Converse and Smoothness Theorems for Erdős Weights in $L_p (0 < p \leq \infty)$

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Abstract

We prove Converse and Smoothness theorems of polynomial approximation in weighted $L_p$ spaces with norm $\|fW\|_{L_p(\mathbb{R})} (0 < p \leq \infty)$ for Erdős weights on the real line. In particular we prove characterization theorems involving Realization functionals and thereby establish some interesting properties of our weighted modulus of continuity.

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1 Introduction and Statement of Results

Let $W := \exp (-Q)$ where $Q : \mathbb{R} \to \mathbb{R}$ is even and is of faster than polynomial growth at infinity. Then $W$ is called an Erdős weight.

Archetypal examples of such weights are:

(a) $W_{k, \alpha}(x) := \exp (-\exp_k(|x|^\alpha)) \quad \alpha > 1, \ k \geq 1 \quad (1.1)$
where $\exp_k ( ) = \exp (\exp (\ldots (\exp ( ))))$ denotes the $k$th iterated exponential.

(b) \[
W_{A,\beta}(x) := \exp \left( - \exp \left( \log \left( A + x^2 \right)^\beta \right) \right)
\]

where $\beta > 1$ and $A$ is large enough.

For more on the subject, we refer the reader to [16,18] and the references cited therein.

Recently, we investigated Jackson theorems for large classes of Erdős weights in $L_p (0 < p \leq \infty)$ [2]. More precisely, we estimated how fast

\[
E_n[f]_{W,p} := \inf_{P \in \mathcal{P}_n} \| (f - P) W \|_{L_p(\mathbb{R})} \to 0, \; n \to \infty.
\]

Here $E_n[f]_{W,p}$ is the error of best weighted approximation for suitable $f : \mathbb{R} \to \mathbb{R}$ and $\mathcal{P}_n$ denotes the class of polynomials of degree at most $n$.

Direct and converse theorems for rates of approximation is an extensively researched and widely studied subject. For weights on $\mathbb{R}$, analogues of Jackson-Bernstein theorems were initiated by Dzrbasjan, but were more intensively studied by Freud in the 1960’s – 1970’s [10,11,23]. Since then, their ideas have been generalized and extended by many. See [2,7,8,9,12,19] and the references cited therein.

In this paper, we investigate converse theorems of polynomial approximation for Erdős weights. To state our results, we need a suitable class of weights and various quantities.

Throughout, $C, C_1, C_2, \ldots$, will denote positive constants independent of $n, x$ and $P \in \mathcal{P}_n$ not necessarily the same in different occurrences. We write $C \neq C(L)$ to mean that the constant is independent of $L$.

Moreover, for real sequences $A_n$ and $B_n \neq 0$, $A_n = O(B_n)$, $A_n \sim B_n$ and $A_n = o(B_n)$ will mean respectively that there exist constants $C_1, C_2, C_3 > 0$ independent of $n$ such that $A_n/B_n \leq C_1$, $C_2 \leq A_n/B_n \leq C_3$ and $\lim_{n \to \infty} |A_n/B_n| = 0$. Similar notation will be used for functions and sequences of functions.

We shall say that a function

\[
f : (a, b) \to (0, \infty)
\]
is quasi-increasing if \( \exists C > 0 \) such that
\[
a < x < y < b \implies f(x) \leq Cf(y).
\]

We need a suitable class of weights:

**Definition 1.1.** Let \( W(x) := \exp [-Q(x)] \) where \( Q : \mathbb{R} \to \mathbb{R} \) is even and continuous satisfying,

(a) \( x Q'(x) \) is strictly increasing in \( (0, \infty) \) with
\[
\lim_{x \to 0^+} x Q'(x) = 0.
\]

(b) The function
\[
T(x) := \frac{x Q'(x)}{Q(x)}
\]

is quasi-increasing in \( (C, \infty) \) for some \( C > 0 \) and
\[
\lim_{x \to \infty} T(x) = \infty.
\]

(c) Assume
\[
\frac{y Q'(y)}{x Q'(x)} \leq C_1 \left( \frac{Q(y)}{Q(x)} \right)^{C_3} y \geq x \geq C_2.
\]

for some \( C_1, C_2, C_3 > 0 \). Then \( Q \) is called the external field associated with \( W \) and we write \( W \in \mathcal{E}_1 \).

**Some Remarks.**

(a) The function \( T \) serves as a measure of the regularity of growth of \( Q \). In particular, it is not difficult to show that (1.4) forces \( Q \) to be of faster than polynomial growth at infinity.

(b) We need the condition that \( x Q'(x) \) be strictly increasing in order to ensure the existence of the Mhaskar-Rakhmanov-Saff number, \( a_u \) defined as the positive root of the equation
\[
u = \frac{2}{\pi} \int_0^1 \frac{a_u t Q'(a_u t) \, dt}{\sqrt{1 - t^2}} \quad u > 0.
\]
For those unfamiliar, the quantity $(PW), P \in \mathcal{P}_n$ “lives” most of the time in $[-a_n, a_n]$. We refer the interested reader to [17,21,26] for more on $a_n$ and its “cousin” $q_n$, the Freud number. For a different perspective on discrete sets and to concave external fields, we refer the reader to [4,5]. For Erdős weights, $a_n$ has the effect that although $Q(x)$ might grow very rapidly for large $x$, $Q(a_u)$ does not exceed a positive power of $u$. For example, for $W_{k,a}$, $a_u$ grows like $(\log u)^{\frac{1}{3}}$ where $\log_\kappa( ) = \log (\log (\ldots (\log( ))))$ denotes the $k$th iterated logarithm.

(c) (1.5) is a weak regularity condition on $T$, for one has typically for each $\varepsilon > 0$,

$$T(x) = O(\log Q'(x))^{1+\varepsilon}, \quad x \to \infty.$$  \hspace{1cm} (1.7)

For example, for $W_{k,a}(x)$,

$$T(x) = \alpha x^\alpha \left[ \prod_{j=1}^{k-1} \exp_j(x^\alpha) \right],$$

so that $C_3$ can be made arbitrarily close to 1. This is also the case for $W_{A,\beta}$.

We proceed to define our modulus of continuity and realization functional as in [1,2,3].

For $h > 0$, an interval $J$, $r \geq 1$ and $f : \mathbb{R} \to \mathbb{R}$ we define

$$\Delta_h^r(f, x, J) := \left\{ \begin{array}{ll} \sum_{i=0}^{r} \left( \begin{array}{l} r \vspace{0.1cm} \end{array} \right) (-1)^i f \left( x + \frac{rh}{2} - ih \right), & x \pm \frac{rh}{2} \in J \\
0, & \text{otherwise} \end{array} \right\}$$  \hspace{1cm} (1.8)

to be the $r$th symmetric difference of $f$. If $J$ is not specified, it can be taken as $\mathbb{R}$.

Following ideas of [9], to reflect endpoint effects in our approximation, we need our increment $h$ in (1.8) to depend on $x$ and in particular on the function,

$$\Phi_{t}(x) := \left[ 1 - \frac{|x|}{\sigma(t)} \right]^\frac{1}{2} + T(\sigma(t))^{-\frac{1}{2}}, \quad x \in \mathbb{R},$$

where

$$\sigma(t) := \inf \left\{ a_u : \frac{a_u}{u} \leq t \right\}$$  \hspace{1cm} (1.10)
and $t > 0$ but is typically small enough.

An easy way to understand $\sigma$ is to see it as the inverse of the map

$$u : \longrightarrow \frac{a_u}{u}$$

which decays to zero as $u \longrightarrow \infty$. Clearly $\sigma$ is decreasing.

We may then define our weighted modulus of continuity for $0 < p \leq \infty$ and $r \geq 1$ by:

$$w_{r,p}(f, W, t) := \sup_{0 < h \leq t} \left\| W \left( \Delta_h^r \Phi_t(x) (f) \right) \right\|_{L_p(|x| \leq \sigma(2t))} + \inf_{R \text{ of } \deg \leq r-1} \left\| (f - R)W \right\|_{L_p(|x| \geq \sigma(4t))}.$$  \hspace{1cm} (1.11)

Further, we define its averaged “cousin”,

$$\overline{w}_{r,p}(f, W, t) := \left( \frac{1}{t} \int_0^t \left\| W \left( \Delta_h^r \Phi_t(x) (f) \right) \right\|_{L_p(|x| \leq \sigma(2t))} dh \right)^{\frac{1}{p}} + \inf_{R \text{ of } \deg \leq r-1} \left\| (f - R)W \right\|_{L_p(|x| \geq \sigma(4t))}.$$  \hspace{1cm} (1.12)

(if $p = \infty$ we set $\overline{w}_{r,p} = w_{r,p}$).

Clearly $\overline{w}_{r,p}(f, W, t) \leq w_{r,p}(f, W, t)$.

Some remarks concerning our modulus.

(a) Although at first difficult to assimilate, we see that the definition of $\sigma$ in (1.10) is natural, as at least for purposes of approximation by polynomials of degree $\leq n$, we may think of $t = \frac{a_n}{n}$ (recall $t$ is small) so that $\sigma(t)$ grows like $a_n$. Following [9], our modulus consists of two parts. The “main” part involves $r$th symmetric differences over the interval $[-a_{\frac{n}{2}}, a_{\frac{n}{2}}]$. The “tail” involves an error of weighted polynomial approximation over the remainder of $\mathbb{R}$ and is necessary because of the inability of $(P_n W)$ to approximate beyond $[-a_{\frac{n}{2}}, a_{\frac{n}{2}}]$. Its presence ensures that at least for $f \in \mathcal{P}_{r-1}$,

$$w_{r,p}(f, W, t) \equiv 0.$$

For converse saturation type results, we refer the reader to [3].
(b) We note that the function $\Phi_t$ describes the improvement in the degree of approximation near $\pm a_\frac{1}{2}$, in much the same way that $\sqrt{1-x^2}$ does for weights on $[-1, 1]$.

(c) We observe that unlike the moduli in [8,9], our modulus $w$ is not necessary monotone increasing in $t$. This created severe difficulties in our analysis. The results of [2] show that under additional assumptions on $W$ it is possible to replace our modulus by one that is increasing in $t$ however for $E$ this is an open question.

In [2], we proved the following Jackson theorems:

**Theorem 1.2.** Let $W \in E_1$, $r \geq 1$ and $0 < p \leq \infty$. Then for all $f : \mathbb{R} \to \mathbb{R}$ for which $fW \in L_p(\mathbb{R})$ (and for $p = \infty$, we require $f$ to be continuous, and $fW$ to vanish at $\pm \infty$), we have for $n \geq C$,

$$ E_n[f]_{w,p} \leq C_1w_{r,p} \left(f, W, C_2\frac{a_n}{n}\right) \leq C_1w_{r,p} \left(f, W, C_2\frac{a_n}{n}\right) \quad (1.13) $$

where the $C_j$ $j = 1, 2$ are independent of $f$ and $n$.

Moreover, given $\lambda(n) \in \left[\frac{1}{5}, 1\right]$.

$$ E_n[f]_{w,p} \leq C_1w_{r,p} \left(f, W, C_2\lambda(n)\frac{a_n}{n}\right) \leq C_1w_{r,p} \left(f, W, C_2\lambda(n)\frac{a_n}{n}\right). \quad (1.14) $$

**Some remarks**

(a) The result above indicated a Nikolskii-Timan-Brudnyi effect whereby, as in weights on $[-1, 1]$, we have better approximation towards the endpoints of the Mhaskar-Rakhmanov-Saff interval.

(b) We remark that with a little extra effort, we may replace $C$ in $(1.13)$ by $r - 1$ (cf. [3]).

In establishing our converse theorems, we need the notion of the K-functional. While K-Functionals were introduced in the context of interpolation of spaces, one of their most important applications has been in the analysis of moduli of continuity, and in converse theorems in approximation
theory. J. Peetre first made the connection between his K-Functional and the modulus of continuity in 1968. His ideas have been generalized and extended by many including Ditzian, Freud, Hristov, Ivanov, Lubinsky, Mhaskar and Totik. We refer the reader to [8,9,10,11,12] and the references cited therein.

The Ditzian-Totik r-th order K-Functional has the form

$$K_{r,p}^*(f, W, t') := \inf_{g^{(r-1)} \text{ locally absolutely continuous}} \left\{ \| (f - g) W \|_{L_p(\mathbb{R})} + t' \| g^{(r)} W \|_{L_p(\mathbb{R})} \right\}.$$  \hfill (1.15)

Here, $t > 0$, $r \geq 1$ and $p \geq 1$.

We may think of the second term of (1.15) measuring the smooth part of $f$ and the first part measuring the distance of $f$ to that smooth part [9]. The idea, following a general technique of Ditzian, Hristov and Ivanov [9], is to prove inequalities of the form

$$w_{r,p}^*(f, W, \alpha t) \leq C_1 K_{r,p}^*(f, W, t') \leq C_2 w_{r,p}^*(f, W, t)$$  \hfill (1.16)

for a suitable modulus $w_{r,p}^*(f, \cdot)$. Here $\alpha > 0$ is fixed in advance, $C_1, C_2 > 0$, and $t$ is small enough.

