Pointwise bounds of orthogonal expansions on the real line via weighted Hilbert transforms.

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Abstract

We study pointwise bounds of orthogonal expansions on the real line for a class of exponential weights of smooth polynomial decay at infinity. As a consequence of our main results, we establish pointwise bounds for weighted Hilbert transforms which are of independent interest.

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1 Introduction and Statement of Results

1.1 Background: Fourier Series/Orthogonal expansions

In this paper, we study pointwise approximation of measurable functions $f : \mathbb{R} \to \mathbb{R}$, by orthogonal expansions on the real line for a class of exponential weights of smooth polynomial decay at infinity. As a consequence of our main results, we establish pointwise bounds for weighted Hilbert transforms which are of independent interest.

To set the scene for our investigations, a *weight* w will be a positive function on $\mathbb{R}$ with $x^n w(x) \in L_1(\mathbb{R}) := L_1$, $n = 0, 1, ...$

Given w as above, we may form an orthonormal/Fourier expansion

$$f \to \sum_{j=0}^{\infty} b_j p_j,\ b_j := \int_{\mathbb{R}} f p_j w^2,\ j \geq 0$$

for any measurable function $f : \mathbb{R} \to \mathbb{R}$ for which

$$\int_{\mathbb{R}} |f(x)x^j| w^2(x) dx < \infty, \qquad j = 0, 1, ... \tag{1.1}$$

Here, see [11], $p_n := p_n(w^2)$, $n \geq 0$ are the unique orthonormal polynomials of degree n satisfying

$$\int_{\mathbb{R}} p_n(x)p_m(x)w^2(x)dx = \delta_{m,n}, \qquad m, n \geq 0 \tag{1.2}$$

where

$$\delta_{m,n} := \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

For $n \geq 1$ and f satisfying (1.1), we set:

$$S_n[f] := \sum_{j=0}^{n-1} b_j p_j. \tag{1.3}$$

Our focus in this paper is to study pointwise bounds for the partial sums given by (1.3) in suitable weighted spaces on the line, which in turn, allows for further investigations concerning pointwise convergence with rates of convergence. For orthonormal expansions on finite intervals, there are many well known mean convergence results starting with those of Riesz and continuing with results on Chebyshev, Jacobi and generalized Jacobi weights. We do not review these aforementioned results here but refer the reader to [18], [20], [29] and the many references cited therein for a comprehensive account of this vast and interesting subject. The first significant results dealing with mean convergence of orthonormal expansions on the line are due to Askey and Wainger for the Hermite weight $w(x) = \exp(-x^2)$, see [1]. Thereafter, followed related results of Muckenhoupt, see [24], [25], Mhaskar and Xu, see [23] and Jha and Lubinsky, see [14].

The subject of pointwise convergence of orthonormal expansions on the line is not cited much in the literature. Indeed, the only results that are known to this author are sufficiency results for pointwise convergence (without rates of convergence). See Remark 3 below. The main idea in this paper is to derive pointwise bounds for weighted Hilbert transforms on the line, which are sharp enough, to allow us to obtain pointwise bounds for orthonormal expansions. These later bounds allow for further investigations concerning pointwise convergence with rates of convergence. Our results on weighted Hilbert transforms are of independent interest and so we have chosen to include their discussion in a separate section which can be read, mostly independently, from the rest of the paper. These later results complement earlier results by the author which appeared in [3]. One of the most interesting discoveries that we make in this paper, is to show that unlike in the case of $L_p(1 < p < \infty)$, the weighted orthonormal operators given by (1.1) are not uniformly bounded operators from weighted L_∞ to weighted L_∞. Indeed, as operators defined from a strict Sobolev subspace of weighted L_∞ to weighted L_∞, they have a norm bounded above by const $\log(n)$ for some absolute positive constant. Both these later facts seem consistent with classical results in Fourier series and with what is known concerning growth of weighted Lagrange constants, see for example [6] and the references cited therein.

The outline of this paper will thus be as follows:

- Section 1.2-1.3: Here we will introduce some needed notation for the remainder of the paper.
- Section 1.4: In this section, we state our main results, namely Theorems 1 and 2 and some important remarks.
- Sections 2-3: In this section, we prove Theorems 1 and 2.
- Section 4: Finally in this last section, we address pointwise bounds for Hilbert Transforms namely Theorems 3 and 4.

1.2 Notation:

Here and throughout, let us agree that henceforth C will denote a positive constant independent of x, y, j, k, n, t, u, f and p_n which will in general take on different values at different times. Moreover, for any two sequences $\{b_n\}$ and $\{c_n\}$ of nonzero real numbers, we shall write $b_n = O(c_n)$ if

$$b_n \leq Cc_n, \qquad n \to \infty$$

and $b_n \sim c_n$ if

$$b_n = O(c_n) \qquad \text{and} \qquad c_n = O(b_n).$$

Given $u, x \in \mathbb{R}$, let

$$\Delta_u(f)(x) := f(x+u) - f(x)$$

denote a difference operator of a measurable $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.1) and for $1 < p \leq \infty$ and such f we shall set, whenever finite:

$$||fw||_p := \begin{cases} \sup_{x\in\mathbb{R}}|fw|(x), & p = \infty \\ \left(\int_{\mathbb{R}} |(fw)(x)|^p dx\right)^{1/p}, & 1 < p < \infty. \end{cases}$$

For a fixed $\alpha \in \mathbb{R}$, we will also define

$$u_\alpha(y) := (1 + |y|)^\alpha,\ y \in \mathbb{R}. \tag{1.4}$$

1.3 A class of admissible weights.

Definition 1 A weight function $w = \exp(-Q) : \mathbb{R} \to (0, \infty)$ will be called *admissible* if each of the following conditions below is satisfied:

(a) $Q := \log(1/w)$ is continuously differentiable, even and satisfies $Q(0) = 0$;

(b) Q' is nondecreasing in $\mathbb{R}$ with

$$\lim_{x\to\infty} Q(x) = \lim_{x\to-\infty} Q(x) = \infty.$$

Assume that there exists $\eta > 1$ with

$$\eta < \frac{xQ'(x)}{Q(x)} \leq C,\ x \in \mathbb{R}\backslash\{0\}. \tag{1.5}$$

(c) For every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x \in \mathbb{R}\backslash\{0\}$,

$$\int_{x-\delta|x|}^{x+\delta|x|} \frac{|Q'(s) - Q'(x)|}{|s-x|^{3/2}} ds \leq \varepsilon |Q'(x)|.$$

Definition 1, as presented first in [16], defines a very general class of even weights for which our results hold. The weak regularity and smoothness conditions on w above are needed, in particular for bounds on p_n, and its zeroes and are used heavily in our proofs, see Sections 1.2 and 1.3 below. Note that Definition 1 does not require Q'' to exist. Instead we require only a mild local Lipshitz 1/2 condition on Q'. (1.5) forces Q to grow as a polynomial at $\pm\infty$. We note, as an easily absorbed example, that

$$w_\alpha(x) := \exp\left(-|x|^\alpha\right), \, \alpha > 1, \qquad x \in \mathbb{R} \tag{1.6}$$

is an admissible weight.

