Another look at an old paper of Geza Freud

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Abstract. For a class of smooth, even exponential weights w with varying rates of decay on the real line and $(-1,1)$, we adapt an old method of Geza Freud to investigate the dependence of the greatest zero of the orthogonal polynomials $p_n(w^2)$ on the associated recurrence coefficients, $\alpha_n(w^2)$. Applications are given to mean and uniform convergence of weighted orthonormal expansions.

§1. Introduction

Denote by I the real line or $(-1,1)$ and let w be a non negative weight on I with $x^n w(x) \in L^1$, $n = 0, 1, \ldots.$ The idea of this paper arose from an old paper of Geza Freud [7] who studied the dependence of the greatest zero of $p_n(w^2)$, the nth orthonormal polynomial for w^2, on the associated recurrence coefficients. More precisely, given w as above, we recall, see [8], that $p_n(w^2)$ admits the representation

$$p_n(w^2, x) := \gamma_n x^n + \cdots, \quad \gamma_n := \gamma_n(w^2) > 0$$

and satisfies

$$\int_I p_n(w^2, x) p_n(w^2, x) w^2(x) dx = \delta_{m,n} \qquad m, n \geq 0.$$

We denote by

$$x_{n,n}(w^2) < x_{n-1,n}(w^2) \cdots < x_{2,n}(w^2) < x_{1,n}(w^2)$$

the n simple zeros of $p_n(w^2)$ in I and write the three term recurrence of $p_n(w^2)$ in the form

$$x p_n(w^2, x) = \alpha_{n+1} p_{n+1}(w^2, x) + \alpha_n p_{n-1}(w^2, x), \qquad n \geq 0.$$

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Here, $p_{-1}(w^2)=0$, $p_0(w^2)=\left(\int w(x)\,dx\right)^{-1/2}$ and

$$\alpha_n(w^2)=\gamma_{n-1}/\gamma_n>0$$

are the associated recurrence coefficients. In this paper, our class of admissible weights, (see Definition 1.1 below), will include as prime examples Freud type weights such as

$$w_\alpha(x):=\exp\left(-|x|^\alpha\right),\ \alpha>1,\ x\in(-\infty,\infty), \tag{1.1}$$

Erdős type weights such as

$$w_{k,\beta}(x):=\exp\left(-\exp_k\left(|x|^\beta\right)\right),\ \beta>0,\ k\geq 1,\ x\in(-\infty,\infty), \tag{1.2}$$

and generalized Pollaczek weights of the form

$$w_{0,\gamma}(x):=\exp\left(-(1-x^2)^{-\gamma}\right),\ \gamma>0 \tag{1.3}$$

and

$$w_{k,\gamma}(x):=\exp\left(-\exp_k(1-x^2)^{-\gamma}\right),\ \gamma>0,\ k\geq 1,\ x\in(-1,1). \tag{1.4}$$

Here and throughout, $\exp_k$ denotes the kth iterated exponential. Freud weights are characterized by their smooth polynomial decay at infinity and Erdős weights by their faster than smooth polynomial decay at infinity. Generalized Pollaczek weights decay strongly near ± 1 as exponentials and are of faster decay than classical Jacobi weights. They violate the well known Szegő condition for orthogonal polynomials, [8, Chapter 5, pg 208]. For such later admissible weights w, the asymptotic behavior of the recurrence coefficients is expressed in terms of the scaled endpoints of the equilibrium measure for $\exp(-2Q)$, $a_n=(a_n)_{n=1}^{\infty}$, where $a_u=a_u(\exp(-2Q))$ is the positive root of the equation

$$u=\frac{2}{\pi}\int_0^1\frac{a_utQ'(a_ut)}{\sqrt{1-t^2}}\,dt.$$

If I^+, denotes the interval $(0,\infty)$ or $(0,1)$, then the number a_u is, as a real valued function of u, uniquely defined, strictly increasing in I^+ and it is well known, see [12], that

$$\lim_{n\to\infty}\frac{\alpha_n}{a_n}=1/2$$

and

$$\lim_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n}=1.$$

To state our main results, we require some additional notation. To this end, let us agree that henceforth C will denote a positive constant which may take on different values at different times, Moreover, for any two sequences b_n and

c_n of non zero real numbers, we shall write $b_n = O(c_n)$ if there exists a positive constant C, independent of n, such that

$$b_n \leq C c_n, \; n \to \infty$$

and $b_n \sim c_n$ if

$$b_n = O(c_n), \text{ and } c_n = O(b_n).$$

Similar notation will be used for functions and sequences of functions.