Unfortunately, $K^* \equiv 0$ in $L_p$ ($0 < p < 1$) [7], so we need the notion of a realization functional, a concept attributed to Hristov and Ivanov. Our realization functional has the form:

$$K_{r,p} (f, W, t') := \inf_{P \in P_n} \left\{ \| (f - P) W \|_{L_p(\mathbb{R})} + t' \| P^{(r)} \Phi^r_t W \|_{L_p(\mathbb{R})} \right\},$$  \hfill (1.17)

where $t > 0$, $0 < p \leq \infty$, and $r \geq 1$ are chosen in advance and

$$n = n(t) := \inf \left\{ k : \frac{a_k}{k} \leq t \right\}.$$  \hfill (1.18)

Further define the ordinary K-Functional by

$$K_{r,p}^*(f, W, t') := \inf_{g^{(r-1)} \text{ locally absolutely continuous}} \left\{ \| (f - g) W \|_{L_p(\mathbb{R})} + t' \| g^{(r)} W \|_{L_p(\mathbb{R})} \right\}.$$  \hfill (1.19)
We begin with our main equivalence result:

**Theorem 1.3.** Let $W \in \mathcal{E}_1$, $L, \alpha > 0$, $r \geq 1, 0 < p \leq \infty$ and $f$ as in Theorem 1.2. Assume that there is a Markov-Bernstein inequality of the form

$$\left\| R_n \Phi_{a_n} W \right\|_{L_p(\mathbb{R})} \leq C \frac{n}{a_n} \left\| R_n W \right\|_{L_p(\mathbb{R})} 0 < p \leq \infty, \ R_n \in \mathcal{P}_n, \quad (1.20)$$

where $C \neq C(n, R_n)$. Then $\exists C_1, C_2, C_3 > 0$ independent of $f$ and $t$ such that for $t \in (0, t_0)$,

(a) $w_{r,p} (f, W, Lt) \leq C_1 K_{r,p} (f, W, t') \leq C_2 w_{r,p} (f, W; C_3 t)$. \quad (1.21)

Moreover, uniformly for $t$ and $f$,

(b) $w_{r,p} (f, W, t) \sim \overline{w}_{\kappa,p} (f, W, t) \sim K_{r,p} (f, W; t') \quad (1.22)$

and

(c) $w_{r,p} (f, W, \alpha t) \sim w_{r,p} (f, W, t)$. \quad (1.23)

Note that the constant in the $\sim$ relation (1.23) depends on $\alpha$. For the exact dependence, we refer the interested reader to [3].

**Remark.**

(a) The Markov inequality (1.20) is true for $W \in \mathcal{E}_1$ and its proof can be found for example in the forthcoming book of Levin and Lubinsky [15]. For this reason, we dispense with the proof here and assume the result. We refer the interested reader to [8, 19] where similar assumptions were made.

(b) (1.20) was proved for $p = \infty$ in [18] and for $0 < p < \infty$ in [20] under additional conditions on $Q$, namely conditions on $Q''$ which are satisfied for $W_{k,\alpha}$ and $W_{A,\beta}$ given by (1.1) and (1.2).
(c) We finally note that for $p \geq 1$, the methods of [9] should enable one to avoid assuming (1.20) altogether. However, as it is needed in the later Corollaries, we do not pursue this idea further here.

Theorem 1.3 allows us to deduce a simpler Jackson theorem to Theorem 1.2:

**Corollary 1.4.** Assume the hypotheses of Theorem 1.3. Then we have for $n \geq C_1$,

$$E_n[f]_{W,p} \leq C_2 \tilde{w}_{r,p} \left( f, W, \frac{a_n}{n} \right) \leq C_2 w_{r,p} \left( f, W, \frac{a_n}{n} \right). \quad (1.24)$$

Here, $C_2$ is independent of $f$ and $n$.

We note that the point of this Corollary is that we have removed the constant from inside the modulus in (1.13) and (1.14).

We have the following converse theorems:

**Theorem 1.5.** Assume the hypotheses of Theorem 1.3. Let $q = \min \{1, p\}$. For $0 < t < C$, determine $n = n(t)$ by (1.18) and let $l = \lceil \log_2 n \rceil = \text{the largest integer} \leq \log_2 n$. Then we have,

$$w_{r,p}(f, W, t) \leq C_1 t^r \left[ \sum_{k=-1}^{l} \left( l - k + 1 \right) \frac{q^k}{a_{2k}} \right]^{\frac{r}{2}} \frac{E_{2k}[f]_{W,p}}{2^{k-1}}. \quad (1.25)$$

where $C_1 \neq C_1(f, t)$ and where we set $E_{2-1} = E_{20}$.

We deduce

**Corollary 1.6.** Assume the hypotheses of Theorem 1.3. Then for every $0 < \alpha < r$ the following are equivalent:

(a) $$w_{r,p}(f, W, t) = O(t^\alpha), \quad t \rightarrow 0^+. \quad (1.26)$$

(b) $$K_{r,p}(f, W, t^r) = O(t^\alpha), \quad t \rightarrow 0^+$$
Remark. We remark that a different characterization appears in [3] where \( \alpha \) is allowed to equal \( r \).

Finally, we obtain estimates of our modulus in terms of \( f^{(r)} \) and deduce the equivalence of the K-Functional with the realization functional for \( p \geq 1 \).

**Corollary 1.7.** Let \( 1 \leq p \leq \infty \) and assume the hypotheses of Theorem 1.3.

(a) If \( f^{(r)} W \in L_p(\mathbb{R}) \), we have for \( t \in (0, C_2) \),

\[
 w_{r,p}(f, W, t) \leq C_1 t^r \left\| f^{(r)} \Phi_y W \right\|_{L_p(\mathbb{R})},
\]

Here \( C_j \neq C_j(f, t), \, j = 1, 2 \).

(b) We have for \( t \in (0, C_3) \),

\[
 1 \leq K_{r,p}^*(f, W, t)/K_{r,p}(f, W, t) \leq C_4.
\]

Here \( C_j \neq C_j(f, t), \, j = 3, 4 \).

Remark. We remark that (1.28) is false for \( 0 < p < 1 \). Indeed set for \( \varepsilon \in \left(0, \frac{1}{2}\right) \)

\[
 f_\varepsilon(x) :=
 \begin{cases} 
 0, & x \in [-1, 0] \\
 \varepsilon^{-1} x, & x \in (0, \varepsilon] \\
 1, & x \in (\varepsilon, 1]
 \end{cases}
\]

Then \( f^W \in L_p(0 < p < 1) \), \( f \) is of compact support and so it is easy to see that for fixed \( t > 0 \), there exists \( C = C(t, W) > 0 \) such that

\[
 w_{r,p}(f_\varepsilon, W, t) > C
\]

and

\[
 \left\| f_\varepsilon' \Phi_y W \right\|_{L_p(\mathbb{R})} \longrightarrow 0, \, \varepsilon \longrightarrow 0^+.
\]
An important note on the structure of this paper.

Sections 2 and 3, establish some machinery, required for the entire paper. This includes, in particular, an extension of the Markov-Bernstein inequality (1.20). Many of the proofs are technical and serve merely as tools for the proofs of our main results. Thus, we suggest the reader skip these sections at first and return to them at the end of the paper. In Section 4, we prove a theorem required for the lower bound in Theorem 1.3, whereby we approximate polynomials of degree \( n, n \geq 1 \) by those of degree \( r - 1, r \geq 1 \). This technique, although similar to that used in [8], is new for Erdős weights on \( \mathbb{R} \) and \([-1, 1]\) and we believe it to be of independent interest. In Section 5 we prove Theorem 1.3 and Corollary 1.4 and in Section 6 we prove Theorem 1.5 and Corollaries 1.6 and 1.7.

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2 Technical lemmas

Lemma 2.1. Let \( W \in \mathcal{E}_1 \). Then

(a) Given \( A \geq 0 \), the functions \( Q'(u)u^{-A} \) and \( Q(u)u^{-A} \) are quasi-increasing and increasing respectively for large enough \( u \).

(b) \( a_u \) is uniquely defined for \( u \in (0, \infty) \). Furthermore, it is a strictly increasing function of \( u \).

(c) We have for \( u \) large enough and \( \alpha > 0 \)

\[
\begin{align*}
\text{(i)} & \quad a_u Q'(a_u) \sim u T'(a_u)^{\frac{3}{2}}. \\
\text{(ii)} & \quad Q(a_u) \sim u T(a_u)^{-\frac{1}{2}}. \\
\text{(iii)} & \quad T(a_{\alpha u}) \sim T(a_u). \\
\text{(iv)} & \quad Q(a_{\alpha u}) \sim Q(a_u). \\
\text{(v)} & \quad Q'(a_{\alpha u}) \sim Q'(a_u).
\end{align*}
\]

\hspace{1cm} (2.1)
(d) If $\alpha > 1$ we have
\[
\left| \frac{a_{\alpha u}}{a_u} - 1 \right| \sim T(a_u)^{-1}
\]
from which it follows in particular that $\forall \beta > 0,$
\[
\frac{a_{\beta u}}{a_u} \to 1, \quad u \to \infty.
\]  

(e) For some $C_j, \ j = 1, 2, 3$ and $s \geq r \geq C_3$
\[
\left( \frac{s}{r} \right)^{C_j r} \leq \frac{Q(s)}{Q(r)} \leq \left( \frac{s}{r} \right)^{C_j T(s)}.
\]

(f) There exists $\epsilon > 0$ such that
\[
T(a_u) = O \left( u^{(2-\epsilon)} \right).
\]
Moreover, $\forall \delta > 0$
\[
a_u = o \left( u^{\delta} \right), \quad u \to \infty.
\]

(g) $\exists C_j, \ j = 1, 2, 3$ such that for $v \geq u \geq C_3$
\[
\left( \frac{a_v}{a_u} \right) \leq C_1 \left( \frac{v}{u} \right)^{\frac{C_3}{T_u}}.
\]
and
\[
\left( \frac{a_v}{v} \right) \left/ \left( \frac{a_u}{u} \right) \right. \leq C_1 \left( \frac{v}{u} \right)^{\frac{C_3}{T_u}}^{-1}.
\]
In particular, given $\epsilon > 0$, we have for $v \geq u \geq C_3$
\[
\left( \frac{a_v}{a_u} \right) \leq C_1 \left( \frac{v}{u} \right)^{\epsilon},
\]
\[
\left( \frac{a_v}{v} \right) \left/ \left( \frac{a_u}{u} \right) \right. \leq C_1 \left( \frac{v}{u} \right)^{\epsilon-1}.
\]
Proof. Firstly, (a), (b), (c) [(i) – (iii)], (2.3), (2.4), (2.5) and (2.6) are part of Lemmas 2.1 and 2.2 in [2]. The rest of (2.2) follows from (2.1). (2.7) will follow using [(a)], as given $A > 0$

\[ C (a_u)^4 \leq Q (a_u) \sim u T (a_u)^{-\frac{3}{2}} \]

\[ \implies \frac{(a_u)^4}{u} \to 0, \quad u \to \infty. \]

It remains to show (g). Now by (2.1) and then (2.5)

\[ C_1 \frac{v}{u} \geq \frac{v T (a_u)^{\frac{3}{2}}}{u T (a_u)^{\frac{2}{3}}} \sim \frac{Q (a_u)}{Q (a_u)} \geq \left( \frac{a_v}{a_u} \right)^{C_2 T (a_u)}, \]

which implies

\[ \left( \frac{a_v}{a_u} \right) \leq C_3 \left( \frac{v}{u} \right)^{\frac{C_4}{T (a_u)}}. \]

So we have (2.8) and then (2.9 – 2.11) also follow. □

Lemma 2.2. Let $W \in \mathcal{E}_1$.

(a) Let $t > 0$ be small enough. Then there exists, $u$ such that

\[ t = \frac{a_u}{u}. \quad (2.12) \]

(b) Let $\epsilon > 0$. Then for $u$ large enough

\[ \sigma \left( \frac{a_u}{u} \right) = a_{v(u)}, \quad (2.13) \]

where

\[ u \left( 1 - \epsilon \right) \leq v(u) \leq u. \]

(c) Let $a > 1$. There exists $C_1, C_2 > 0$ such that for $\frac{t}{a} \leq t \leq s$ and $s \leq C_1$

\[ 1 \leq \frac{\sigma(t)}{\sigma(s)} \leq 1 + \frac{C_2}{T (\sigma(t))}. \quad (2.14) \]

Further, for $t$ small enough, we have for some $\epsilon > 0$,

\[ T (\sigma(t)) = O \left( \frac{\sigma(t)}{t} \right)^{2-\epsilon}. \quad (2.15) \]

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(d) Recall the definition (1.9) and let $\beta \in (0, \infty)$. Then we have for some $C_1 > 0$ and $\forall x \in \mathbb{R}$
\[ \Phi_t^\beta(x) \geq C_1 T(\sigma(t))^{-\frac{\beta}{2}}. \tag{2.16} \]
Further if $m \leq n$ and $n$, $m \geq C_2$, then
\[ \sup_{x \in \mathbb{R}} \frac{\Phi_{an}^\beta(x)}{\Phi_{mn}^\beta(x)} \leq C_3 \sqrt{\log \left( \frac{2 + n}{m} \right)}. \tag{2.17} \]
for some $C_3 > 0$ independent of $n$, $m$ and $x$.

(e) Given $a > 1$, there exists $C_1 > 0$ independent of $s$, $t$ and $x$ such that for $0 < s < C_1$ and $\frac{s}{a} \leq t \leq s$
\[ \Phi_s(x) \sim \Phi_t(x), \ x \in \mathbb{R}. \tag{2.18} \]

(f) Uniformly for $n \geq 1$ and $x \in \mathbb{R}$,
\[ \Phi_{an}^\beta(x) \sim \sqrt{1 - \frac{|x|}{a_n}} + T(a_n)^{-\frac{\beta}{2}}. \tag{2.19} \]
Further given $\beta > 0$, we have for some $C_1 > 0$ and for all $x \in \mathbb{R}$,
\[ \Phi_{an}^\beta(x) \geq C_1 T(a_n)^{-\frac{\beta}{2}}. \tag{2.20} \]

**Proof.** (2.16) follows from the definition of $\Phi_t$. (2.12), (2.13), (2.14), (2.17), (2.18) and (2.19) are part of Lemmas 3.1 and 7.1 in [2]. (2.20) follows from (2.19). Finally to prove (2.15), we may by (2.12) put $t = \frac{a_n}{u}$ for some $u \geq u_\omega$. Then using Lemma 2.1 (b), (2.13) and (2.6) gives for some $\varepsilon > 0$
\[ T(\sigma(t)) \leq T(a_n) = O \left( u^{2-\varepsilon} \right) = O \left( \frac{\sigma(t)}{t} \right)^{\frac{2-\varepsilon}{2}}. \]
We have an infinite-finite range inequality:

**Lemma 2.3.** Let $W \in \mathcal{E}_1$, $0 < p \leq \infty$ and $s > 1$. Then for some $C_1, C_2, C_3 > 0$ and $\forall P \in \mathcal{P}_n$, $n \geq 1$,
\( \|PW\|_{L_p(\mathbb{R})} \leq C_1 \|PW\|_{L_p(-a_{sn}, a_{sn})}. \) (2.21)

\[ \|PW\|_{L_p(|x| \geq a_{sn})} \leq C_2 \exp \left[ -C_3 nT(a_n)^{-\frac{1}{2}} \right] \|PW\|_{L_p(-a_{sn}, a_{sn})}. \] (2.22)