1.3.1 The numbers a_u and A_n

In analyzing admissible weights w^2 and their associated orthogonal polynomials on the line, an important role is played by the scaled endpoints $\pm a_u$ of the support of the equilibrium measure for w^2 and the asymptotic behavior of the recurrence coefficients A_n for $p_n(w^2)$. More precisely:

Given $u > 0$, we define the real number a_u by the positive root of the equation

$$u = \frac{2}{\pi} \int_0^1 \frac{a_u t Q'(a_u t)}{\sqrt{1-t^2}} dt, \, u > 0.$$

It is known, see [16], that a_u is uniquely defined, strictly increasing in $(0, \infty)$ with

$$\lim_{u \to \infty} a_u = \infty$$

and of polynomial growth at ∞. For example, for the weight given by (1.6), it is known that

$$a_u \sim u^{1/\alpha}.$$

For this paper, we will need the important and well established fact, see [16], that

$$||Pw||_\infty = ||Pw||_{\infty[-a_n, a_n]}$$

and

$$||(Pw)(x)||_{\infty(|x| \geq s a_n)} \leq \exp(-Cn)||Pw||_{\infty[-a_n, a_n]} \tag{1.7}$$

for every fixed $s > 1$ and for every polynomial P of degree at most $n \geq 1$.

It is also well known, see [11], that the orthonormal polynomials given by (1.2) admit the representation

$$p_n(x) = \gamma_n x^n + \cdots, \quad \gamma_n := \gamma_n(w^2) > 0$$

and satisfy the three term recurrence

$$xp_n(x) = A_n(w^2)p_{n+1}(x) + A_{n-1}(w^2)p_{n-1}(x),\ x \in \mathbb{R},\ n \geq 0. \tag{1.8}$$

Here, $p_{-1} = 0$, $p_0 = \left(\int w^2(x)\,dx\right)^{-1/2}$ and

$$A_n := A_n(w^2) = \gamma_{n-1}/\gamma_n > 0,\ n \geq 0.$$

For each $n \geq 0$, the numbers A_n are called the recurrence coefficients for p_n and satisfy the following relations, see [16]:

$$\lim_{n\to\infty} \frac{A_n}{a_n} = 1/2 \tag{1.9}$$

and

$$\lim_{n\to\infty} \frac{A_{n+1}}{A_n} = 1. \tag{1.10}$$

1.4 Our Main Results

1.4.1 Orthogonal expansions

We are ready to state our main results. Our first result deals with necessary conditions for pointwise boundedness:

Theorem 1(a): Necessity Let w be an admissible weight and $B, b \in \mathbb{R}$ with $b < B$. Then for

$$\begin{aligned} &\sup_{n\in\Omega}\{|S_n[f]wu_b|(x)\} \\ &\leq C||fwu_B||_\infty \end{aligned} \tag{1.11}$$

to hold for some infinite subsequence $\Omega \subseteq \mathbb{N}$, for all $x \in \mathbb{R}$ and for all measurable f satisfying (1.1) for which the right hand side of (1.11) is finite, it is necessary that

$$B > 0 \tag{1.12}$$

and

$$a_n^{b-\min\{B,1\}} n^{1/6} C_{B,n} = O(1),\ n \geq 1 \tag{1.13}$$

where

$$C_{B,n} := \begin{cases} 1, & B \neq 1 \\ \log n, & B = 1. \end{cases}$$

Theorem 1(b): Sufficiency Let w be an admissible weight, $b \leq 0$ and assume (1.12) and (1.13). Let $x \in \mathbb{R}$ and assume moreover that

$$\frac{A_{n+1}}{A_n} = 1 + O\left(\frac{1}{n}\right), \qquad n \to \infty. \tag{1.14}$$

Then there exists an infinite subsequence $\Omega \subseteq \mathbb{N}$ so that for $n \in \Omega$, (1.12) and (1.13) are sufficient for

$$\begin{aligned} &|(S_n[f]wu_b)(x)| \\ &\leq C\left[\log n||fwu_B||_\infty + \int_{-1}^{1}\left|\frac{[wu_b\Delta_y(f)](x)}{y}dy\right|\right] \end{aligned} \tag{1.15}$$

to hold for all measurable $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.1) for which the right hand side of (1.15) is finite. If in addition to (1.14), we assume

$$\frac{A_n}{a_n} = \frac{1}{2}\left[1 + O\left(\frac{1}{n^{2/3}}\right)\right], \qquad n \to \infty, \tag{1.16}$$

then for $n \geq 1$

$$\begin{aligned} &|(S_n[f]wu_b)(x)| \\ &\leq C\left[\log n||fwu_B||_\infty + \int_{-1}^{1}\left|\frac{[wu_b\Delta_y(f)](x)}{y}dy\right|\right] \end{aligned} \tag{1.17}$$

for all measurable $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.1) for which (1.17) is finite.

Remark 1: L_p analogues of Theorem 1

Analogues of Theorem 1 in $L_p(1 < p < \infty)$ are contained in [1], [24], [25], [23] [14] and [3]. More precisely, in [1], [24] and [25], Askey, Waigner and Muckenhoupt proved an L_p analogue of Theorem 1 for the Hermite weight ((1.6) with $\alpha = 2$). Subsequently in [23], Mhaskar and Xu generalized Muckenhoupts result to a larger class of weights on the line and in [14, Theorem 1.2], Jha and Lubinsky obtained necessary and sufficient conditions for mean convergence. We note that in this later paper, the authors assumed both (1.14) and (1.16) for their sufficiency. The sharpest form of [14, Theorem 1.2] under the weakest conditions on w was proved recently by the author in [3, Theorem 2] and is contained in the following:

Theorem A Let w be admissible, $b, B \in \mathbb{R}$ with $b \leq B$, $1 < p < \infty$ and $C_{B,n}$ as defined by Theorem 1(a). Then for

$$\sup_{n\in\Omega}\{||S_n[f]wu_b||_p\} \leq C||fwu_B||_p \tag{1.18}$$

to hold for some infinite subsequence $\Omega \subseteq \mathbb{N}$ and for all measurable $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.1) such that the right hand side of (1.18) is finite, it is necessary that

$$b < 1 - 1/p,\ B > -1/p. \tag{1.19}$$

In addition, it is necessary that if $p < 4/3$ then
I:

$$a_n^{\max\{b,-1/p\}-B}n^{1/6(4/p-3)}C_{B,n} = O(1)$$

and II: if $p = 4/3$ or 4 then $b < B$, and if $p > 4$ then

$$a_n^{b-\min\{B,1-1/p\}} n^{1/6(1-4/p)} C_{B,n} = O(1).$$

Moreover, if (1.14) holds, then (1.19), I and II are also sufficient for (1.18) to hold. If in addition to (1.14), (1.16) holds, then

$$\sup_{n\geq 1}\{||S_n[f]wu_b||_p\} \leq C||fwu_B||_p.$$

In particular, assuming (1.14), (1.16), (1.19), I and II, we have

$$\lim_{n\to\infty} ||(S_n[f]-f)wu_b||_p = 0 \tag{1.20}$$

for all continuous $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\lim_{|x|\to\infty} |fwu_{B+\delta}|(x) = 0$$

for some $\delta > 1$.

Remark 2: The assumptions (1.14) **and** (1.16) The additional assumptions (1.14) and (1.16) used in Theorem 1 and Theorem A, are needed to obtain matching lower and upper bounds for the difference $(p_n - p_{n-2})w^2$ near $\pm a_n$ which are essential in our proof. The sharpness of these estimates near $\pm a_n$, do not follow from well known estimates on $p_n w^2$. (See [16, Theorem 1.8], [10, Theorem 1] and (3.5) below which hold without these later requirements). (1.14) and (1.16) were used for the first time in their present form in [14, Theorem 1.2] and in [3, Theorem 1.3]. Indeed, in [3, Theorem 1.3], it was shown that (1.14) implies (1.16) for a subsequence of $\mathbb{N}$. Notice that (1.14) and (1.16) are stronger than (1.9) and (1.10). For the example given by (1.6) it is known, (see [15]), that both (1.14) and (1.16) are satisfied. See also [4] and [10] for related examples.

Remark 3: Pointwise and mean boundedness; the essential differences

We note a fundamental difference in the estimates on the right hand side of (1.15) and (1.17), namely the appearance of the extra integral and $\log(n)$ terms. The integral term arises since, (see Section 4, Theorems 3 and 4 and [3, Theorems 1.6(a-b)]), the weighted Hilbert transform $H[;w]$ is a bounded map from a strict Sobolev subspace of weighted L_∞ to weighted L_∞, and bounded from weighted L_p to weighted L_p for $1 < p < \infty$. Thus the second integral term in the pointwise case, forces more smoothness on the function f which is not needed in L_p. This is consistent with classical results in Fourier series, see [28]. The extra $\log(n)$ term at first appears unatural when one compares to $L_p(1 < p < \infty)$. However, it arises naturally in both upper and lower pointwise bounds for weighted Lebesgue constants which are closely related to orthonormal expansions, see [6], [7] and the references cited therein. Actually one of the main objectives in this paper is to show that pointwise and mean bounds for weighted orthonormal expansions are indeed very different and should not be mixed in general. This again, is quite consistent with what is known concerning weighted Lebesgue constants.

Concerning convergence alone, an old result of Nikol'skii, see [20], implies that if f is absolutely continuous, w admissible and $wf' \in L_1$ then $(f - S_n(f))w$ converges to 0 pointwise as $n \to \infty$. Freud and Mhaskar in [13] and [21] proved an interesting generalization of this result as follows. Suppose $Q \in C^2$ is even and convex, Q'' is increasing in $(0, \infty)$ and satisfies Definition 1(b), f is of weighted bounded variation on compact intervals on $\mathbb{R}$ and x is a point of continuity of f then

$$|(f - S_n[f])(x)w(x)\exp(-CxQ'(x))| \to 0,\ n \to \infty.$$

Extensions of this later result to other even exponential weights have recenty been given in [19]. Theorem 1 gives pointwise bounds for the orthonormal operators given by (1.1), which in turn allow for the investigation of pointwise convergence with rates of polynomial approximation.

As an immediate corollary of Theorem 1, we have:

Theorem 2 Let w be admissible, $x \in \mathbb{R}$, $b \le 0$ and assume (1.12)-(1.14). Then there exists an infinite subsequence $\Omega \subseteq \mathbb{N}$ such that for $n \in \Omega$

$$|S_n[f]wu_b|(x) \le C\left[\log n ||fwu_B||_\infty + ||f'wu_b||_\infty\right] \tag{1.21}$$

holds for measurable $f : \mathbb{R} \to \mathbb{R}$ for which the right hand side of (1.21) is finite. Moreover, if (1.16) holds, then for $n \ge 1$,

$$|S_n[f]wu_b|(x) \le C\left[\log n ||fwu_B||_\infty + ||f'wu_b||_\infty\right] \tag{1.22}$$

holds for all measurable functions $f : \mathbb{R} \to \mathbb{R}$ for which the right hand side of (1.22) holds.

Theorems 1-2 allow for pointwise convergence with rates of convergence. These investigations and results will appear in a forthcoming paper.

2 Proof of Necessity of Theorem 1

In this section, we present the proof of the necessity of Theorem 1. Our first Lemma is a beautiful application of duality theory whose ideas can be traced back to Pollard, Freud and Nevai. See also [14, Lemma 3.1]. We provide full details for the reader's convenience.

Lemma 1 Let w be admissible, $B, b, x \in \mathbb{R}$, $1 \le p \le \infty$ and let

$$q := \begin{cases} \frac{p}{p-1}, & 1 < p < \infty \\ \infty, & p = 1, \\ 1, & p = \infty \end{cases}$$

Suppose there exists C such that for some infinite subsequence $\Omega \subseteq \mathbb{N}$ and for all $x \in \mathbb{R}$,

$$\sup_{n\in\Omega}|S_n[f]wu_b|(x) \leq C\,[||fwu_B||_p] \tag{2.1}$$

holds for every measurable $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.1) for which the right hand side of (2.1) is finite. Then

$$\sup_{n\in\Omega}\{||p_n wu_{-B}||_q||p_n wu_b||_\infty\} = O(1). \tag{2.2}$$

Proof Let $x \in \mathbb{R}$ and choose $\Omega \subseteq \mathbb{N}$ so that (2.1) holds. Let $n \in \Omega$ and write

$$S_n[f] = \sum_{j=0}^{n-1} b_j p_j$$

where

$$b_j = \int_{\mathbb{R}} (fp_j w^2)(y)dy,\ j = 0, 1, 2...$$

Using (2.1), we then have,

$$\begin{aligned} &|b_n||p_n wu_b|(x) = |(S_{n+1} - S_n)(f)wu_b|(x) \\ &\leq 2C\,[||fwu_B||_p] \end{aligned} \tag{2.3}$$

for every measurable $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.1) for which the right hand side of (2.3) is finite.