Our class of weights w will then be assumed to be *admissible* in the sense of the following definition, see [11].

Definition 1.1 Let $w = \exp(-Q)$ with

$$Q : I \to (0, \infty)$$

even.

(a) Assume that Q'' is continuous in $I+$ and Q'', $Q' \geq 0$ in $I+$.

(b) The function

$$T(x) := 1 + \frac{xQ''(x)}{Q'(x)}, \; x \in I+$$

satisfies for large enough x or x close enough to ± 1

$$T(x) \sim \frac{xQ'(x)}{Q(x)}.$$

Moreover T satisfies either:

(b1) There exist $A > 1$ and $B > 1$ such that

$$A \leq T(x) \leq B, \; x \in I+ .$$

(b2) T is increasing in $I+$ with $\lim_{x \to 0+} T(x) > 1$. If $I = (-\infty, \infty)$,

$$\lim_{|x| \to \infty} T(x) = \infty$$

and if $I = (-1, 1)$, for x close enough to ± 1,

$$T(x) \geq \frac{A}{1 - x^2}$$

for some $A > 2$.

(b3) For every $\varepsilon > 0$,

$$\frac{T(y)}{T(x)} = O\left(\frac{Q(y)}{Q(x)}\right)^{\varepsilon}$$

for $y \geq x$, $x, y \in I$, y large enough or close enough to 1.

Then w shall be called an *admissible* weight.

Canonical examples of admissible weights are those listed in (1.1) – (1.4).

We shall prove:

Theorem 1.2 *Let w be an admissible weight. If*

$$\frac{\alpha_{n+1}}{\alpha_n} = 1 + O\left(\frac{1}{(nT(a_n))^{2/3}}\right), \ n \to \infty \tag{1.5}$$

then there exists a subsequence $n = n_j$ of natural numbers such that

$$\frac{\alpha_n}{a_n} = \frac{1}{2}\left[1 + O\left(\frac{1}{(nT(a_n))^{2/3}}\right)\right], \ j \to \infty. \tag{1.6}$$

The following important remark suffices:

Remark 1.3

(a) Firstly it is well known that the sequences a_n and $T(a_n)$ satisfy uniformly for $n \geq C$:

(A) Freud:

$$\frac{1}{C} n^{1/B} \leq a_n \leq C n^{1/A}$$

(B) Erdös: For every $\varepsilon > 0$,

$$a_n = O(n^{\varepsilon}), \ T(a_n) = O(n^{\varepsilon}).$$

(C) Generalized Pollaczek: For every $\varepsilon > 0$,

$$1 - \varepsilon \leq a_n \leq 1, \ T(a_n) = O(n^{2-\varepsilon}).$$

(D)

$$\left|\frac{a_{n+1}}{a_n} - 1\right| \sim \frac{1}{nT(a_n)}$$

Thus (1.6) immediately implies (1.5) for every sufficiently large n. It is the other direction which is new and non trivial for admissible weights w. We refer the reader to two applications of our result, (see Theorems 1.4 and 1.6 below).