**Proof.** This is Lemma 2.3 in [2]. \( \square \)

Note that (2.6) shows that for large \( n \),

\[ nT(a_n)^{-\frac{1}{2}} \geq nC_3, \text{ some } C_3 > 0. \]

**Lemma 2.4.** Let \( W \in \mathcal{E}_1 \), \( t \in (0, t_0) \) and \( \beta > 0 \). Put for \( u \) large enough

\[ t = \frac{\beta a_u}{u}. \]

Set

\[ n := n(t) = \inf \left\{ k : \frac{a_k}{k} \leq \frac{\beta a_u}{u} \right\}. \] (2.23)

Then

(a)

\[ \frac{a_n}{n} \leq \frac{\beta a_u}{u} < \frac{a_{n-1}}{n-1}. \] (2.24)

(b)

\[ \frac{a_n}{n} \leq \frac{\beta a_u}{u} < 2 \frac{a_n}{n}. \] (2.25)

(c)

\[ u \sim n. \] (2.26)
Proof. (2.24) follows from the definition of $n$. (2.25) follows from (2.24) as

$$a_{n-1} < a_n.$$  

To show (2.26), we first show that $\exists \alpha > 0$ such that

$$u \leq \alpha n. \quad (2.27)$$

Suppose first that $u \geq n$. Using (2.24) and Lemma 2.1 (g), there exists $C > 0$ such that

$$\frac{1}{\beta} \leq \frac{a_u}{u} / \frac{a_n}{n} \leq C \left(\frac{u}{n}\right)^{-\frac{1}{2}}$$

which implies (2.27). Suppose $u \leq n$. Then (2.27) follows with $\alpha = 1$. So it suffices to show that $\exists C_1 > 0$ such that

$$u \geq C_1 n.$$  

Well, if $n - 1 \geq u$ by (2.24) and Lemma 2.1 (g), there exists $C_2 > 0$ such that

$$\beta \leq \frac{a_{n-1}}{n-1} / \frac{a_u}{u} \leq C_2 \left(\frac{n-1}{u}\right)^{-\frac{1}{2}}$$

which implies

$$u \geq C_3 n$$

for some $C_3 > 0$. Further, if $u \geq n - 1$ we are done. \(\square\)

We now present two lemmas on differences.

Lemma 2.5. Let $W \in \mathcal{E}_1$.

(a) Recall the difference operator $\Delta_\beta^r$ defined by (1.8). Then we have $\forall x \in \mathbb{R}, \forall P \in \mathcal{P}_{r-1}, \ r \geq 1, \ \beta \in \mathbb{R}$ and $t > 0$

(i)\[\Delta_{h\Phi_\beta^r(x)}^r P(x) \equiv 0. \quad (2.28)\]

(ii)\[r! \left(h\Phi_\beta^r(x)\right)^r = \Delta_{h\Phi_\beta^r(x)}^r x^r.\]
(b) Let $L, s > 0$. Then uniformly for $u \geq 1$ and $|x|, |y| \leq a_u s$ such that

$$
|x - y| \leq L \frac{a_u}{u} \sqrt{1 - \left( \frac{|y|}{a_us} \right)} \quad \text{or} \quad |x - y| \leq L \frac{a_u}{u} T (a_u)^{-\frac{1}{2}},
$$

we have

$$
W(x) \sim W(y). \quad (2.29)
$$

(c) Let $L, M > 0$. For $t \in (0, t_0), \ |x|, \ |y| \leq \sigma (Mt)$ such that

$$
|x - y| \leq L t \Phi_t(x)
$$

we have (2.29) and

$$
\Phi_t(x) \sim \Phi_t(y), \quad (2.30)
$$

**Proof.** This is Lemma 3.2 in [2].

**Lemma 2.6.** Let $W \in \mathcal{E}_1, \; 0 < \delta < 1; \; L, \; M > 0 \; \text{and} \; 0 < p \leq \infty$.

(a) Let $s \in (0, 1)$ and $[a, b]$ be contained in one of the ranges

$$
|x| \leq \sigma(t) \left[ 1 - \left( \frac{s}{2\delta \sigma(t)} \right)^2 \right] \quad (2.31)
$$

or

$$
|x| \geq \sigma(t) \left[ 1 + \left( \frac{s}{2\delta \sigma(t)} \right)^2 \right]. \quad (2.32)
$$

Then

$$
\int_a^b |f(x \pm s \Phi_t(x))| dx \leq \frac{2}{1 - \delta} \int_{-\delta}^{\delta} |f(x)| dx \quad (2.33)
$$

where

$$
\left\{ \frac{a}{b} \right\} := \left\{ \inf \sup \right\} \{ x \pm s \Phi_t(x) : x \in [a, b] \}. \quad (2.34)
$$
(b) Let $r \geq 1$, $t \in \left(0, \frac{1}{Mt}\right)$, $h \in (0, Mt)$ and $[a, b]$ be as above with $s = Mr t$. Define $\bar{a}$ and $\bar{b}$ by (2.34) with $s = Mr t$. Assume moreover that

$$[a, b] \subseteq \left[ -\sigma(Lt), \sigma(Lt) \right].$$

(2.35)

Then for some $C \neq C(a, b, t, g)$

$$\left\| \Delta^t_{\Phi_t(x)} \left(g, x, \mathbb{R}\right) W(x) \right\|_{L_p[a, b]} \leq C \inf_{P \in \mathcal{F}_{r-1}} \left\| W(g - P) \right\|_{L_p[\mathbb{R}, \mathbb{R}]} \leq C \left\| W g \right\|_{L_p[\mathbb{R}, \mathbb{R}]}.$$

(2.36)

**Proof.**

(a) Define $\kappa = \pm 1$ and $u(x) := x + \kappa s \Phi_t(x)$.

We shall assume that $[a, b]$ is contained in the range (2.31) and also $a \geq 0$. The case where $a < 0$ is similar, as is the case when $[a, b]$ is contained in the range (2.32). Then for $x \in [a, b]$,

$$u'(x) = 1 + \frac{\kappa s}{2 \sqrt{1 - \frac{x}{\sigma(t)}}} \left( 1 - \frac{1}{\sigma(t)} \right) \geq 1 - \delta,$$

by (2.31). Hence $u$, is increasing in $[a, b]$ and writing $v := u(x)$ gives

$$\int_a^b |f(x \pm s \Phi_t(x))| \, dx = \int_a^b |f(u(x))| \, dx = \int_{u(a)}^{u(b)} |f(v)| \frac{dv}{du} \, dv, \quad v = u(x)$$

$$\leq \frac{1}{1 - \delta} \int_{u(a)}^{u(b)} \frac{dv}{du} \, dv$$

$$= \frac{1}{1 - \delta} \int_{\bar{a}}^{\bar{b}} |f(x)| \, dx$$

in this case. The extra 2 in (2.33) takes care of having to split $[a, b]$ into two intervals if $a < 0 < b$. 

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(b) Now recall that we have
\[ W(x)\Delta^r_{h\Phi_t(x)}(g(x)) = \sum_{i=0}^{r} \binom{r}{i} (-1)^i W(x) g \left( x + \left( \frac{r}{2} - i \right) h\Phi_t(x) \right). \]

Also (2.29) gives
\[ W(x) \sim W \left( x + \left( \frac{r}{2} - i \right) h\Phi_t(x) \right) \]
uniformly in \( i \) and for \( |x| \leq \sigma(Lt) \) and \( h \leq Mt \). Thus we obtain from part [(a)]
\[
\left\| W(x)\Delta^r_{h\Phi_t(x)}(g(x)) \right\|_{L_p[\alpha, \beta]} \\
\leq C \sup_{0 \leq t \leq T} \int_{a}^{b} \left| gW \right|^p \left( x + \left( \frac{r}{2} - i \right) h\Phi_t(x) \right) dx \\
\leq \frac{2C}{1 - \delta} \int_{a}^{b} \left| gW \right|^p (x) dx.
\]

Note that for \( 0 \leq i \leq r \), (2.31) with \( s = Mr \delta \) gives
\[
|x| \leq \sigma(t) \left( 1 - \left[ \frac{Mr}{2\delta \sigma(t)} \right]^2 \right) \\
\leq \sigma(t) \left( 1 - \left[ \frac{ih}{4\delta \sigma(t)} \right]^2 \right)
\]
so the range restrictions of (a) are satisfied.

Finally note that by (2.28) for \( P \in \mathcal{P}_{r-1} \),
\[ \Delta^r_{h\Phi_t(x)}(P, x, \mathbb{R}) \equiv 0. \]

Hence
\[
\left\| \Delta^r_{h\Phi_t(x)}(g, x, \mathbb{R}) W(x) \right\|_{L_p[\alpha, \beta]} = \left\| \Delta^r_{h\Phi_t(x)}(g - P, x, \mathbb{R}) W(x) \right\|_{L_p[\alpha, \beta]} \\
\leq C \left\| (g - P) W \right\|_{L_p[\alpha, \beta]}.
\]

It remains to take the infimum over \( P \). □

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3 A Markov-Bernstein Inequality

In this section, we prove an extension of the Markov-Bernstein inequality (1.20).

**Theorem 3.1.** Let \( W \in E_1 \) and assume (1.20). Let \( 0 < p \leq \infty \) and define for \( n \geq 1 \),

\[
\Psi_n(x) := \left(1 - \left(\frac{x}{a_n}\right)^2\right)^2 + T(a_n)^{-2}, \ x \in \mathbb{R} \tag{3.1}
\]

Then for \( n \geq C_1, 0 \leq l \leq n \) and \( \forall P \in \mathcal{P}_n \) we have,

\[
\left\|P^{(l+1)} \Psi_n^{l+1/4} W\right\|_{L_p(\mathbb{R})} \leq C_2 \left\{ \frac{n}{a_n} + \frac{l}{a_n} T(a_n)^{\frac{3}{2}} \right\} \left\|P^{(l)} \Psi_n^{l/4} W\right\|_{L_p(\mathbb{R})} \tag{3.2}
\]

\[
\leq C_3 \frac{n}{a_n} [l + 1] \left\|P^{(l)} \Phi_{\frac{l}{2}}^{1/4} W\right\|_{L_p(\mathbb{R})}. \tag{3.3}
\]

Here \( C_j \neq C_j (n, l, P) \) \( j = 2, 3, \ldots, 5 \).

We remark that (3.2) and (3.3) will hold with constants depending on \( l \) if we replace \( \Psi_n^{l/4} \) by \( \Phi_{\frac{l}{2}}^{1/4} \).

More precisely,

\[
\left\|P^{(l+1)} \Phi_{\frac{l}{2}}^{l+1/4} W\right\|_{L_p(\mathbb{R})} \leq C_4 \left\{ \frac{n}{a_n} + \frac{l}{a_n} T(a_n)^{\frac{3}{2}} \right\} \left\|P^{(l)} \Phi_{\frac{l}{2}}^{l/4} W\right\|_{L_p(\mathbb{R})} \tag{3.4}
\]

\[
\leq C_5 \frac{n}{a_n} [l + 1] \left\|P^{(l)} \Phi_{\frac{l}{2}}^{l/4} W\right\|_{L_p(\mathbb{R})} \tag{3.5}
\]

where \( C_j \neq C_j (n, P) \) \( j = 4, 5 \).

We need several lemmas.

**Lemma 3.2.** Let \( s > 1 \) and \( n \geq C_1 \). Then there exist polynomials \( R \) of degree \( o(n) \) such that uniformly for \( |x| \leq a_{sn} \)

\[
R(x) \sim \Phi_{\frac{l}{2}}^{l/4} (x) \sim \Psi_{\frac{l}{2}}^{l/4} (x) \tag{3.6}
\]

and

\[
\left|\frac{R'(x)}{R(x)}\right| \leq C_1 \frac{n}{a_n} \Psi_{\frac{l}{2}}^{l/4} (x). \tag{3.7}
\]
Proof. Let
\[ u(x) := (1 - x^2)^{-\frac{3}{4}}, \quad x \in [-1, 1] \]
be the ultraspherical weight on \((-1, 1)\) and let \(\lambda_n(u, x)\) be the Christoffel function corresponding to \(u\) satisfying
\[ \lambda_n^{-1}(u, x) \in \mathcal{P}_{2n-2}. \]
Then it is known [25, p.36], that given \(A > 0\) we have uniformly in \(n\) and \(|x| \leq 1 - \frac{A}{n^2}\)
\[ \lambda_n(u, x) \sim \frac{1}{n} (1 - x^2)^{-\frac{3}{4}} \tag{3.8} \]
and
\[ |\lambda'_n(u, x)| \leq \frac{C_1}{n} (1 - x^2)^{-\frac{5}{4}}. \tag{3.9} \]
Now choose \(m := m(n) = \text{the largest integer } \leq T(a_n)^{-\frac{1}{2}}\) and put
\[ R(x) := \frac{1}{m^2} \lambda_m^{-2} \left( u, \frac{x}{a_{2sn}} \right), \quad x \in [-a_{sn}, a_{sn}]. \]
Then by (2.6), \(R\) has degree \(o(n)\) and by (2.3), (2.6), (2.19), (3.1) and (3.8) we have uniformly for \(|x| \leq a_{sn},
\[ R(x) \sim \Phi_{a_{sn}}(x) \sim \Psi_n^{\frac{1}{2}}(x). \]
To prove (3.7), we observe much as in [22, p.228] that
\[ \left| \lambda_n^{-1} \left( u, \frac{x}{a_{2sn}} \right) \right| = \frac{\lambda_n' \left( u, \frac{x}{a_{2sn}} \right)}{a_{2sn}^2 \lambda_n^2 \left( u, \frac{x}{a_{2sn}} \right)}, \tag{3.10} \]
so that by (3.8), (3.9) and the definition of \(R\) we have uniformly for \(|x| \leq a_{sn},
\[ \left| R'(x)/R(x) \right| \leq \frac{C_2}{a_n} \left( 1 - \left( \frac{x}{a_{2sn}} \right)^2 \right)^{-1} \]
\[ \leq \frac{C_3}{a_n} \Psi_n^\frac{1}{2}. \quad \square \]
Our next lemma is an infinite-infinite range inequality:

**Lemma 3.3.** Let $W \in \mathcal{E}_1$. Let $0 < p \leq \infty$, $s > 1$ and $\Psi_n$ be as in (3.1). Then for $n \geq C_1$, $\forall P \in \mathcal{P}_n$ and $0 \leq l \leq n$ we have,

$$\left\| PW \Psi_n^{l/4} \right\|_{L_p(\mathbb{R})} \leq C_1 \left\| PW \Psi_n^{l/4} \right\|_{L_p(|x| \leq \alpha_{3n})}.$$  \hspace{1cm} (3.11)

Moreover,

$$\left\| PW \Psi_n^{l/4} \right\|_{L_p(|x| \geq \alpha_{3n})} \leq C_2 \exp \left[ -C_3 n^{C_4} \right] \left\| PW \Psi_n^{l/4} \right\|_{L_p(|x| \leq \alpha_{3n})}.$$ \hspace{1cm} (3.12)

Here, $C_j \neq C_j (n, P, l), \ j = 1, 2$.