Recalling the definition of b_n, (2.3) then implies that we have

$$\begin{aligned} &\left|\int_{\mathbb{R}} (p_n wu_{-B})(y))(fwu_B)(y)dy\right| |p_n wu_b|(x) \\ &\leq 2C\,[||fwu_B||_p] \end{aligned} \tag{2.4}$$

for every measurable $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.1), for which the right hand side of (2.4) is finite. Suppose first that $1 \leq p < \infty$. Then from (2.4), we have

$$\begin{aligned} &\sup_{\{f:\,||fwu_B||_p \leq 1\}} \left|\int_{\mathbb{R}} (p_n wu_{-B})(y))(fwu_B)(y)dy\right| |p_n wu_b|(x) \\ &\leq 2C. \end{aligned} \tag{2.5}$$

As L_q is the dual space of L_p, (2.5) implies that

$$\{||p_n wu_{-B}||_q||p_n wu_b||_\infty\} \leq 2C. \tag{2.6}$$

Taking sups we see that we have (2.2) as required. Suppose next that $p = \infty$. In this case, clearly L_1 is not the dual of L_∞ but duality of the norms does hold ie,

$$\sup_{\{h:\,||h||_\infty \leq 1\}} \left|\int_{\mathbb{R}} gh\right| = ||g||_1$$

for L_1 functions $g : \mathbb{R} \to \mathbb{R}$. Indeed, this last statement can be easily proved by taking $h = \mathrm{sign}(g)$. Thus we deduce that (2.6) persists for $p = \infty$ also. This completes the proof of the Lemma. $\Box$

We are now ready to present the:

Proof of Theorem 1(a) Applying Lemma 1, we have

$$\sup_{n\in\Omega} ||p_n w u_{-B}||_1 ||p_n w u_b||_\infty = O(1). \tag{2.7}$$

Next, we need the following facts: For $n \geq 1$,

$$||p_n w u_{-B}||_1 \sim \left\{ \begin{array}{ll} a_n^{-1/2} + a_n^{1/2-B}, & B > 1 \\ a_n^{-1/2} \log n + a_n^{1/2-B}, & B = 1 \\ a_n^{1/2-B}, & B < 1 \end{array} \right\} \tag{2.8}$$

and

$$||p_n w u_b||_\infty \sim \left\{ \begin{array}{ll} a_n^{-1/2} + a_n^{b-1/2} n^{1/6}, & b \leq 0 \\ a_n^{b-1/2} n^{1/6}, & b > 0. \end{array} \right\} \tag{2.9}$$

The bounds (2.8) and (2.9) follow if we apply the method of [14, Lemma 4.7] together with [10, Theorem 1], [16, Theorem 1.18] and [16, Theorem 13.6]. We will now apply (2.8) and (2.9) together with (2.7) to deduce the necessity of Theorem 1. To see this, suppose that $B \leq 0$. Then $b < 0$ and so an application of (2.7-2.9) implies that

$$n^{1/6} = O(1).$$

This gives an obvious contradiction for large enough n so necessarily $B > 0$. If we assume similarly that $b \geq 1$, then we obtain, on applying (2.7-2.9) that

$$a_n^{b-1} n^{1/6} = O(1).$$

Thus we obtain a contradiction again. Finally if

$$a_n^{b-\min\{B,1\}} n^{1/6} C_{B,n}$$

is unbounded for large n, then a straightforward and tedious application of (2.7-2.9) again gives a contradiction. Thus the necessity of Theorem 1 is established. We remark that actually we have shown the following fact which we use again in the sufficiency of Theorem 1: Given any infinite subsequence $\Omega \subseteq \mathbb{N}$,

$$\sup_{n\in\Omega} ||p_n w u_{-B}||_1 ||p_n w u_b||_\infty \tag{2.10}$$

is bounded and bounded away from 0 iff (1.12) and (1.13) hold. $\Box$

3 Proofs of Theorems 1-2

In this section, we complete the proof of Theorem 1 and present the proof of Theorem 2. We begin with the

Proof of Theorem 1(b) Using [3, Theorem 1.3] and [10, Theorem 1], we know that (1.14) implies (1.16) for an infinite subsequence of $\mathbb{N}$, so without loss of generality we will assume throughout that only (1.14) holds and choose and fix an $\Omega \subseteq \mathbb{N}$ for which (1.16) holds. Let $n \in \Omega$ and assume without loss of generality that n is large enough. We will need the kernel

$$K_n(x,t) := \sum_{j=0}^{n-1} p_j(x)p_j(t),\ t,x \in \mathbb{R}$$

where

$$S_n[f](x) = \int_{\mathbb{R}} K_n(x,t)(fw^2)(t)dt,\ x \in \mathbb{R}.$$

More precisely, we will use Pollards decomposition of K as applied by Askey and Wainger, Muckenhoupt, Mhaskar and Xu and Lubinsky and Jha in [1], [24], [25], [23] and [14]. For a given $t, x \in \mathbb{R}$, write,

$$K_n(x,t) = K_{n,1}(x,t) + K_{n,2}(x,t) + K_{n,3}(x,t)$$

where

$$K_{n,1}(x,t) := \frac{A_n}{A_n + A_{n-1}} p_{n-1}(x)p_{n-1}(t),$$

$$K_{n,2}(x,t) := \frac{A_{n-1}A_n}{A_n + A_{n-1}} p_{n-1}(t)\frac{(p_n - p_{n-2})(x)}{x-t},$$

and

$$K_{n,3}(x,t) = K_{n,2}(t,x).$$

Then setting

$$S_{n,j}[f](x) = \int_{\mathbb{R}} K_{n,j}(x,t)(fw^2)(t)dt,\ j = 1,2,3,$$

we have

$$S_n[f](x) = \sum_{j=1}^{3} S_{n,j}[f](x).$$

Firstly, we see that if $x \in \mathbb{R}$

$$\begin{aligned}|(S_{n,1}[f]wu_b)(x)| &\le |p_{n-1}wu_b|(x)\left|\int_{\mathbb{R}}(p_{n-1}w(t)(fw)(t)dt\right| \\ &\le C||p_{n-1}wu_b||_\infty ||fwu_B||_\infty ||p_{n-1}wu_{-B}||_1.\end{aligned}$$

