results related to this problem have been proved in [8] and [24].

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