We recall that (2.6) shows that for large $n$,

$$nT(a_n)^{\frac{3}{2}} \geq n^{C_5}.$$ \hspace{1cm} (3.13)

**Proof.** First note that by (2.20) and the definition of $\Psi_n$, given $\beta > 0$ we have,

$$\Psi_n^\beta(x) \geq T(a_n)^{-\frac{\beta}{2}}, \ x \in \mathbb{R}.$$ \hspace{1cm} (3.14)

Now write $l = 4j + k$, $0 \leq k < 3$. Then for some $0 < \alpha \leq 3$ and $C_1$ depending on $k$ we have,

$$\left\| PW \Psi_n^{l/4} \right\|_{L_p(|x| \geq \alpha_{3n})} = \left\| PW \Psi_n^{l/4} \right\|_{L_p(|x| \geq \alpha_{3n})} \leq C_1 \left\| PW \Psi_n^{l/4} \right\|_{L_p(|x| \leq \alpha_{3n})}.$$ \hspace{1cm} (3.15)

Now $P x^\alpha \Psi_n^j$ is a polynomial of degree $\leq n + l + 3 \leq 3n$ so by (2.22), we may continue (3.15) as

$$\leq C_2 \exp \left[ -C_3 nT(a_n)^{-\frac{\beta}{2}} \right] \left\| PW \Psi_n^{j/4} \right\|_{L_p(|x| \leq \alpha_{3n})}$$

$$\leq C_4 \exp \left[ -C_3 nT(a_n)^{-\frac{\beta}{2}} \right] a_n^{\alpha T(a_n)^{\frac{3}{2}}} \left\| PW \Psi_n^{j+k/4} \right\|_{L_p(|x| \leq \alpha_{3n})} \quad \text{(by (3.14))}$$

$$\leq C_5 \exp \left[ -C_6 nT(a_n)^{\frac{3}{2}} \right] \left\| PW \Psi_n^{l/4} \right\|_{L_p(|x| \leq \alpha_{3n})}$$
by (2.7) and (3.13). □

We can now give:

**The Proof of Theorem 3.1.** We prove (3.2). Then (3.3) will follow by (2.6). (3.4) and (3.5) will follow as

\[ \Psi_n^{1/4}(x) \sim \Phi_{\frac{a_n}{a_3 n}}(x), \quad x \in \mathbb{R} \]

Put \( s > 1 \) and write \( l = 4j + k, \) \( 0 \leq k \leq 3. \) Put \( Q := P^{(l)}. \) Then

\[ J := \left\| P^{(l+1)} W \Psi_n^{(l+1)/4} \right\|_{L_p(\|x\| \leq a_3 n)} = \left\| Q' W \Psi_n^{(l+1)/4} \right\|_{L_p(\|x\| \leq a_3 n)} \]

\[ = \left\| Q' W \Psi_n^{l+1/4} \right\|_{L_p(\|x\| \leq a_3 n)}. \]

Choose by Lemma 3.2, \( R \) of degree \( o(n) \) such that

\[ R(x) \sim \Psi_n^{1/2}(x) \]

and

\[ \left| R'(x) / R(x) \right| \leq \frac{C_1}{a_n} \Psi_n^{1/2}(x) \]

uniformly for \( \|x\| \leq a_3 n. \)

Then continue this estimate as

\[ J \leq C_2 \left\| Q' W \Psi_n^j R^k \Psi_n^{1/2} \right\|_{L_p(\|x\| \leq a_3 n)} \]

where \( C_2 \) depends only on \( k. \) This is in turn can be can continued as

\[ \leq C_2 \left\| (Q \Psi_n^j R^k)' \Psi_n^{1/2} W \right\|_{L_p(\|x\| \leq a_3 n)} + C_2 \left\| (\Psi_n^j)' R^k Q \Psi_n^{1/2} W \right\|_{L_p(\|x\| \leq a_3 n)} + C_2 \left\| \Psi_n^j (R^k)' Q \Psi_n^{1/2} W \right\|_{L_p(\|x\| \leq a_3 n)} \]

\[ = T_1 + T_2 + T_3. \]

We begin with the estimation of \( T_1 : \)
Note that $Q\Psi_n^j R^k$ is a polynomial of degree $\leq n + l + o(n) \leq 3n$. Thus, we can write

$$T_1 \leq C_7 \frac{n}{a_n} \left\| Q\Psi_n^j R^k W \right\|_{L_p(\mathbb{R})} \quad \text{(by (1.20))}$$

$$\leq C_7 \frac{n}{a_n} \left\| Q\Psi_n^j W \right\|_{L_p(|x| \leq a_{3n})} \quad \text{(by (2.21))}$$

$$\leq C_7 \frac{n}{a_n} \left\| P^{(i)} \Psi_n^\frac{j}{2} W \right\|_{L_p(|x| \leq a_{3n})}$$

$$\leq C_7 \frac{n}{a_n} \left\| P^{(i)} \Psi_n^\frac{j}{2} W \right\|_{L_p(\mathbb{R})} \quad (3.16)$$

Next we estimate $T_2$:

Note that for $|x| \leq a_{3n}$ and by straightforward differentiation, (2.3) gives

$$\left\| (\Psi_n^j) \right\| (x) \leq C_6 \Psi_n (x)^{j - \frac{1}{2} \frac{j}{a_n}}.$$

Thus

$$T_2 \leq C_7 \frac{j}{a_n} \left\| P^{(i)} \Psi_n^\frac{j}{2} W \right\|_{L_p(|x| \leq a_{3n})} \leq C_7 \frac{j}{a_n} \left\| P^{(i)} \Psi_n^\frac{j}{2} W \right\|_{L_p(|x| \leq a_{3n})} \leq C_8 \frac{T(a_n)}{a_n} \left\| P^{(i)} \Psi_n^\frac{j}{2} W \right\|_{L_p(\mathbb{R})} \quad (3.17)$$

by (3.14).

It remains to estimate $T_3$:

Write

$$T_3 \leq C_9 \frac{k}{a_n} \left\| \Psi_n^j R^{k-1} R' Q\Psi_n^\frac{j}{2} W \right\|_{L_p(|x| \leq a_{3n})} \leq C_9 \frac{k}{a_n} \left\| \Psi_n^j R^{k-1} W \right\|_{L_p(|x| \leq a_{3n})} \quad \text{by (3.7)}$$

$$\leq C_9 \frac{k}{a_n} \left\| \Psi_n^j \Psi_n^\frac{k-1}{2} W \right\|_{L_p(|x| \leq a_{3n})}$$

$$\leq C_9 \frac{k}{a_n} \left\| \Psi_n^j \Psi_n^\frac{k-1}{2} W \right\|_{L_p(|x| \leq a_{3n})}$$

$$\leq C_9 \frac{k}{a_n} \left\| \Psi_n^j \Psi_n^\frac{k-1}{2} W \right\|_{L_p(|x| \leq a_{3n})}$$

$$\leq C_9 \frac{k}{a_n} \left\| \Psi_n^j \Psi_n^\frac{k-1}{2} W \right\|_{L_p(\mathbb{R})} \quad (3.18)$$
as in the estimation of \( T_2 \).

Combining (3.16), (3.17) and (3.18) gives

\[
J \leq C_{11} \left\{ \frac{n}{a_n} + \frac{l}{a_n} T(a_n) \right\} \left\| P^{(l)} W \Psi_n^\frac{1}{2} \right\|_{L_p[\mathbb{R}]},
\]

(3.19)

where \( C_{11} \neq C_{11} (n, P, l) \).

Finally by (3.11), (3.19) becomes

\[
\left\| P^{(l+1)} W \Psi_n^\frac{1}{2} \right\|_{L_p[\mathbb{R}]} \leq C_{12} \left\{ \frac{n}{a_n} + \frac{l}{a_n} T(a_n) \right\} \left\| P^{(l)} W \Psi_n^\frac{1}{2} \right\|_{L_p[\mathbb{R}]}
\]

as required where \( C_{12} \neq C_{12} (n, P, l) \). \( \Box \)

4 Approximation of Polynomials of degree \( \leq n \) by those of degree \( \leq r - 1 \).

In this section, we obtain a crucial inequality introduced in a related context in [8], in order to obtain an upper bound for our modulus in terms of our realization-functional. The main idea is to approximate polynomials of degree \( \leq n \) by polynomials of degree \( \leq r - 1 \). Here \( n \geq n_0 \) and \( r \geq 1 \).

We prove:

**Theorem 4.1.** Let \( W \in \mathcal{E}_1 \) and assume (1.20). Let \( r \geq 1, L > 0, 0 < p \leq \infty, P_n \in \mathcal{P}_n \) and \( n \geq C \). Set

\[
P(x) := P_n(x) - \int_{a_n}^x \int_{a_n}^{u_{r-1}} \ldots \int_{a_n}^{u_1} P_n^{(r)}(u) \, du_{r-1} \ldots du_0 \in \mathcal{P}_{r-1}.
\]

(4.1)

Then, \( \exists C_1 > 0, C_1 \neq C_1 (n, P_n, P) \) such that

\[
\left\| W (P_n - P) \right\|_{L_p[\mathcal{A}_{a_n, \infty}]} \leq C_1 \left( \frac{a_n}{n} \right)^r \left\| WP_n^{(r)} \Phi_n^\frac{1}{2} \right\|_{L_p[\mathbb{R}]}. \]

(4.2)

We break the proof down into several steps. We begin with:
Lemma 4.2. Let $W \in \mathcal{E}_1$, $1 \leq p \leq \infty$. Then for $n \geq C$ and \( \forall g \in L_p[a_{L_n}, \infty), \exists C_1 > 0, \exists C_1(g, n) \) such that

\[
\left\| W(x) \int_{a_{L_n}}^{x} g(u) du \right\|_{L_p[a_{L_n}, \infty)} \leq \frac{a_n}{nT(a_n)^{\frac{1}{2}}} \|gW\|_{L_p[a_{L_n}, \infty)} \quad (4.3)
\]

**Proof.** We notice that

\[
W(x)^{\frac{1}{2}} \int_{t}^{x} W(u)^{-\frac{1}{2}} Q'(u) du = 2 \left[ 1 - \left( \frac{W(x)}{W(t)} \right)^{\frac{1}{2}} \right] \leq 2 \quad (4.4)
\]
as $t \leq x$.

Next, notice that for $u \geq a_{L_n}$, and $n$ large enough, we have by Lemma 2.1

\[
Q'(u) \geq C Q'(a_{L_n}) \sim \frac{nT(a_n)^{\frac{1}{2}}}{a_n}, \quad (4.5)
\]
so that for $x \geq a_{L_n}$

\[
\frac{a_n}{nT(a_n)^{\frac{1}{2}}} W(x)^{\frac{1}{2}} \int_{a_{L_n}}^{x} |gW(u)| Q'(u) W^{-\frac{1}{2}}(u) du \geq C_1 W(x)^{\frac{1}{2}} \int_{a_{L_n}}^{x} |gW(u)^{\frac{1}{2}}| du \geq W(x) \left| \int_{a_{L_n}}^{x} g(u) du \right|. \quad (4.6)
\]

Now recalling Jensen’s Inequality for integrals

\[
\left( \int f d\mu \right)^{p} \leq \left( \int |f|^{p} d\mu \right) \left( \int d\mu \right)^{p-1}
\]
valid for $\mu$ measurable functions $f$ and non negative measures $\mu$, gives:

**Case 1.** $p = \infty$. Here (4.6) gives for $x \geq a_{L_n}$

\[
W(x) \left| \int_{a_{L_n}}^{x} g(u) du \right| \leq \frac{a_n}{nT(a_n)^{\frac{1}{2}}} W(x)^{\frac{1}{2}} \|gW\|_{L_\infty[a_{L_n}, \infty)} \int_{a_{L_n}}^{x} Q'(u) W^{-\frac{1}{2}}(u) du \leq C_2 \frac{a_n}{nT(a_n)^{\frac{1}{2}}} \|gW\|_{L_\infty[a_{L_n}, \infty)} \quad (by \ (4.4)).
\]

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Case 2. \( 1 \leq p < \infty \). Here
\[
\left\| W(x) \int_{a_{Ln}}^{x} g(u) du \right\|_{L^p[a_{Ln}, \infty)}
\leq \frac{a_n}{nT(a_n)^\frac{1}{p}} \left[ \int_{a_{Ln}}^{\infty} \left( W(x) \right)^{\frac{1}{p}} \left( \int_{a_{Ln}}^{x} |gW(u)| Q'(u) W^{-\frac{1}{p}}(u) du \right)^p dx \right]^{\frac{1}{p}}
\leq C_3 \frac{a_n}{nT(a_n)^\frac{1}{p}} \left[ \int_{a_{Ln}}^{\infty} 2^{p-1} W(x)^{\frac{1}{p}} \left( \int_{a_{Ln}}^{x} |gW(u)|^p Q'(u) W^{-\frac{1}{p}}(u) du \right)^p dx \right]^{\frac{1}{p}}
\]
by Jensen’s Inequality, with \( d\mu = W(x)^{\frac{1}{p}} Q'(u) W(u)^{-\frac{1}{p}} \) on \([a_{Ln}, x]\) and \( f \, d\mu \leq 2 \) (see (4.4)).