Thus applying (2.10), we learn that

$$|(S_{n,1}[f]w)(x)| \le C||fwu_B||_\infty,\ x \in \mathbb{R}. \tag{3.1}$$

For the estimation of $S_{n,j}$, $j = 2, 3$, we note that for $x \in \mathbb{R}$

$$S_{n,2}[f](x) = \frac{A_{n-1}A_n}{A_n + A_{n-1}}(p_n - p_{n-2})(x)H[fp_{n-1}w^2](x)$$

and

$$S_{n,3}[f](x) = \frac{-A_{n-1}A_n}{A_n + A_{n-1}}p_{n-1}(x)H[f(p_n - p_{n-2})w^2](x).$$

Here, we recall the Hilbert transform $H[.]$ is defined formally for measurable $f : \mathbb{R} \to \mathbb{R}$ by

$$H[f](x) := \lim_{\varepsilon \to 0+} \int_{|t-x| \geq \varepsilon} \frac{f(t)}{t - x} dt$$

where the integral above is understood as a Cauchy-Principal valued integral. See Section 4.

Thus we have for $x \in \mathbb{R}$

$$(S_{n,j}[f]w)(x) = \frac{\pm A_{n-1}A_n}{A_n + A_{n-1}}(\Psi_n w)(x)H[f\Phi_n w^2](x),\ j = 2, 3 \tag{3.2}$$

where

$$\{\Psi_n, \Phi_n\} := \{p_n - p_{n-2}, p_{n-1}\} \tag{3.3}$$

and where we agree that because of the symmetry of the cases $j = 2, 3$, once we have chosen Ψ_n to be one of the functions on the right hand side of (3.3) then Φ_n is chosen as the other function. We need to estimate (3.2). We will proceed in two ways. Firstly if x is bounded away from t, we will bound (3.2) directly for in this case, the Hilbert transform can be estimated easily. If x is close to t, we will need more refined estimates of the Hilbert Transform. Indeed, as we will show, the estimation that we use is enough to also show Theorems 3-4 in Section 4. Let us make some further observations re (3.2). We first recall from [3, Theorem 1.5], that

$$\begin{aligned} &\sup_{y \in \mathbb{R}} |p_n(y) - p_{n-2}(y)|\, w(y) \times \\ &\times \left\{ \left| 1 - \frac{|y|}{a_n} \right| + n^{-2/3} \right\}^{-1/4} \sim a_n^{-1/2}. \end{aligned} \tag{3.4}$$

The importance of (3.4) lies in the fact that for $|y|$ close to a_n, (3.4) improves the bound, see [16, Theorem 1.8] and [10, Theorem 1],

$$\sup_{y \in \mathbb{R}} |p_n(y)|\, w(y) \left\{ \left| 1 - \frac{|y|}{a_n} \right| + n^{-2/3} \right\}^{1/4} \sim a_n^{-1/2} \tag{3.5}$$

by a factor of 1/4 as it should. See also [14, Theorem 1.1]. We will set for simplicity,

$$\psi_n(y) := \left| 1 - \frac{|y|}{a_n} \right| + n^{-2/3},\ y \in \mathbb{R}. \tag{3.6}$$

For what follows, we also find it convenient to split our target function f into pieces which are supported on subintervals of the line. Proceeding, henceforth, let χ denote the indicator function of an interval $E \subseteq \mathbb{R}$ and let us write

$$\begin{aligned} f &= f\chi[-a_n/2, a_n/2] + f\chi(a_n/2, 2a_n) + \\ &+ f\chi(-2a_n, -a_n/2) + f\chi\left(t: |t| \geq 2a_n\right) \\ &= f_{n,1} + f_{n,2} + f_{n,3} + f_{n,4}. \end{aligned} \tag{3.7}$$

Theorem 1(b) will follow from (1.7), (2.10), (3.1) and the following two claims below:

Claim 1 Let $j = 2, 3$, $k = 1, 4$ and suppose that $|x| \leq 4a_n$: Then

$$\begin{aligned} &|(S_{n,j}[f_{n,k}]wu_b)(x)| \leq C\,[\log n ||fwu_B||_\infty \times \\ &\times \{1 + ||p_n wu_b||_\infty ||p_n wu_{-B}||_1\} + \int_{-1}^{1} \left| \frac{[wu_b \Delta_y(f)](x)}{y} dy \right| \Bigg]. \end{aligned} \tag{3.8}$$

Claim 2 Let $j = 2, 3$, $k = 2, 3$ and suppose that $x \in \mathbb{R}$: Then

$$\begin{aligned} &|(S_{n,j}[f_{n,k}]wu_b)(x)| \leq C\,[\log n ||fwu_B||_\infty \times \\ &\times \left\{ ||p_n wu_b||_\infty ||p_n wu_{-B}||_1 + a_n^{b-B} n^{1/6} \right\} + \int_{-1}^{1} \left| \frac{[wu_b \Delta_y(f)](x)}{y} dy \right| \Bigg]. \end{aligned} \tag{3.9}$$

We proceed with the proofs of the claims above.

The Proof of Claim 1 Let $j = 2$. First suppose that $2a_n/3 \leq |x| \leq 3a_n/2$: Then we see that $|x - t| \geq Ca_n$. Thus applying (1.9) to (3.2), we find that

$$\begin{aligned} &|(S_{n,j}[f_{n,k}]wu_b)(x)| \\ &\leq C||fwu_B||_\infty ||p_n wu_b||_\infty ||p_n wu_{-B}||_1. \end{aligned} \tag{3.10}$$

Now we consider the more difficult case $|x| \leq 2a_n/3$ and $3a_n/2 \leq |x| \leq 4a_n$ and let x be chosen in this range. By symmetry we may assume that $x \geq 0$. Assume first that $x > D$ for some large enough and fixed $D > 0$. We will make heavy use of the fact that as $k = 1, 4$, we have $\psi_n(t) \sim 1$ for t in the support of $f_{n,k}$ and $\psi_n(x) \sim 1$. By symmetry, it is also enough to assume that $k = 1$ for the other case, $k = 4$, is similar. Thus using (3.4), (3.2) becomes

$$|S_{n,j}[f_{n,k}]wu_b|(x) \leq Ca_n^{1/2} u_b(x) \left| \int_{-a_n/2}^{a_n/2} \frac{(fwp_{n-1}w)(t)}{t - x} \right|. \tag{3.11}$$