(b) Fortunately the hypotheses on the recurrence coefficients given by (1.5) and (1.6) are not always vacuous ones. On the other hand results on second order terms for admissible weights are difficult to prove and rather scarce. At the time of writing the following hold for large n:

(A) Let $w = \exp(-Q)$ where Q is an even polynomial of fixed degree. Then see [5],

$$\frac{\alpha_n}{a_n} = \frac{1}{2}\left[1 + O\left(\frac{1}{n^2}\right)\right].$$

(B) Let $w = W\exp(-Q)$ where Q is an even polynomial of fixed degree with non negative coefficients and $W(x) = |x|^\rho$ for some real $\rho > -1$. Then see [3] and [13],

$$\frac{\alpha_n}{a_n} = \frac{1}{2}\left[1 + O\left(\frac{1}{n}\right)\right].$$

Note that when $\rho \neq 0$, w is not always admissible.

(C) Let $w = w_\alpha$ given by (1.1). Then see [10]

$$\frac{\alpha_n}{a_n} = \frac{1}{2}\left[1 + O\left(\frac{1}{n}\right)\right]$$

(D) Let $m \geq 1$ and $w = w_{2m,1}$ given by (1.2). Then see [3],

$$\frac{\alpha_n}{a_n} = \frac{1}{2}\left[1 + O\left(\frac{T(a_n)}{n}\right)\right].$$

As a consequence of Theorem 1.2, we now state:

Theorem 1.4 *Let w be an admissible Freud weight and assume that the recurrence coefficients α_n satisfy*

$$\frac{\alpha_{n+1}}{\alpha_n} = 1 + O\left(\frac{1}{n}\right), \; n \to \infty. \tag{1.7}$$

Then there exists N_0 and a infinite set of natural numbers Ω such that uniformly for $1 \leq n \leq N_0$ and $n \geq N_0$, $n \in \Omega$,

$$\sup_{x\in(-\infty,\infty)} \left|p_{n+1}(w^2, x) - p_n(w^2, x)\right| w(x) \times$$

$$\times \left\{\left|1 - \frac{|x|}{a_n}\right| + n^{-2/3}\right\}^{-1/4} \sim a_n^{-1/2}. \tag{1.8}$$

Remark 1.5 In [9, Theorem 1.1], (1.8) is established for every $n \geq 1$ assuming in addition to (1.7) the assumption that

$$\frac{\alpha_n}{a_n} = \frac{1}{2}\left[1 + O\left(\frac{1}{n^{2/3}}\right)\right], \; n \to \infty. \tag{1.9}$$

For the Hermite weight it appears in earlier papers of Askey and Waigner, see [1] and Muckenhoupt, see [14] and [15]. The importance of (1.8) lies in the fact that for $|x|$ close to a_n, it improves the well known bound (see [11]):

$$\sup_{x\in(-\infty,\infty)} \left|p_n(w^2,x)\right| w(x) \left\{\left|1-\frac{|x|}{a_n}\right| + n^{-2/3}\right\}^{1/4} \sim a_n^{-1/2} \tag{1.10}$$

by a factor of 1/4 as it should. Its main application is to orthogonal expansions. More precisely, given an admissible Freud weight, w, we may form an orthonormal expansion

$$f(x) \to \sum_{j=0}^{\infty} b_j p_j(x),\ b_j := \int_{(-\infty,\infty)} f p_j w,\ j \geq 0$$

for any measurable function $f : (-\infty,\infty) \to (-\infty,\infty)$ for which

$$\int_{(-\infty,\infty)} |f(x)x^j| w(x)dx < \infty,\ j = 1, 2, 3...$$

To obtain sufficient conditions for weighted mean convergence, it is well known that one needs bounds such as in Theorem 1.4 and bounds for the weighted Hilbert transform $H[f.]$. Here we recall that

$$H[f](x) := \lim_{\varepsilon\to 0^+} \int_{|x-t|\geq\varepsilon} \frac{f(t)}{x-t} dt$$

and provided $b < 1 - 1/p$, $B > -1/p$, $b \leq B$,$1 < p < \infty$,

$$||H[f]u_b||_{L_p((-\infty,\infty))} \leq C||fu_B||_{L_p((-\infty,\infty))}$$

where $u_b(x) := (1 + |x|)^b$ and C is independent of f. Applying Theorem 1.2 and the methods of [14], [15] and [9], we obtain:

Theorem 1.6 *Let w be admissible, $1 < p < \infty$, $b, B \in (-\infty,\infty)$ and assume that*

$$b < 1 - 1/p,\ B > -1/p,\ b \leq B$$

and (1.5) holds. Then there exists a subsequence $n = n_j$ of natural numbers such that for $j \geq C$

$$||s_n[f]wu_b||_{L_p((-\infty,\infty))} \leq C||fwu_B||_{L_p((-\infty,\infty))} \tag{1.11}$$

which implies that

$$\lim_{j\to\infty} ||(s_n[f] - f)wu_b||_{L_p((-\infty,\infty))} = 0.$$

The remainder of this paper is devoted to the proofs of Theorem 1.2, Theorem 1.4 and a short section on uniform convergence of orthonormal expansions in light of recent work of the author and K. Diethelm, see [4].

§2. Proofs of Theorems 1.2 and 1.4

In this section, we prove Theorems 1.2 and 1.4.

We begin with:

The Proof of Theorem 1.2 Let $n \geq C$, and define

$$m = m(n) = [n^{1/3}(T(a_n))^{1/3}]$$

where $[x]$ denotes the greatest integer $\leq x$. It follows using the method of [2, Lemma 2.1b] that uniformly for $r = 1, ..., m$ and $n \geq 1$,

$$T(a_{n+r}) \sim T(a_n).$$

Armed with this identity, we apply (1.5) repeatedly and deduce that there exists $N = N(m)$ such that for $n \geq N$,

$$\alpha_{n+r} \geq \left(1 - \frac{D}{(nT(a_n))^{2/3}}\right)\alpha_n,\ r = 1, 2, ..., m. \tag{2.1}$$

Here $D > 0$ does not depend on n or m so we fix it. Now set

$$\varepsilon = \varepsilon(n) = \frac{D}{(nT(a_n))^{2/3}}.$$

A careful adaption of the proof of [7, Theorem 6] then shows that

$$x_{1,n}(w) \geq 2(1-\varepsilon)\alpha_n \cos\frac{\pi}{m+1}$$

$$\geq 2(1-\varepsilon n)\alpha_n(1 - C/m^2)$$

$$\geq 2\left(1 - \frac{C}{(nT(a_n))^{2/3}}\right)\alpha_n\left(1 - \frac{C}{(nT(a_n))^{2/3}}\right). \tag{2.2}$$

Now recall that

$$\left|\frac{x_{1,n}}{a_n} - 1\right| = O\left(\frac{1}{n^{2/3}T(a_n)^{2/3}}\right). \tag{2.3}$$

Thus (2.2) and (2.3) give:

$$\frac{\alpha_n}{a_n} \leq \frac{1}{2}\left[1 + O\left(\frac{1}{(nT(a_n))^{2/3}}\right)\right], \qquad n \geq C. \tag{2.4}$$

Another careful inspection of [7, Theorem 7], reveals that we have

$$\lim_{n\to\infty} \frac{x_{1,n}}{\max_{k\leq n}\alpha_k} \leq 2.$$

Thus we may apply this with (2.3) and obtain:

$$\frac{\max_{k\le n}\alpha_k}{a_n}\ge\frac{1}{2}\left(1-\frac{1/C}{(nT(a_n))^{2/3}}\right). \tag{2.5}$$

Choosing an increasing sequence n_r with

$$\max_{k\le n_r}\alpha_k=\alpha_{n_r} \tag{2.6}$$

and applying (2.4)-(2.6), we obtain the theorem.

We now present the:

Proof of Theorem 1.4 We sketch the important ideas of the proof. The remaining technical details are very similar to [9, Theorem 1.1] and so we refer to reader to that paper for these.

Step 1: We reduce the proof to one important case.

(a) By symmetry we may assume that $x\ge 0$.

(b) It suffices to prove (1.8) for n sufficiently large.

(c) Using infinite finite inequalities and well known approximation arguments, see [9, The Proof of Theorem 1.1], it suffices to prove (1.8) for

$$0\le x\le a_n\left(1-\frac{C_1}{n^{2/3}}\right)$$

for some $C_1>0$.