Then
\[
\int_{a_{Ln}}^{\infty} W(x)^{\frac{1}{p}} \left( \int_{a_{Ln}}^{x} |gW(u)|^p Q'(u) W^{-\frac{1}{p}}(u) du \right) dx
= \int_{a_{Ln}}^{\infty} |gW(u)|^p \left[ \int_{u}^{\infty} W(x)^{\frac{1}{p}} Q'(u) dx \right] W^{-\frac{1}{p}}(u) du
\leq C_4 \int_{a_{Ln}}^{\infty} |gW(u)|^p \left[ \int_{u}^{\infty} W(x)^{\frac{1}{p}} Q'(x) dx \right] W^{-\frac{1}{p}}(u) du \quad \text{(as} \, x > u)\]
\leq C_5 \| gW \|^p_{L^p(a_{Ln}, \infty)}.
\]

We are now in the position to give:

**The Proof of Theorem 4.1 for \( 1 \leq p \leq \infty \).**

We will repeatedly make use of (2.20):

\[
\Phi_{2n}(x) \geq CT(a_n)^{-\frac{1}{2}}, \quad \forall x \in \mathbb{R}.
\]

(4.7)

Firstly, if \( r = 1 \), Lemma 4.2 with \( g = P_{n}' \) gives
\[
\left\| W(x) \int_{a_{Ln}}^{x} P_{n}'(u_o) du_o \right\|_{L^p[a_{Ln}, \infty)} \leq C_1 \frac{a_n}{nT(a_n)^\frac{1}{p}} \left\| P_{n}' W \right\|_{L^p(\mathbb{R})}
\leq C_2 \frac{a_n}{n} \left\| P_{n}' \Phi_{2n}(x) W \right\|_{L^p(\mathbb{R})} \quad \text{(by} \, (4.7)).
\]

Now apply (4.1). If \( r = 2 \), we apply Lemma 4.2 with
\[
g(u_1) = \int_{a_{Ln}}^{u_1} P^{(2)}(u_o) du_o
\]
to give
\[
\left\| W(x) \int_{a_{L_n}}^{u_1} \int_{a_{L_n}}^{u_1} P_n^{(2)}(u_0) \, du_0 \, du_1 \right\|_{L_p(a_{L_n}, \infty)}
= \left\| W(x) \int_{a_{L_n}}^{x} g(u_1) \, du_1 \right\|_{L_p(a_{L_n}, \infty)}
\leq C_3 \frac{a_n}{nT(a_n)^{\frac{1}{2}}} \left\| gW \right\|_{L_p(a_{L_n}, \infty)}
= C_3 \frac{a_n}{nT(a_n)^{\frac{1}{2}}} \left\| W \int_{a_{L_n}}^{u_1} P_n^{(2)}(u_0) \, du_0 \right\|_{L_p(a_{L_n}, \infty)}
\leq C_4 \left( \frac{a_n}{nT(a_n)^{\frac{1}{2}}} \right)^2 \left\| P_n^{(2)}W \right\|_{L_p(\mathbb{R})}
\leq C_5 \left( \frac{a_n}{n} \right)^2 \left\| P_n^2 \Phi_{2n}(x)W \right\|_{L_p(\mathbb{R})}.
\]
Applying now (4.1), and an induction argument on $r$ gives the result. □

We now treat the more complicated case, $0 < p < 1$. For this case we need two lemmas.

**Lemma 4.3.** Let $W \in \mathcal{E}_1$ and assume (1.20). Let $0 < p < 1$, $r \geq 1$, $R_n \in \mathcal{P}_n$, $R \in \mathcal{P}_{r-1}$ and $n \geq C$. Set for $x \in \mathbb{R}$, and $L > 0$
\[
g_n(x) := (R_n - R)(x)
\]
and
\[
J_n(x) := \left\| p_nW(u)^{1-p} \left( \frac{W(x)}{W(u)} \right)^{\frac{1}{2}} \right\|_{L_\infty[a_{L_n}, x]}^{\frac{1}{2-p}}. \tag{4.8}
\]
Then
\[
\int_{a_{L_n}}^{\infty} J_n(x) \, dx \leq C_1 \left[ \sum_{j=1}^{r-1} \left( \frac{a_n}{nT(a_n)^{\frac{1}{2}}} \right)^{(j-1)p} \left\| W(R_n^{(j)} - R^{(j)}) \right\|_{L_\infty(a_{L_n}, \infty)}^p \right.
+ \left( \frac{a_n}{nT(a_n)^{\frac{1}{2}}} \right)^{(r-1)p} \left\| WR_n^{(r)} \right\|_{L_\infty(\mathbb{R})}^p \right]. \tag{4.9}
\]
Here $C_1 \neq C_1 (n, R_n, R)$.

**Proof.** Write

$$J_n(x) = \left\| \left| g'_n W(u) \right|^p \left( \frac{W(x)}{W(u)} \right)^{\frac{p}{2(1-p)}} \right\|_{L_{\infty}[a_{L_n}, x]}$$

and set

$$\tau := \frac{\delta a_n}{nT(a_n)^{\frac{1}{2}}}$$

where $\delta > 0$ is chosen small enough so that for $n \geq 1$ and $\forall S \in \mathcal{P}_n$,

$$\|S'W\|_{L_p(\mathbb{R})} \leq \left(2\delta^{-1}\right) \frac{nT(a_n)^{\frac{1}{2}}}{a_n} \|SW\|_{L_p(\mathbb{R})}.$$  \hspace{1cm} (4.10)

(See (1.20) and (2.20)).

Now given $x \geq a_{L_n}$, we set

$$k_o := k_o(x) = \max \{ k_o : x - (k+1)\tau \geq a_{L_n} \}$$

and write

$$J_n(x) \leq I_1 + I_2,$$

where

$$I_1 := \max_{0 \leq k \leq k_o} \left\| \left| g'_n W(u) \right|^p \left( \frac{W(x)}{W(u)} \right)^{\frac{p}{2(1-p)}} \right\|_{L_{\infty}[x-(k+1)\tau, x-k\tau]} \hspace{1cm} (4.11)$$

and

$$I_2 := \left\| \left| g'_n W(u) \right|^p \left( \frac{W(x)}{W(u)} \right)^{\frac{p}{2(1-p)}} \right\|_{L_{\infty}[a_{L_n}, x-(k_o+1)\tau]}.$$

First we observe that for $u \in [x-(k+1)\tau, x-k\tau]$

$$\frac{W(x)}{W(u)} \leq \exp \left( Q(x-k\tau) - Q(x) \right).$$

Further, as $x-k\tau \geq a_{L_n} > 0$

\[
Q(x) - Q(x-k\tau) \geq C_1 k\tau Q'(x-k\tau) \geq C_2 k\tau Q'(a_{L_n}) \geq C_3 \frac{nT(a_n)^{\frac{1}{2}}}{a_n nT(a_n)^{\frac{1}{2}}} \frac{\delta a_n k}{a_n} \\
= C_3 k\delta
\]

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by (2.1) of Lemma 2.1. So

$$
\left( \frac{W(x)}{W(u)} \right)^{\frac{p}{p(1-p)}} \leq \alpha^k, \quad u \in [x - (k + 1)\tau, x - k\tau]
$$

where $\alpha \in (0, 1)$ is independent of $x, u, k$. Thus we may write

$$
I_1 + I_2 \leq \max_{0 \leq k \leq k_o} \alpha^k \|g'_n W\|_{L^p_{\infty}[x-(k+1)\tau, x-k\tau]}^p + \alpha^{k_o} \|g'_n W\|_{L^p_{\infty}[a_{L_n}, x-(k_o+1)\tau]}^p
$$

$$
\leq \sum_{k=0}^{k_o(x)} \alpha^k \|g'_n W\|_{L^p_{\infty}[x-(k+1)\tau, x-k\tau]}^p + \alpha^{k_o} \|g'_n W\|_{L^p_{\infty}[a_{L_n}, x-(k_o+1)\tau]}^p.
$$

Then

$$
\int_{a_{L_n}}^{\infty} J_n(x) \, dx = \sum_{m=0}^{\infty} \int_{a_{L_n} + m\tau}^{a_{L_n} + (m+1)\tau} J_n(x) \, dx
$$

$$
\leq \sum_{m=0}^{\infty} \int_{a_{L_n} + m\tau}^{a_{L_n} + (m+1)\tau} \left[ \sum_{k=0}^{k_o(x)} \alpha^k \|g'_n W\|_{L^p_{\infty}[x-(k+1)\tau, x-k\tau]}^p + \alpha^{k_o} \|g'_n W\|_{L^p_{\infty}[a_{L_n}, x-(k_o+1)\tau]}^p \right] \, dx.
$$

We observe that

$$
\int_{a_{L_n} + m\tau}^{a_{L_n} + (m+1)\tau} \|g'_n W\|_{L^p_{\infty}[x-(k+1)\tau, x-k\tau]}^p \, dx
$$

$$
= \int_{a_{L_n} + (m-k)\tau}^{a_{L_n} + (m-k-1)\tau} \|g'_n W\|_{L^p_{\infty}[x, x+\tau]}^p \, dx
$$

and since

$$
x \in [a_{L_n} + (m - k - 1)\tau, a_{L_n} + (m - k)\tau] \quad \Rightarrow \quad m \geq k_o \geq m - 1,
$$

we have

$$
\int_{a_{L_n}}^{\infty} J_n(x) \, dx \leq \sum_{m=0}^{\infty} \sum_{k=0}^{m-1} \int_{a_{L_n} + (m-k-1)\tau}^{a_{L_n} + (m-k)\tau} \alpha^k \|g'_n W\|_{L^p_{\infty}[x, x+\tau]}^p \, dx
$$

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Here
\[ I_3 := \sum_{s=0}^{\infty} \int_{a_{LN}+s\tau}^{a_{LN}+(s+1)\tau} \|g'_n W\|_{L^p[a_{LN}, x]}^p \]  
(4.13)
and
\[ I_4 := \int_{a_{LN}}^{a_{LN}+\tau} \|g'_n W\|_{L^p[a_{LN}, x]}^p \]  
(4.14)

We begin by estimating \( I_3 \). Observe that \( g'_n \) is a polynomial of degree \( \leq n - 1 \) for \( u \in [x, x + \tau] \), so expanding it in a Taylor series about \( x \) gives
\[
|g'_n(u)|^p = \left| \sum_{j=1}^{n} \frac{g^{(j)}(x)(u-x)^{j-1}}{(j-1)!} \right|^p \\
\leq \sum_{j=1}^{n} \left| g^{(j)}(x) \right|^p (u-x)^{j-1} \\
(\text{by the inequality, } (a+b)^\alpha \leq a^\alpha + b^\alpha, \ 0 < \alpha < 1, \ a, b \in \mathbb{R}) \\
\leq \sum_{j=1}^{n} \left| R^{(j)}_n(x) - R^{(j)}_n(x) \right|^p (u-x)^{j-1} + \sum_{j=1}^{n} \left| R^{(j)}_n(x) \right|^p (u-x)^{j-1}.
\]

Thus using
\[ W(u) \leq W(x), \ u \in [x, x + \tau], \]  
(4.15)
the definition of \( \tau \) and (4.10) gives
\[
I_3 \leq C_5 \left[ \sum_{j=1}^{\tau} \left\| (R^{(j)}_n - R^{(j)}_n) W \right\|_{L^p[a_{LN}, \infty]}^p \tau^{(j-1)p} \right]
\]
\[ + \tau^{(r-1)p} \sum_{j=r}^{n} \left\| R_{n}^{(j)} W \right\|_{L_{p}[a_{L_{n}}, \infty]}^{p} \tau^{(j-r)p} \]

\[ \leq C_{6} \left[ \sum_{j=1}^{r-1} \left( \frac{a_{n}}{nT(a_{n})^{1/2}} \right)^{(j-1)p} \left\| (R_{n}^{(j)} - R^{(j)}) W \right\|_{L_{p}[a_{L_{n}}, \infty]}^{p} \right]^{(r-1)p} \]

\[ + \tau^{(r-1)p} \sum_{j=r}^{n} \left( \frac{nT(a_{n})^{1/2}}{2\delta a_{n}} \right)^{(j-r)p} \left\| R_{n}^{(r)} W \right\|_{L_{p}(\mathbb{R})}^{p} \]

\[ \leq C_{7} \left[ \sum_{j=1}^{r-1} \left( \frac{a_{n}}{nT(a_{n})^{1/2}} \right)^{(j-1)p} \left\| (R_{n}^{(j)} - R^{(j)}) W \right\|_{L_{p}[a_{L_{n}}, \infty]}^{p} \right]^{(r-1)p} \]

\[ + \left( \frac{a_{n}}{nT(a_{n})^{1/2}} \right)^{(r-1)(1-p)} \left\| R_{n}^{(r)} W \right\|_{L_{p}(\mathbb{R})}^{p} \]  \quad (4.16)

To estimate \( I_{4} \) we proceed in a similar way to that of \( I_{3} \), except that we use (2.29) instead of (4.15), which we may in view of the definition of \( \tau \), (2.3) and (2.6). Combining our estimates for \( I_{3} \) and \( I_{4} \) give the lemma. \( \square \)

**Lemma 4.4.** Let \( W \in \mathcal{E}_{1} \) and assume (1.20). Let \( 0 < p < 1, \quad r \geq 1, \quad L > 0, \quad R_{n} \in \mathcal{P}_{n}, \quad R \in \mathcal{P}_{r-1} \) satisfying,

\[ (R_{n} - R)(a_{L_{n}}) = 0. \]

Then for \( n \geq C \) there exists \( C_{1} \neq C_{1}(n, R_{n}, R) \) such that

\[ \left\| W (R_{n} - R) \right\|_{L_{p}[a_{L_{n}}, \infty]} \leq C_{1} \left[ \left( \frac{a_{n}}{nT(a_{n})^{1/2}} \right) \left\| W (R_{n}^{t} - R^{t}) \right\|_{L_{p}[a_{L_{n}}, \infty]}^{p} \right]^{(r-1)(1-p)} \]

\[ \times \left[ \sum_{j=1}^{r-1} \left( \frac{a_{n}}{nT(a_{n})^{1/2}} \right)^{(j-1)(1-p)} \left\| (R_{n}^{(j)} - R^{(j)}) W \right\|_{L_{p}[a_{L_{n}}, \infty]}^{p} \right]^{(r-1)(1-p)} \]

\[ + \left( \frac{a_{n}}{nT(a_{n})^{1/2}} \right)^{(r-1)(1-p)} \left\| R_{n}^{(r)} W \right\|_{L_{p}(\mathbb{R})}^{p} \]  \quad (4.17)
Proof. Set 
\[ g_n(x) := (R_n - R)(x) \]
satisfying \( g_n(a_{LN}) = 0 \) and write
\[ g_n(x) = \int_{a_{LN}}^{x} g_n'(u) du. \]