Let us suppose that D is so large so that

$$\frac{1}{Q'(y + \delta)} < \delta$$

for some small enough $0 < \delta < 1$ and for all $y > D$. Then define

$$\varepsilon = \varepsilon(n, x) := \min\left\{ \frac{1}{2Q'(x+\delta)}, \frac{1}{n} \right\}.$$

We write for some fixed $\beta > 1$:

$$\begin{aligned}
&\int_{-a_n/2}^{a_n/2} \frac{(fwp_{n-1}w)(t)}{t-x} \qquad (3.12)\\
&= \left(\int_{|t|>2x} + \int_{-2x}^{0} + + \int_{0}^{x/\beta} + \int_{x/\beta}^{x-\varepsilon} + \right.\\
&\left. + \int_{x+\varepsilon}^{2x} + \int_{x-\varepsilon}^{x+\varepsilon} \right) \frac{(fwp_{n-1}w)(t)}{t-x} dt\\
&= \sum_{i=1}^{6} I_i(x).
\end{aligned}$$

Keeping in mind (3.11), let us now proceed to estimate each of the terms in (3.12).

I_1: Using (3.5), we see that we have

$$\begin{aligned}
&a_n^{1/2} |I_1|(x)\\
&\leq C a_n^{1/2} ||fwu_B||_\infty ||p_{n-1}w||_{\infty[-a_n/2, a_n/2]} \int_{\mathbb{R}} \frac{u_{-B}(t)}{|t|} dt\\
&\leq C ||fwu_B||_\infty.
\end{aligned}$$

I_2: Using (3.5), we see that we have

$$\begin{aligned}
&a_n^{1/2} |I_2|(x)\\
&\leq C a_n^{1/2} ||fwu_B||_\infty ||p_{n-1}w||_{\infty[-a_n/2, a_n/2]} \int_{x}^{3x} \frac{du}{u}\\
&\leq C ||fwu_B||_\infty.
\end{aligned}$$

I_3: Using (3.5), we see that we have

$$\begin{aligned}
&a_n^{1/2} |I_3|(x)\\
&\leq C a_n^{1/2} ||fwu_B||_\infty ||p_{n-1}w||_{\infty[-a_n/2, a_n/2]} \int_{0}^{x/\beta} \frac{1}{x-t} dt\\
&\leq C ||fwu_B||_\infty.
\end{aligned}$$

I_4: Using (3.5), we see that we have

$$\begin{aligned}
&a_n^{1/2}|I_4|(x) \\
&\leq Ca_n^{1/2}||fwu_B||_\infty||p_{n-1}w||_{\infty[-a_n/2,a_n/2]}\int_{x/\beta}^{x-\varepsilon}\frac{u_{-B}(t)}{x-t}dt \\
&\leq C||fwu_B||_\infty u_{-B}(x/\beta)\log\left(\frac{x}{\varepsilon}\right) \\
&\leq C\log n||fwu_B||_\infty.
\end{aligned}$$

I_6: Much as in the previous estimate we see that we have

$$\begin{aligned}
&a_n^{1/2}|I_6|(x) \\
&\leq C||fwu_B||_\infty u_{-B}(x+\varepsilon)\log\left(\frac{x}{\varepsilon}\right) \\
&\leq C\log n||fwu_B||_\infty.
\end{aligned}$$

Thus we may summarize the calculations above in the following: For $i = 1-4, 6$ we have

$$a_n^{1/2}|I_i|(x) \leq C\log n||fwu_B||_\infty. \tag{3.13}$$

Finally, we deal with I_5: Here is this case, we will write:

$$\begin{aligned}
&|I_5|(x) \leq \left|\int_{x-\varepsilon}^{x+\varepsilon}\left(\frac{f(t)-f(x)}{t-x}\right)(p_{n-1}w^2)(t)dt\right| + \\
&+|f(x)|\left|\int_{x-\varepsilon}^{x+\varepsilon}\frac{(p_{n-1}w^2)(t)-(p_{n-1}w^2)(x)}{t-x}dt\right| \\
&= I_{5,1}(x)+I_{5,2}(x).
\end{aligned}$$

$|I_{5,1}|(x)$: Here, we see that using (3.5) we have

$$\begin{aligned}
&a_n^{1/2}u_b(x)|I_{5,1}|(x) \\
&\leq Ca_n^{1/2}||p_{n-1}w||_{\infty[-a_n/2,a_n/2]}\times \\
&\times\int_{x-\varepsilon}^{x+\varepsilon}(wu_b)(x)\left|\frac{f(t)-f(x)}{t-x}\right|dt \\
&\leq C\int_{-1}^{1}\left|\frac{(wu_b)(x)\Delta_u(f)(x)}{u}\right|du.
\end{aligned}$$

Note also that we use that $w(t)\sim w(x)$ for $t\in[x-\varepsilon,x+\varepsilon]$. Finally, we estimate

$|I_{5,2}|(x)$: Here we have using the mean value theorem, (3.5) and a Markov-Bernstein inequality, see [16, Theorem 1.15] that

$$\begin{aligned}
&a_n^{1/2}|I_{5,2}|(x)\\
&\leq C||fwu_B||_\infty||Q'||_{[x-\varepsilon,x+\varepsilon]}\varepsilon+\\
&+Ca_n^{1/2}||fwu_B||_\infty||(p_{n-1}w)'||_\infty\varepsilon\\
&\leq C_1||fwu_B||_\infty+\\
&+C_2a_n^{1/2}||fwu_B||_\infty\frac{n}{a_n}||p_{n-1}w||_\infty\varepsilon\\
&\leq C_3||fwu_B||_\infty[1+\frac{n}{a_n}\varepsilon]\\
&\leq C_4||fwu_B||_\infty.
\end{aligned}$$

This last estimate establishes Claim 1 for this range of x. We observe that if $1 \leq x < D$, then the estimates for $I_1(x)$ go through as before. $I_2(x)$ follows without change and the estimates for the remaining integrals are easier for in this case w and x are uniformly bounded. If $0 \leq x < 1$, then we write

$$\begin{aligned}
&\int_{-a_n/2}^{a_n/2}\frac{(fwp_{n-1}w)(t)}{t-x}dt\\
&=\int_{-a_n/2}^{x-1}+\int_{x-1}^{x+1}+\int_{x+1}^{a_n/2}\frac{(fwp_{n-1}w)(t)}{t-x}dt
\end{aligned}$$

For the first and last integrals, t is bounded away from x, and for the second integral, we proceed as above, but the proof is easier since both x and w are uniformly bounded. Thus we have established Claim 1 for $j = 2$. It remains to notice that the proof of Claim 1 for $j = 3$ is identical to the proof for $j = 2$ if p_{n-1} is replaced by $p_n - p_{n-2}$ and visa versa. This follows by using (3.4) and (3.5). The proof of Claim 1 is completed.