(d) For $0\le x\le a_{n/2}$,

$$\left|1-\frac{|x|}{a_n}\right|+n^{-2/3}\sim 1$$

so the result follows easily from (1.10).

Step 2 Without loss of generality we may thus assume that

$$a_{n/2}\le x\le a_n\left(1-\frac{C_1}{n^{2/3}}\right).$$

for some $C_1>0$ which will be chosen later. Define for $y\ge 0$

$$\tau_n(y):=\frac{1}{\alpha_n^2}\sum_{k=0}^{n-1}(\alpha_{k+1}^2-\alpha_k^2)p_k^2(y).$$

The Dombrowski-Fricke identity, see [6], gives for $0\le y\le 2\min\{\alpha_n,\alpha_{n+1}\}$,

$$|p_{n+1}(y)-p_{n-1}(y)|w(x)\le\left\{2\tau_{n+1}(y)w^2(y)\right\}^{1/2}+\left\{2\tau_n(y)w^2(y)\right\}^{1/2}. \tag{2.7}$$

Applying Theorem 1.2, we deduce that there exists $D > 0$ and an infinite set of natural numbers Ω such that (2.7) holds for all

$$0 \leq y \leq a_n \left(1 - Dn^{-2/3}\right), \; n \in \Omega.$$

Choose $C_1 = D$ in the above and assume as we may that (2.7) holds for x and $n \in \Omega$.

Step 3 Estimation of $\tau_n(x)$, $n \in \Omega$: We write

$$\tau_n(x) := \frac{1}{\alpha_n^2} \left\{ \sum_{k=0}^{[n/4]} + \sum_{k=[n/4]+1}^{n-1} \right\} (\alpha_{k+1}^2 - \alpha_k^2) p_k^2(y)$$

$$= \tau_{n,1}(x) + \tau_{n,2}(x).$$

Now applying (1.7), we have that

$$\tau_{n,2}(x) w^2(x) \leq C \frac{1}{n} w^2(x) \lambda_n(x)^{-1}$$

where λ is the Christoffel function for w^2. But for this range of x, it is well known that

$$\lambda_n(x)^{-1} w^2(x) \sim \frac{n}{a_n} \sqrt{\left|1 - \frac{|x|}{a_n}\right| + n^{-2/3}}.$$

Thus

$$\tau_{n,2}(x) w^2(x) \leq C a_n^{-1} \sqrt{\left|1 - \frac{|x|}{a_n}\right| + n^{-2/3}}.$$

Moreover, using infinite-finite range inequalities, see [9, Theorem 1.1], yields that

$$\tau_{n,1}(x) w^2(x) = O\left(\exp(-Cn)\right).$$

Combining the above estimates for both $\tau_{n,1}$ and $\tau_{n,2}$ then yield the theorem.

§3. Uniform convergence of weighted orthonormal expansions for Freud weights

We close with a brief discussion on the extension of Theorem 1.6 to $p = \infty$. Define for continuous $f : (-\infty, \infty) \to (-\infty, \infty)$

$$L_{\infty,w} := \left\{ f : \lim_{|x| \to \infty} fw(x) = 0 \right\}$$

and

$$L'_{\infty,w} := \left\{ f \in L_{\infty,w} : \|fw\|_{L_\infty(I)} + \|f'w\|_{L_\infty(I)} < \infty \right\}.$$

Then it follows from [4, Theorem 1.1] that $H[\cdot; w^2]$ is a bounded map from $L'_{\infty,w}$ to $L_{\infty,w}$. Moreover, it is also well known that

$$S_n[f] = \alpha_n \left\{ p_n(x) H[fp_{n-1}](x) - p_{n-1} H[fp_n] \right\}.$$

Using these two tools above, the author is able to study uniform convergence of orthogonal expansions for Freud weights and prove analogous results to Theorem 1.6. These results will appear in a forthcoming paper.

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