Then
\[
\Delta = \| W(R_n - R) \|_{L^p[a_{LN}, \infty)} = \| W g_n \|_{L^p[a_{LN}, \infty)}
\]
\[
= \left[ \int_{a_{LN}}^{\infty} \left( \int_{a_{LN}}^{x} g_n' W(u) \frac{W(x)}{W(u)} du \right)^p dx \right]^{\frac{1}{p}}
\]
\[
\leq \left[ \int_{a_{LN}}^{\infty} \left( |g_n' W(u)|^{1-p} \left( \frac{W(x)}{W(u)} \right)^{\frac{1}{2}} \right)^p d\|_{L^p[a_{LN}, \infty)}
\]
\[
\times \left( \int_{a_{LN}}^{x} |g_n' W(u)|^{p} \left( \frac{W(x)}{W(u)} \right)^{\frac{1}{2}} d\right)^{\frac{1}{p}}.
\]  \hspace{5cm} (4.18)

Now apply Hölder’s Inequality with \( r = \frac{1}{1-p}, \sigma = \frac{1}{p} \) satisfying \( r^{-1} + \sigma^{-1} = 1 \) to give
\[ \Delta \leq I_1 I_2 \]
where
\[
I_1 := \left( \int_{a_{LN}}^{\infty} \left| g_n' W(u) \right|^{1-p} \left( \frac{W(x)}{W(u)} \right)^{\frac{1}{2}} dx \right)^{\frac{p}{1-p}} \left( \int_{a_{LN}}^{\infty} \left( \frac{W(x)}{W(u)} \right)^{\frac{1}{2}} dx \right)^{\frac{1}{p}} \right)^{\frac{1-p}{p}} \]  \hspace{5cm} (4.19)

and
\[
I_2 := \left( \int_{a_{LN}}^{\infty} \int_{a_{LN}}^{x} |g_n' W(u)|^{p} \left( \frac{W(x)}{W(u)} \right)^{\frac{1}{2}} dudx \right).
\]  \hspace{5cm} (4.20)

Now by (4.8) we may write
\[
I_1 = \left( \int_{a_{LN}}^{\infty} J_n(x) dx \right)^{\frac{1-p}{p}}
\]
\[
\leq C \left[ \sum_{j=1}^{r-1} \left( \frac{a_n}{nT(a_n)^{\frac{3}{2}}} \right)^{(j-1)(1-p)} \right]^{1-p} \times \left\| W \left( R_n^{(j)} - R^{(j)} \right) \right\|_{L^p[a_{Ln}, \infty)} \\
+ \left( \frac{a_n}{nT(a_n)^{\frac{3}{2}}} \right)^{(r-1)(1-p)} \left\| WR_n^{(r)} \right\|_{L^p(\mathbb{R})} \right]^{1-p}
\]
(4.21)

(by Lemma 4.3).

Also
\[
I_2 = \int_{a_{Ln}}^{\infty} \left| g_n(u) \right| W(u)^p \int_{u}^{\infty} \left( \frac{W(x)}{W(u)} \right)^{\frac{1}{2}} \, dx \, du.
\]

Now if \( x \geq u \geq a_{Ln} \), Lemma 2.1 gives
\[
Q'(x) \geq C_1 Q'(a_{Ln}) \geq C_2 \frac{nT(a_n)^{\frac{3}{2}}}{a_n}
\]
so that
\[
I_2 \leq C_3 \frac{a_n}{nT(a_n)^{\frac{3}{2}}} \int_{a_{Ln}}^{\infty} \left| g_n'(u) \right| W(u)^p \left[ W(u)^{-\frac{1}{2}} \int_{u}^{\infty} W(x)^{\frac{1}{2}} Q'(x) \, dx \right] \, du
\]
\[
\leq C_4 \frac{a_n}{nT(a_n)^{\frac{3}{2}}} \int_{a_{Ln}}^{\infty} \left| g_n'(u) \right| W(u)^p \, du.
\]

This gives
\[
I_2 \leq C_4 \frac{a_n}{nT(a_n)^{\frac{3}{2}}} \left\| (R_n^r - R^r) W \right\|_{L^p[a_{Ln}, \infty)}^{p}.
\]
(4.22)

Combining our estimates for \( I_1 \) and \( I_2 \) give the result. \( \square \)

We are now in the position to give:

**The Proof of Theorem 4.1 for 0 < p < 1.** Let \( P_n \in \mathcal{P}_n \) and \( P \in \mathcal{P}_{r-1} \) be given by (4.1). We first note that if \( 0 \leq l < r \),
\[
\left( P_n^{(l)} - P^{(l)} \right)(a_{Ln}) = 0.
\]

Thus applying (4.17) to \( P_n^{(l)} \) with \( r \) in (4.17) replaced by \( r - l \) gives
\[
\left\| W \left( P_n^{(l)} - P^{(l)} \right) \right\|_{L^p[a_{Ln}, \infty)}
\]

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\[
\begin{align*}
&\leq C_1 \left[ \frac{a_n}{nT(a_n)^{\frac 1 2}} \left\| W \left( P_{n}^{(l+1)} - P^{(l+1)} \right) \right\|_{L_p[a, \infty]}^p \right] \\
&\times \left[ \sum_{j=l+1}^{r-1} \left( \frac{a_n}{nT(a_n)^{\frac 1 2}} \right)^{(j-l-1)(1-p)} \left\| W \left( P_{n}^{(j)} - P^{(j)} \right) \right\|_{L_p[a, \infty]}^{1-p} \right] \\
&+ \left( \frac{a_n}{nT(a_n)^{\frac 1 2}} \right)^{(r-l-1)(1-p)} \left\| W \left( P_{n}^{(r)} \right) \right\|_{L_p[a, \infty]}^{1-p}. \quad (4.23)
\end{align*}
\]

We show that for \( k = r-1, r-2, ..., 0 \)
\[
\left\| W \left( P_{n}^{(k)} - P^{(k)} \right) \right\|_{L_p[a, \infty]} \leq C_3 \left( \frac{a_n}{nT(a_n)^{\frac 1 2}} \right)^{r-k} \left\| WP_{n}^{(r)} \right\|_{L_p}\cdot \quad (4.24)
\]

Firstly, if \( k = r-1 \), (4.23) with \( l = r-1 \) gives
\[
\left\| W \left( P_{n}^{(r-1)} - P^{(r-1)} \right) \right\|_{L_p[a, \infty]} \leq C_4 \left( \frac{a_n}{nT(a_n)^{\frac 1 2}} \right)^{(r-k)(1-p)} \left\| WP_{n}^{(r)} \right\|_{L_p}\cdot
\]

Assume now that (4.24) holds for \( r-1, ..., k+1 \). We prove (4.24) for \( k \).
Substituting (4.24) with \( r-1, .. k+1 \) into (4.23) with \( l = k \) gives
\[
\begin{align*}
&\left\| W \left( P_{n}^{(k)} - P^{(k)} \right) \right\|_{L_p[a, \infty]} \leq C_5 \left[ \frac{a_n}{nT(a_n)^{\frac 1 2}} \right]^p \left( \frac{a_n}{nT(a_n)^{\frac 1 2}} \right)^{(r-k-1)p} \left\| WP_{n}^{(r)} \right\|_{L_p}^p \\
&\times \left[ \sum_{j=k+1}^{r-1} \left( \frac{a_n}{nT(a_n)^{\frac 1 2}} \right)^{(j-k-1)(1-p)} \left( \frac{a_n}{nT(a_n)^{\frac 1 2}} \right)^{(r-j)(1-p)} \right] \\
&\times \left\| WP_{n}^{(r)} \right\|_{L_p}^{1-p} + \left( \frac{a_n}{nT(a_n)^{\frac 1 2}} \right)^{(r-k-1)(1-p)} \left\| WP_{n}^{(r)} \right\|_{L_p}^{1-p} \right\|_{L_p}^{1-p} \\
&\leq C_6 \left( \frac{a_n}{nT(a_n)^{\frac 1 2}} \right)^{r-k} \left\| WP_{n}^{(r)} \right\|_{L_p}^{1-p}. \quad (4.24)
\end{align*}
\]

Thus (4.24) holds for all \( k \). In particular, we have
\[
\left\| W \left( P_{n} - P \right) \right\|_{L_p[a, \infty]} \leq C_7 \left( \frac{a_n}{nT(a_n)^{\frac 1 2}} \right)^r \left\| WP_{n}^{(r)} \right\|_{L_p}^{r}. \quad (35)
\]
\[ \leq C_8 \left( \frac{a_n}{n} \right)^r \left\| WP^{(r)} \Phi_{\frac{\pi}{n}} (x) \right\|_{L_p(\mathbb{R})} \cdot \Box \]

5 Equivalence of Modulus and Realization Functional

In this section we prove Theorem 1.3 which establishes the fundamental equivalence of our modulus of continuity and its corresponding realization-functional. We also deduce Corollary 1.4. Throughout for \(0 < p \leq \infty\) we set:
\[ q := \min \{1, p\} . \]

We begin by quickly recalling the definitions of our moduli and realization functional. See (1.11), (1.12) and (1.17). Let \( r \geq 1, 0 < t \leq C \) and let \( n = n(t) \) be determined by (1.18). Then we have

\begin{equation}
(1) \quad u_{r,p}(f, W, t) := \sup_{0 < h \leq t} \left\| W \left( \Delta^r_{ht} (x) (f) \right) \right\|_{L_p([|x| \leq (2t)])} + \inf_{R \text{ of } \deg \leq \tau - 1} \left\| (f - R) W \right\|_{L_p([|x| \geq \sigma (4t)])} \tag{5.1}
\end{equation}

\begin{equation}
(2) \quad \overline{w}_{r,p}(f, W, t) := \left[ \frac{1}{t} \int_0^t \left\| W \left( \Delta^r_{ht} (x) (f) \right) \right\|_{L_p([|x| \leq (2t)])}^p \, dh \right]^{\frac{1}{p}} + \inf_{R \text{ of } \deg \leq \tau - 1} \left\| (f - R) W \right\|_{L_p([|x| \geq \sigma (4t)])} \tag{5.2}
\end{equation}

where we set \( \overline{w} = w \) for \( p = \infty \) and

\begin{equation}
(3) \quad K_{r,p} (f, W, t^*) := \inf_{P \in \mathcal{P}_n} \left\{ \left\| (f - P) W \right\|_{L_p(\mathbb{R})} + t^* \left\| P^{(r)} \Phi_{\frac{\pi}{n}} (x) W \right\|_{L_p(\mathbb{R})} \right\} . \tag{5.3}
\end{equation}
We begin with our lower bound.

**Lemma 5.1.** Let \( W \in \mathcal{E}_1 \), assume (1.20) and let \( L > 0 \) be fixed. Let \( r \geq 1, \ 0 < p \leq \infty \) and \( 0 < t < C \). Then there exists \( C_1 \neq C_1 (f, t) \) such that
\[
 w_{r,p} (f, W, L; t) \leq C_1 K_{r,p} (f, W, t') .
\] (5.4)

**Proof.** Let \( q = \min \{1, p\} \). Then by (2.12), there exists \( u \) such that \( 4Lt = \frac{a_u}{u} \). Now let \( n = n(t) \) be determined by (1.18) and recall it has the form
\[
n = \inf \left\{ k : \frac{a_k}{k} \leq t \right\} .
\]
Thus by (2.25) and (2.18) we have
\[
 \frac{a_n}{2n} \leq \frac{t}{2} < \frac{a_n}{n} \quad \text{(5.5a)}
\]
and
\[
 \Phi_1 (x) \sim \Phi_{\frac{a_u}{u}_n} (x) \sim \Phi_{Lt} (x) \quad \forall x \in \mathbb{R}, \quad \text{(5.5b)}
\]
where the constants in the \( \sim \) relation are independent of \( t \) and \( x \). Also by (2.13) and (2.26), \( \exists \beta > 0 \) such that
\[
 \sigma (4Lt) = \sigma (\frac{a_u}{u}) \geq a_{\frac{a_u}{u}} \geq a_{\beta n} . \quad \text{(5.6)}
\]
Choose \( P \in \mathcal{P} \) such that
\[
 \| (f - P)W \|_{L_p (\mathbb{R})} + t' \| \Phi_r^{(r)} W \|_{L_p (\mathbb{R})} \leq 2K_{r,p} (f, W, t') . \quad \text{(5.7)}
\]
We show that
\[
 \sup_{0 < h \leq L} \| W (\Delta^r h \Phi_{Lt} (x) (f)) \|_{L_p (|x| \leq \sigma (2Lt))} \leq C_6 K_{r,p} (f, W, t') \quad \text{(5.8)}
\]
and
\[
 \inf_{r \text{ of } \deg \leq r - 1} \| (f - R)W \|_{L_p (|x| \geq \sigma (tL))} \leq C_2 K_{r,p} (f, W, t') . \quad \text{(5.9)}
\]

This then gives (5.4) using the definition (5.1).

We begin with:

**The Proof of (5.9).** We appeal to Theorem 4.1 and choose for our given \( P, S \in \mathcal{P}_{r-1} \) as in (4.1), so that (4.2) holds. Next we recall Lemma 3.1 from [8]: Let \( W \) be an even weight. Then for \( f \) satisfying \( fW \in L_p(\mathbb{R}) \) and for \( \xi > 0 \),

\[
\inf_{R \text{ of } \deg \leq r-1} \| (f - R)W \|_{L_p(|x| \geq \xi)} \leq 2^{4/q-3} \left[ \inf_{R \text{ of } \deg \leq r-1} \| (f - R)W \|_{L_p(x \geq \xi)} + \inf_{R \text{ of } \deg \leq r-1} \| (f - R)W \|_{L_p(x \leq -\xi)} \right].
\]

We apply the above with \( \xi := \sigma (4Lt) \). In particular, we estimate

\[
\inf_{R \text{ of } \deg \leq r-1} \| (f - R)W \|_{L_p(x \geq \sigma(4Lt))}^q.
\]

The other term can be handled similarly. Thus

\[
\inf_{R \text{ of } \deg \leq r-1} \| (f - R)W \|_{L_p(x \geq \sigma(4Lt))}^q \leq \| (f - S)W \|_{L_p(x \geq \sigma(4Lt))}^q
\]

\[
\leq \| (f - P)W \|_{L_p(x \geq \sigma(4Lt))}^q + \| (P - S)W \|_{L_p(x \geq \sigma(4Lt))}^q
\]

\[
\leq C_3 (K_{r,p} (f, W, t'))^q + C_4 t^r \| P^{(r)}W \|_{L_p(\mathbb{R})}^q
\]

(by (5.6) and (5.7))

\[
\leq C_5 (K_{r,p} (f, W, t'))^q.
\]

Hence (5.9).