We now proceed with Claim 2:

The Proof of Claim 2 Assume first that $j = 2$. First suppose that $|x| \leq a_n/4$ or $|x| \geq 3a_n$: Then we see that $|x-t| \geq Ca_n$. Thus as in Claim 1, we easily see that again we have

$$\begin{aligned}
&|(S_{n,j}[f_{n,k}]wu_b)(x)|\\
&\leq C||fwu_B||_\infty||p_nwu_b||_\infty||p_nwu_{-B}||_1.
\end{aligned} \tag{3.14}$$

Next suppose that $a_n/4 \leq |x| \leq 3a_n$ and as before we may only consider the case $k = 2$. We observe that the situation we have here is different to Claim 1 since here, aprori, the sequence $\psi_n(t)$ is not always uniformly bounded in n in the support of $f_{n,k}$ which is the interval $[a_n/2, 2a_n]$. Indeed, $\psi_n(x)$ is also not always uniformly bounded in n either. We write, using (3.4) and (3.5),

$$|S_{n,j}[f_{n,k}]wu_b|(x) \leq Ca_n^{1/2}\psi_n(x)^{1/4}u_b(x)\left|\int_{a_n/2}^{2a_n}\frac{(fwp_{n-1}w)(t)}{t-x}dt\right|. \tag{3.15}$$

We now consider a transformation in (3.15) given by the maps:

$$t :\to a_n(1+n^{-2/3}T),\ x :\to a_n(1+n^{-2/3}X).$$

Let $A_n := [-2n^{2/3}, 2n^{2/3}]$ and for a function $h : \mathbb{R} \to \mathbb{R}$, let

$$h_n(T) := h(t)\chi_{[-1/2n^{2/3}, n^{2/3}]}(T),\ n \geq 1.$$

Then we see that (3.15) can be rewritten as:

$$|S_{n,j}[f_{n,k}]wu_b|(x) \leq Ca_n^{1/2+b}n^{-1/6}\left|\int_{A_n} \frac{w_n(T)f_n(T)(p_{n-1}w)_n(T)}{T-X}u_{1/4}(X)dT\right|. \tag{3.16}$$

We may assume, without loss of generality that $X > D$ for some fixed and large enough D. Let $0 < \varepsilon := \varepsilon(n,X) < 1$ be chosen as in Claim 1 and let $\beta > 1$ be fixed. Split:

$$\begin{aligned}
&\int_{A_n} \frac{w_n(T)f_n(T)(p_{n-1}w)_n(T)}{T-X}u_{1/4}(X)dT \qquad (3.17)\\
&= \left(\int_{|T|>2X} + \int_{-2X}^{0} + + \int_{0}^{X/\beta} + \int_{X/\beta}^{X-\varepsilon} + \right.\\
&\left. + \int_{X+\varepsilon}^{2X} + \int_{X-\varepsilon}^{X+\varepsilon}\right) \frac{w_n(T)f_n(T)(p_{n-1}w)_n(T)}{T-X}u_{1/4}(X)dT\\
&= \sum_{i=1}^{6} I_i(X).
\end{aligned}$$

Now we may proceed in a similar way to the proof of Claim 1 and deduce after mapping back that

$$a_n^{1/2+b}n^{-1/6}|I_i(X)| \leq Ca_n^{b-B}n^{1/6}||fwu_B||_{\infty(\mathbb{R})},\ i = 1-4, 6. \tag{3.18}$$

Finally, we deal with I_5: Here as in Claim 1, we will write:

$$\begin{aligned}
|I_5|(X) &= \left|\int_{X-\varepsilon}^{X+\varepsilon} \left(\frac{f_n(T)-f_n(X)}{T-X}\right)(p_{n-1}w^2)_n(T)dT\right| u_{1/4}(X) +\\
&+ |f_n(X)|u_{1/4}(X)\left|\int_{X-\varepsilon}^{X+\varepsilon} \frac{(p_{n-1}w^2)_n(T)-(p_{n-1}w^2)_n(X)}{T-X}dT\right|\\
&= I_{5,1}(X) + I_{5,2}(X).
\end{aligned}$$

$|I_{5,1}|(X)$: Here, we see that, we have using (3.5) that

$$\begin{aligned}
&a_n^{1/2+b}n^{-1/6}|I_{5,1}|(X)\\
&\leq Ca_n^b n^{-1/6}\int_{X-\varepsilon}^{X+\varepsilon} w_n(X)\left|\frac{f_n(T)-f_n(X)}{T-X}\right| u_{1/4}(X)u_{-1/4}(T)dT\\
&\leq C\int_{x-\frac{\varepsilon a_n}{n^{2/3}}}^{x+\frac{\varepsilon a_n}{n^{2/3}}}(wu_b)(x)\left|\frac{f(t)-f(x)}{t-x}\right|dt\\
&\leq C\int_{-1}^{1}(wu_b)(x)\left|\frac{\Delta_u(f)(x)}{u}\right|du.
\end{aligned}$$

Finally, we estimate:

$|I_{5,2}|(X)$: Here we have using the mean value theorem, (3.5), the Markov-Bernstein inequality, see [16, Theorem 1.15], and mapping back that

$$\begin{aligned}
&a_n^{1/2+b}n^{-1/6}|I_{5,2}|(X)\\
&\leq Ca_n^{b-B}||fwu_B||_\infty||Q'||_{[x-\varepsilon,x+\varepsilon]}\varepsilon+\\
&+Ca_n^{1/2+b-B}||fwu_B||_\infty||(p_{n-1}w)'||_\infty\varepsilon\\
&\leq C_1a_n^{b-B}||fwu_B||_\infty+\\
&+C_2a_n^{1/2+b-B}||fwu_B||_\infty\frac{n}{a_n}||(p_{n-1}w)||_\infty\varepsilon\\
&\leq C_3n^{1/6}a_n^{b-B}||fwu_B||_\infty[1+\frac{n}{a_n}\varepsilon]\\
&\leq C_4n^{1/6}a_n^{b-B}||fwu_B||_\infty.
\end{aligned}$$

This last estimate establishes Claim 2 for $j=2$. A similar proof justifies the claim for $j=3$. The proof of Theorem 1(b) is complete. □

We complete this section with the:

Proof of Theorem 2 This follows as an immediate application of Theorem 1. □.