Next we proceed with:

**The Proof of (5.8).** Let \( 0 < h \leq Lt \) and write

\[
\| W \left( \Delta_{h\Phi_L(x)}(f) \right) \|_{L_p(|x| \leq \sigma(2Lt))}^q
\]

\[
\leq \| W \left( \Delta_{h\Phi_L(x)}(f - P) \right) \|_{L_p(|x| \leq \sigma(2Lt))}^q
\]

\[
+ \| W \left( \Delta_{h\Phi_L(x)}(P) \right) \|_{L_p(|x| \leq \sigma(2Lt))}^q
\]

\[= I_1 + I_2.\]
We first deal with the estimation of $I_1$. Note that given $A > 0$,

$$|x| \leq \sigma(2Lt)$$

implies

$$1 - \frac{|x|}{\sigma(tL)} \geq 1 - \frac{\sigma(2Lt)}{\sigma(tL)} \geq \frac{C_7}{T(\sigma(Lt))} \geq \left(\frac{At}{\sigma(tL)}\right)^2$$

by (2.14) and (2.15) provided $t$ is small enough. Thus (2.31) and (2.35) are satisfied so that by (2.36),

$$I_1 \leq C_6 \|W(f - P)\|_{L_p([\mathbb{R}])}^q \leq C_7 K_{r,p} (f, W, t^n)^q$$

(5.10)

by (5.7).

To deal with the estimation of $I_2$ we observe first much as in [8] that for

$$S(w) := \sum_{l=0}^{r-1} \frac{P^{(l)}(x)(w - x)^l}{l!} \in \mathcal{P}_{r-1}$$

we have by (2.28) that $\Delta_{\Phi_{t,L}(x)} S = 0$.

Thus expanding $P\left( x + \left( \frac{r}{2} - k \right) h\Phi_t(x) \right)$, $0 \leq k \leq r$, in a power series about $x$ gives

$$\Delta_{\Phi_{t,L}(x)} P(x) = \sum_{k=0}^{r} \binom{r}{k} (-1)^k P\left( x + \left( \frac{r}{2} - k \right) h\Phi_{tL}(x) \right)$$

$$= \sum_{k=0}^{r} \binom{r}{k} (-1)^k \left[ \sum_{l=0}^{r-1} + \sum_{l=r}^{n} \frac{\left( \frac{r}{2} - k \right) h\Phi_{tL}(x)\right]}{l!} \frac{P^{(l)}(x)}{l!}$$

$$= \sum_{k=0}^{r} \binom{r}{k} (-1)^k \sum_{l=r}^{n} \frac{\left( \frac{r}{2} - k \right) h\Phi_{tL}(x)\right]}{l!} \frac{P^{(l)}(x)}{l!},$$

so that

$$I_2 \leq C_8 \sum_{k=0}^{r} \binom{r}{k} \sum_{l=r}^{n} \frac{\left( \frac{r}{2} h \right)^{(l-r)q}}{l!q} \left\| P^{(l)}(x) \Phi_{tL}^l W \right\|_{L_p(x: |x| \leq \sigma(2Lt))}^q$$

$$\leq C_9 2^{rq}h^q \sum_{l=r}^{n} \frac{\left( \frac{r}{2} h \right)^{(l-r)q}}{l!q} \left\| P^{(l)}(x) \Phi_{tL}^l W \right\|_{L_p(x: |x| \leq \sigma(2Lt))}^q. \quad (5.11)$$
Now by repeated applications of Theorem 3.1, we have by using (5.5),
\[
\left\| P^{(r)} \Phi_{tL}^r W \right\|_{L_p([\mathbb{R}])} \\
\leq C_{10}^r \left\| P^{(r)} \Phi_{tL}^r W \right\|_{L_p([\mathbb{R}])} C_{11}^{l-r} \prod_{j=r}^{l-1} \left( \frac{n}{a_n} + \frac{j}{a_n} T(a_n)^{\frac{1}{2}} \right) 
\]  
(5.12)

where \(C_j, j = 10, 11\) are independent of \(n, x, l, L\) and \(h\). Now we observe using (2.6) that given \(\varepsilon > 0\), we have for \(n\) large enough and \(r \leq l \leq n\)
\[
\prod_{j=r}^{l-1} \left( \frac{n}{a_n} + \frac{j}{a_n} T(a_n)^{\frac{1}{2}} \right) \leq C_{12} \varepsilon^{l-r} \left( \frac{n}{a_n} \right)^{l-r} l! 
\]  
(5.13)

Here it is important that \(C_{12}\) does not depend on \(l, n, h\) or \(L\) and that \(C_{10}\) and \(C_{11}\) above are independent of \(\varepsilon\).

We may now substitute (5.13) into (5.12) so that (5.11) becomes
\[
I_2 \leq C_{13} h^{rq} \left\| P^{(r)} \Phi_{tL}^r W \right\|_{L_p([\mathbb{R}])} q \sum_{l=r}^{n} \left[ \left( \frac{hC_{16}}{2} \right)^{(l-r)q} l! \right] \\
\leq C_{14} t^{rq} \left\| P^{(r)} \Phi_{tL}^r W \right\|_{L_p([\mathbb{R}])} q \sum_{k=0}^{\infty} \left[ \frac{1}{2} \right]^k \\
\leq C_{15} t^{rq} \left\| P^{(r)} \Phi_{tL}^r W \right\|_{L_p([\mathbb{R}])} q \leq C_{16} K_{t, p} (f, W; t^r)^{q}. 
\]  
(5.14)

Thus combining (5.10) and (5.14) and taking \(s\) over \(0 \leq h \leq L t\) gives (5.8).

We proceed with the upper bound. This is more difficult than the lower bound and does not follow as easily using for example the methods of [8]. The crux is establishing the following quasi monotonicity type property of \(\overline{w}\).

**Lemma 5.2.** There exists \(C_j, j = 1, 2\) and \(0 < \varepsilon_0 < 1\) such that if \(0 < \lambda < \varepsilon_0\) and \(0 < s, t < C_1\) with
\[
\lambda \leq \frac{s}{l} \leq \varepsilon_0
\]  
(5.15)
we have
\[ \overline{w}_{r,p}(f, W, s) \leq C_2 \overline{w}_{r,p}(f, W, t). \]  
It is important that the \( C_j, j = 1, 2 \) and \( \varepsilon_0 \) do not depend on \( f, s \) and \( t \) but depend on \( \lambda \).

**Remark.** We remark that the above property is by no means obvious as recall our modulus is not necessarily monotone increasing. We prove it for \( p < \infty \) as the case \( p = \infty \) is much easier.

**Proof.** Let us write
\[
\overline{w}_{r,p}(f, W, s)^p \leq \frac{2^p}{s} \left[ \int_s^0 \left\| W \left( \Delta_{h\Phi_s(x)}(f) \right) \right\|_{L_p(|\xi| \leq \sigma(3\xi))}^p \right.
\left. + \left\| W \left( \Delta_{h\Phi_s(x)}(f) \right) \right\|_{L_p(|\xi| \leq \sigma(2\xi))}^p \right] dh
\]
\[
+ 2^p \inf_{r \text{ of } \deg \leq r-1} \| (f - R)W \|_{L_p(|\xi| \geq \sigma(4\xi))}^p.
\]
\[
= I_1 + I_2. \tag{5.17}
\]

Firstly, by choice of \( s \) and \( t \), \( \frac{s}{t} \leq 1 \) so that
\[ \sigma(4s) \geq \sigma(4t) \]
(recall \( \sigma \) is decreasing). Thus
\[
I_2 \leq 2^p \inf_{r \text{ of } \deg \leq r-1} \| (f - R)W \|_{L_p(|\xi| \geq \sigma(4\xi))}^p
\]
\[
\leq 2^p \overline{w}_{r,p}(f, W, t). \tag{5.18}
\]

Next we estimate \( I_1 \):
Write \( I_1 \leq I_3 + I_4 \), where
\[
I_3 := \frac{2^p}{s} \int_s^0 \left\| W \left( \Delta_{h\Phi_s(x)}(f) \right) \right\|_{L_p(|\xi| \leq \sigma(3\xi))}^p dh
\]
and
\[
I_4 := \frac{2^p}{s} \int_s^0 \left\| W \left( \Delta_{h\Phi_s(x)}(f) \right) \right\|_{L_p(|\xi| \leq \sigma(2\xi))}^p dh.
\]
We begin with the estimation of \( I_4 \). To this end we make use of Lemma 2.6. Much as in the proof of Lemma 5.1, we have
\[
I_4 \leq C_1 \inf_{r \text{ of } \deg \leq r-1} \| (f - R)W \|_{L_p(|\xi| \geq \sigma(4\xi))}^p \leq C_1 \overline{w}_{r,p}(f, W, t)^p. \tag{5.19}
\]
Here we used that for small $t$,
\[
\inf\{x - Mrs\Phi_s(x) : \sigma(3t) \leq x \leq \sigma(2s)\} \\
= \sigma(3t) - Mrs\Phi_s(\sigma(3t)) \\
\geq \sigma(3t) - C\ell\Phi_t(\sigma(3t)) \\
\geq \sigma(3t) - CtT(\sigma(t))^{\frac{n}{3}} \\
\geq \sigma(3t) + o(1/T(\sigma(t))) \geq \sigma(4t)
\]
by (2.3), (2.6), (2.14), (2.18) and as $\Phi_s$ is decreasing in $[0, \sigma(2s)]$.

It remains to estimate $I_3$:

As $s$ and $t$ are small enough, we can use (2.12), (2.19) and (2.30) to obtain a large enough positive integer $n$ such that $\frac{6}{n} < \frac{1}{n} \sim s$ and then divide $J := [-\sigma(3t), \sigma(3t)]$ into $O(1/s)$ intervals $J_k$ such that

$$|J_k| \leq s\Phi_s(x), \quad x \in J_k.$$

Formally, we do this by choosing a partition

$$-\sigma(3t) = \tau_0 < \tau_1 \ldots < \tau_n = \sigma(3t)$$

with

$$\frac{\int_{\tau_k}^{\tau_{k+1}} \Phi_{s}^{-1}(x)dx}{\int_{\tau_k}^{\tau_{k+1}} \Phi_{s}^{-1}(x)dx} = \frac{1}{n}, \quad 0 \leq k \leq n$$

and set:

$$J_k = [\tau_k, \tau_{k+1}].$$

Then if $|J_k|$ denotes the length of $J_k$ we have,

(1) \hspace{1cm} \Phi_s(x) \sim \Phi_s(y), \quad x, y \in J_k \hspace{1cm} (5.20)

and

(2) \hspace{1cm} W(x) \sim W(y), \quad x, y \in J_k.
Here the constants in the $\sim$ relation are independent of $x, y, s, k$.

Then

$$I_3 = \frac{2^p}{s} \int_0^s \left\| W \left( \Delta_{h\Phi_s}(x), f \right) \right\|_{L^p(1 \leq |z| \leq 3\varepsilon)}^p dh$$

$$\leq C_2 \sum_k W^p(t_k) \int_{s_k}^1 \frac{1}{s} \int_0^t \left| \Delta_{h\Phi_s}(x) \right|^p dh dx$$

$$= C_2 \sum_k W^p(t_k) \int_{s_k}^1 \frac{1}{s} \int_0^t \left| \Delta_{h\Phi_s}(x) \right|^p \frac{\Phi_1(x)}{\Phi_s(x)} du dx.$$

Now we may rewrite (2.17) for the given $s$ and $t$ as

$$\sup_{x \in \mathbb{R}} \frac{s \Phi_s(x)}{t \Phi_1(x)} \leq C \frac{s}{t} \sqrt{\log \left( 2 + \frac{t}{s} \right)}$$

for some $C \neq C(s, t)$. It follows that

$$\sup_{x \in \mathbb{R}} \frac{s \Phi_s(x)}{t \Phi_1(x)} \leq 1$$

if

$$s/t \leq \varepsilon_0$$

where $\varepsilon_0$ is independent of $s$ and $t$. Then if $\lambda < \varepsilon_0$, we have for $\lambda \leq s/t \leq \varepsilon_0$,

$$C_3 \leq \frac{\Phi_s(x)}{\Phi_1(x)} \leq C_4 \forall x \in \mathbb{R}$$

where $C_3$ and $C_4$ are independent of $s, t$ and $\varepsilon_0$. Then

$$I_3 \leq C_5 \sum_k W^p(t_k) \int_{s_k}^1 \frac{1}{s} \int_0^t \left| \Delta_{h\Phi_s}(x) \right|^p du dx$$

$$\leq C_6 \frac{1}{t} \int_0^t \left\| W \left( \Delta_{h\Phi_s}(x) \right) \right\|_{L^p(1 \leq |z| \leq 2t)}^p dh$$

$$\leq C_6 \mathcal{W}_{r,p}(f, W, t)^p.$$  \hspace{1cm} (5.21)

Combining our estimates (5.18), (5.19) and (5.21) give the lemma. \hfill \Box

**Lemma 5.3.** Let $W \in \mathcal{E}_1$ and assume (1.20). Let $r \geq 1$ and $0 < p \leq \infty$. Then for $0 < t < C_1$, there exists $C_2$, $C_3 \neq C_2$, $C_3 (f, t)$ such that

$$K_{r,p} (f, W, t') \leq C_2 \mathcal{W}_{r,p} (f, W, C_3 t).$$ \hspace{1cm} (5.22)\)
Proof. Put $\frac{t}{2} = \frac{a_u}{u}$ for some $u \geq u_0$ and let $n = n(t)$ be determined by (1.18), so that
\[
n = \inf \left\{ k : \frac{a_k}{k} \leq \frac{2a_u}{u} \right\}
\]
and
\[
\frac{1}{2} \frac{a_n}{n} \leq \frac{a_u}{u} < \frac{a_n}{n}.
\]
(5.23)

Now it is easy to see that for large enough $u$ and the given $n$,
\[
t = 2 \frac{a_u}{u} = \frac{a_n}{n} \lambda(n) C
\]
for some $\lambda(n) \in \left[ \frac{1}{2}, 1 \right]$ and $C > 0$ independent of $n$. We then apply (1.14), and choose $P \in \mathcal{P}_n$ such that
\[
\|(f-P)W\|_{L_p(\mathbb{R})} \leq C_1 \|w_{r,p}(f, W, C_2 t)\|
\]
for some $C_1, C_2 \neq C_1, C_2 (f, t)$.