4 Pointwise bounds on weighted Hilbert transforms

In this last section, we record new results concerning pointwise bounds on weighted Hilbert transforms namely Theorems 3-4 whose proofs are hidden in Theorem 1 but which in our opinion, are of independent interest.

4.1 Hilbert transforms

We recall that the *Hilbert transform* is defined formally for measurable $f:\mathbb{R}\to\mathbb{R}$ by

$$H[f](x):=\lim_{\varepsilon\to 0+}\int_{|t-x|\geq\varepsilon}\frac{f(t)}{t-x}dt \qquad (4.1)$$

where the integral above is understood as a Cauchy-Principal valued integral. It is known, see [24], that if $b < 1 - 1/p$, $B > -1/p$, $b \leq B$ and $1 < p < \infty$, we have

$$||H[f]u_b||_{L_p(\mathbb{R})} \leq C||fu_B||_{L_p(\mathbb{R})}, \tag{4.2}$$

provided the right hand side of (4.2) is finite. Indeed, relations such as (4.2) are essential in studying boundedness and convergence of orthonormal expansions. This is mainly due to the following identity which follows from the Christoffel-Darboux formula for orthonormal polynomials, see [11].

$$S_n[f] = A_n \left\{ p_n H[fp_{n-1}] - p_{n-1} H[fp_n] \right\}. \tag{4.3}$$

For further results on weighted Hilbert transforms for admissible weights in L_p, we refer the reader to the survey [6] and the references cited therein. For pointwise convergence of orthonormal expansions, it thus seems natural to look for L_∞ analogues of (4.2). For large classes of weights (not necessarily admissible) these analogues have recently been investigated in [8] and [3, Theorems 1.6(A-B)].

We have:

Theorem 3 Let $f : \mathbb{R} \to \mathbb{R}$ be measurable, $B > 0$. Then for $x \in \mathbb{R}$

$$\begin{aligned} &|H[f]|(x) \\ &\leq C \left[||fu_B||_\infty + \int_{-1}^{1} \left| \frac{\Delta_u(f)(x)}{u} \right| du \right] \end{aligned} \tag{4.4}$$

provided the right hand side of (4.4) is finite.

Finally, we present a pointwise analogue of [24, pg 441] which is also new. We have:

Theorem 4 Let $f : \mathbb{R} \to \mathbb{R}$ be measurable and supported in $[-A, A]$ for some $A > 0$ and let $x \in [-A, A]$. Then:

$$\begin{aligned} &\left| H[fu_{-1/4}]u_{1/4} \right|(x) \\ &\leq C \left[A^{1/4}||f||_\infty + u_{1/4}(x) \int_{x-1}^{x+1} \left| \frac{(fu_{-1/4})(t)}{t-x} dt \right| \right] \end{aligned} \tag{4.5}$$

provided the right hand side of (4.5) is finite.

We begin with the

Proof of Theorem 3 This follows easily if we apply the technique used in the proof of Claim 1 in Theorem 1 as follows. First replace $fwp_{n-1}w$ by f, $\pm a_n/2$ by $\pm\infty$ and choose ε small enough but fixed. Each of the integrals is now estimated in exactly the same way. Note that in this case $I_{5,2}$ will be identically zero.

Finally we present the

Proof of Theorem 4 Fix $x \in [-A, A]$, choose $0 < \eta \leq \frac{A}{2}$ and suppose that $x > \eta$. Much as in the proof of Theorem 3, let $\varepsilon > 0$ be fixed and small enough. Then, let us write for some $\beta = \beta(\varepsilon)$

$$H[fu_{-1/4}](x)u_{1/4}(x) = \tag{4.6}$$

$$= u_{1/4}(x)\left(\int_{2x\leq|t|\leq A} + \int_{-2x}^{0} + \int_{0}^{x/\beta} + \int_{x/\beta}^{x-\varepsilon}\right. \tag{4.7}$$

$$\left. + \int_{x+\varepsilon}^{2x} + \int_{x-\varepsilon}^{x+\varepsilon}\right) u_{1/4}(x)\frac{(fu_{-1/4})(t)}{t-x}dt$$

$$= \sum_{i=1}^{6} I_i(x). \tag{4.8}$$

The essential idea for each integral is to use the $u_{-1/4}$ term to estimate the integral and then the $u_{1/4}$ factor gives the factor $A^{1/4}$. Proceeding henceforth, one obtains

$$|I_1(x)| \leq C||f||A^{1/4}\int_{2\eta\leq|t|\leq A}\frac{1}{|t|(1+|t|)^{1/4}}dt \leq C||f||A^{1/4}.$$

$$|I_2(x)| \leq C||f||A^{1/4}\int_{-2x}^{0}\frac{1}{x-t}dt \leq C||f||A^{1/4}.$$

$$|I_3(x)| \leq C||f||A^{1/4}\int_{0}^{x/\beta}\frac{1}{x-t}dt \leq C||f||A^{1/4}.$$

$$\begin{aligned}
&|I_4(x)| \leq C_1||f||A^{1/4}|x|^{-1/4}\log(x) \\
&+C_2A^{1/4}||f||_A|x|^{-1/4}\log(1-\frac{1}{\beta}) \\
&+C_3A^{1/4}||f||_A|x|^{-1/4}\log\left(\frac{1}{\varepsilon}\right) \\
&\leq C_4||f||_A|x|^{-1/4}\left[\log(x)+\log\left(\frac{1}{\varepsilon}\right)\right]
\end{aligned}$$

where C_j, $j = 1, ..., 4$ are positive constants independent of f, x and n. Also,

$$\begin{aligned}
&|I_6(x)| \leq \\
&\leq CA^{1/4}||f||_A|x|^{-1/4}\left[\log(x)+\log\left(\frac{1}{\varepsilon}\right)\right].
\end{aligned}$$

A careful choice of $\varepsilon = \varepsilon(x)$ then gives the claim in this case. When $0 \leq x \leq \eta$ the proof of (4.5) follows in an easier way by splitting as

$$\begin{aligned}
&H[fu_{-1/4}](x)u_{1/4}(x) = \\
&= u_{1/4}(x)\left(\int_{-A}^{x-\eta} + \int_{x+\eta}^{A+\eta} + \int_{x-\eta}^{x+\eta}\right)\frac{(fu_{-1/4})(t)}{t-x}dt.
\end{aligned}$$

The proof of Theorem 4 is complete. □

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