We show that for some $C_3 \neq C_3 (f, t)$,
\[
t^r \left\| \mathcal{P}(r) \Phi^r W \right\|_{L_p(\mathbb{R})} \leq C_3 \|w_{r,p}(f, W, C_2 t)\|
\]
(5.25)
for then by (5.24),
\[
K_{r,p}(f, W, t^r) = \inf_{R \in \mathcal{P}_n} \left\{ \|(f-R)W\|_{L_p(\mathbb{R})} + t^r \left\| \mathcal{P}(r) \Phi^r W \right\|_{L_p(\mathbb{R})} \right\}
\leq \|(f-P)W\|_{L_p(\mathbb{R})} + t^r \left\| \mathcal{P}(r) \Phi^r W \right\|_{L_p(\mathbb{R})}
\leq (C_1 + C_3) \|w_{r,p}(f, W, C_2 t)\|.
\]

Thus we show (5.25).

Now let $\delta > 0$ be a small enough positive number and put $s := \delta t$. It is sufficient at this point of the proof to choose $\delta$ small enough so that by Lemma 5.2,
\[
\|w_{r,p}(f, W, s)\| \leq C_1 \|w_{r,p}(f, W, C_2 t)\|.
\]
(5.26)
Later, we will need to choose $\delta$ smaller still.
Let us recall much as in Lemma 5.1 that we have for $0 < h \leq s$

$$\Delta_h^{r} \Phi_s(x)P(x) = \sum_{k=0}^{r} \binom{r}{k} (-1)^k \sum_{l=r}^{n} \frac{\left[\left(\frac{r}{2} - k\right) h \Phi_s(x)\right]^l}{l!} P^{(l)}(x).$$  (5.27)

Applying (5.27) to $x^r \in P_r$ and using (2.28) gives

$$(r!)^{-1} \Delta_h^{r} \Phi_s(x)x^r = (h \Phi_s(x))^r = \sum_{k=0}^{r} \binom{r}{k} (-1)^k \frac{\left[\left(\frac{r}{2} - k\right) h \Phi_s(x)\right]^r}{r!}.$$  (5.28)

We now combine (5.27) and (5.28) together with (3.5) to give much as in (5.14),

$$\left\| W \Delta_h^{r} \Phi_s(x)P(x) - W (h \Phi_s(x))^r P^{(r)}(x) \right\|_{L_p(|x| \leq \sigma(2s))}^q \leq C_5 h^r \left\| WP^{(r)} \Phi_s(x) \right\|_{L_p(|x| \leq \sigma(2s))}^q \sum_{l=r+1}^{n} \left( C_6 \frac{n}{2} \frac{h}{a_n} \frac{(l-r)q}{l!} \right)$$

where $C_6$ is independent of $t, n, h, P_n$ and $l$.

Now by (2.26), (2.4) and (5.23) we can choose $\alpha > 3$ independent of $t, n, h, P_n$, $l$ and $C_2$ such that $a_u < a_{an}$. Further (if necessary) we make $\delta$ in the definition of $s$ smaller still so that

$$\delta < \min \left( \frac{1}{8\alpha}, \frac{1}{2} \right)$$

and

$$2s \leq \frac{t}{4\alpha} \leq \frac{a_{an}}{\alpha_n}.$$  (5.30)

This gives

$$\sigma(2s) \geq \sigma \left( \frac{t}{4\alpha} \right) \geq \sigma \left( \frac{a_{an}}{\alpha_n} \right) \geq a_{\xi_n}$$

for some fixed $3 < \xi < \alpha$.

It follows that we obtain using (5.31), (2.18) and (3.12),

$$\left\| W \Delta_h^{r} \Phi_s(x)P(x) - W (h \Phi_s(x))^r P^{(r)}(x) \right\|_{L_p(|x| \leq \sigma(2s))}^q \leq \frac{1}{2} h^r \left\| WP^{(r)} \Phi_s(x) \right\|_{L_p(|x| \leq \sigma(2s))}^q$$

(5.32)
provided \( \frac{a}{n} h \leq \Delta \), where \( \Delta \) is a fixed positive small number independent of \( t, h, n, P_n \) and \( l \).

Now by (5.30) and (5.23), it is easy to see that \( \Delta s \leq \Delta a \) so that \( \forall 0 < h \leq \Delta s \) we have

\[
\left\| W^{\Delta h \Phi_s (x)} P(x) \right\|_{L_p([|x| \leq \sigma (2s))} \geq h^{\Delta s} \left\| W^{\Phi_s (x)} \right\|_{L_p([|x| \leq \sigma (2s))}^p
\]

\[
- \left\| W^{\Delta h \Phi_s (x)} P(x) - W^{h \Phi_s (x)} \right\|_{L_p([|x| \leq \sigma (2s))}^p
\]

\[
\geq \frac{1}{2} h^{\Delta s} \left\| W P^{\Delta h \Phi_s (x)} P^r (x) \right\|_{L_p([|x| \leq \sigma (2s))}^p
\]

(by (3.32))

\[
\geq C_7 h^{\Delta s} \left\| W^{\Delta h \Phi_s (x)} P^r (x) \right\|_{L_p([|x| \leq \sigma (2s))}^p
\]  

(5.33)

by (3.12). Now raising (5.33) to the \( p/q \)th powers, integrating for \( h \) from 0 to \( \Delta s \) using the fact that \( \Phi_s (x) \sim \Phi_1 (x) \), \( x \in \mathbb{R} \) (see (2.18)) and assuming that \( \Delta < 1 \) as we may, gives

\[
\left\| W P^{\Delta h \Phi_s (x)} P^r (x) \right\|_{L_p([|x| \leq \sigma (2s))}^p
\]

\[
\leq \frac{C_8}{s} \int_0^{\Delta s} \left\| W^{\Delta h \Phi_s (x)} P(x) \right\|_{L_p([|x| \leq \sigma (2s))}^p dh
\]

\[
\leq \frac{C_8}{s} \int_0^{\Delta s} \left\| W^{\Delta h \Phi_s (x)} (P - f) \right\|_{L_p([|x| \leq \sigma (2s))}^p dh
\]

\[
+ \frac{C_8}{s} \int_0^{\Delta s} \left\| W^{\Delta h \Phi_s (x)} f(x) \right\|_{L_p([|x| \leq \sigma (2s))}^p dh
\]

\[
\leq C_9 \left\{ \left\| W (P - f) \right\|_{L_p([|x| \leq \sigma (2s))}^p + \overline{w}_{r,p} (f, W, \sigma) \right\}
\]

(by (2.36))

\[
\leq C_{10} \overline{w}_{r,p} (f, W, C_{2t})
\]

by (5.26) and (5.24). Thus we have (5.25) and the lemma. \( \square \)

We now combine Lemmas 5.1 and 5.3 to give

**The Proof of Theorem 1.3.** We have for any \( L > 0 \) and \( 0 < t < t_0 \),

\[
\overline{w}_{r,p} (f, W, Lt) \leq w_{r,p} (f, W, Lt) \leq C_1 K_{r,p} (f, W, t')
\]

\[
\leq C_2 \overline{w}_{r,p} (f, W, C_{3t}) \leq C_{2r} w_{r,p} (f, W, C_{3t})
\]  

(5.34)
where $C_3$ is independent of $L$, $f$ and $t$ while $C_1$ and $C_2$ are independent of $f$ and $t$ but depend on $L$.

Fix $M > 0$ and choose $L = MC_3$ and $s = C_3t$ to deduce that

$$w_{r,p}(f, W, Ms) \leq C_2 w_{r,p}(f, W, s)$$

and so we have the upper bound in (1.23). Similarly (5.34) gives

$$\bar{w}_{r,p}(f, W, Ms) \leq C_2 \bar{w}_{r,p}(f, W, s).$$

Then (5.34) gives

$$w_{r,p}(f, W, s) \sim \bar{w}_{r,p}(f, W, s) \sim K_{r,p}(f, W, s')$$

with constants independent of $f$ and $s$. The proof of the lower bound of (1.23) is similar and easier. □

6 The Proofs of Theorem 1.5 and Corollaries 1.6 and 1.7

We begin with:

The Proof of Theorem 1.5. For each $n \geq 0$, choose $P_n^*$ to be the best approximant to $f$ satisfying

$$\| (f - P_n^*) W \|_{L_p(\mathbb{R})} = E_n[f] W_p.$$ 

Here, we set $P_{n+1}^* = P_0^*$. Now let $t > 0$ be small enough and define $n$ by (1.18). Put $l = \lfloor \log_2 n \rfloor$ is the largest integer $\leq \log_2 n$ so that $2^l \leq n < 2^{l+1}$.

Then by Theorem 1.3 and Corollary 1.4

$$w_{r,p}(f, W, \frac{a_n}{n})^q \leq C_1 K_{r,p} \left( f, W, \left( \frac{a_n}{n} \right)^r \right)^q$$

$$\leq C_2 \left[ \| (f - P_n^*) W \|_{L_p(\mathbb{R})}^q + \left( \frac{a_n}{n} \right)^r \| P_n^* \|_{L_p(\mathbb{R})}^q \right]$$

$$\leq C_3 \left[ E_2[f] W_p^q + \left( \frac{a_n}{n} \right)^r \sum_{k=1}^{L-1} \| P_{2^k+1}^* - P_{2^k}^* \|_{L_p(\mathbb{R})}^q \right]$$
$$\leq C_4 \left[ E_{2^i} [f]^q_{W, p} \right. \\
+ \left( \frac{a_n}{n} \right)^{rq} \sum_{k=1}^{l-1} \left\| P_{2^{k+1}}^* - P_{2^k}^* \right\|_{L_p(\mathbb{R})}^q \left[ \phi_{2^{k+1}}^{2^{k+1}} \left( \log \left( \frac{2^{l-k}}{a_{2k}} \right) \right)^{\frac{q}{2}} \left( \frac{2^{k}}{a_{2k}} \right)^{\frac{q}{r}} \left( \frac{a_{2k}}{a_{2k+1}} \right)^{\frac{q}{r}} \left\| P_{2^{k+1}}^* - P_{2^k}^* \right\|_{L_p(\mathbb{R})}^q \right]$$

as $r \geq 1$ and by (2.17). This can be continued as

$$\leq C_5 \left[ E_{2^i} [f]^q_{W, p} + \left( \frac{a_n}{n} \right)^{rq} \sum_{k=1}^{l-1} \left( l - k + 1 \right)^{\frac{q}{r}} \left( \frac{2^{k}}{a_{2k}} \right)^{\frac{q}{r}} \left\| P_{2^{k+1}}^* - P_{2^k}^* \right\|_{L_p(\mathbb{R})}^q \right]$$

by (1.20).

We can continue this as

$$\leq C_6 \left[ E_{2^i} [f]^q_{W, p} + \left( \frac{a_n}{n} \right)^{rq} \sum_{k=1}^{l-1} \left( l - k + 1 \right)^{\frac{q}{r}} \left( \frac{2^{k}}{a_{2k}} \right)^{\frac{q}{r}} E_{2^k} [f]^q_{W, p} \right]$$

$$\leq C_7 \left( \frac{a_n}{n} \right)^{rq} \left[ \sum_{k=1}^{l} \left( l - k + 1 \right)^{\frac{q}{r}} \left( \frac{2^{k}}{a_{2k}} \right)^{\frac{q}{r}} E_{2^k} [f]^q_{W, p} \right]. \quad (6.1)$$

Now by (2.25) we have that $t \sim \frac{a_n}{n}$. Also by (2.18),

$$\Phi_{\gamma}(x) \sim \Phi_{\frac{a_n}{n}}(x), \quad x \in \mathbb{R}$$

so that by Theorem 1.3

$$K_{r, p} (f, W, t^r) \sim K_{r, p} \left( f, W, \left( \frac{a_n}{n} \right)^{r} \right)$$

and

$$w_{r, p}(f, W, t) \sim w_{r, p} \left( f, W, \frac{a_n}{n} \right). \quad (6.2)$$

Thus (6.2) becomes

$$w_{r, p}(f, W, t)^q \leq C_8 t^{rq} \left[ \sum_{k=1}^{l} \left( l - k + 1 \right)^{\frac{q}{2}} \left( \frac{2^{k}}{a_{2k}} \right)^{\frac{q}{r}} E_{2^k} [f]^q_{W, p} \right]$$

where $C_8 \neq C_8 (f, t)$. \quad \Box

We deduce

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The Proof of Corollary 1.6. Suppose first that 

\[ w_{r,p}(f, W, t) = O(t^a), \quad t \to 0^+ \]

Then in particular

\[ w_{r,p}(f, W, \frac{a_n}{n}) = O\left(\left(\frac{a_n}{n}\right)^a\right), \quad n \to \infty, \]

so that by Corollary 1.4

\[ E_n[f]_{W,p} = O\left(\left(\frac{a_n}{n}\right)^a\right). \]

Next suppose \( E_n[f]_{W,p} = O\left(\left(\frac{a_n}{n}\right)^a\right) \). Let \( 0 < \varepsilon < 1 \). Then, by (1.25)

\[
\begin{align*}
    w_{r,p}(f, W, \frac{a_n}{n}) &\leq C_1 \left(\frac{a_n}{n}\right)^r \left[ \sum_{k=1}^{l} (l - k + 1)^{\frac{ra}{2}} \left(\frac{2^k}{a_{2k}}\right)^{(r-a)q} \right]^{\frac{1}{q}} \\
    &\leq C_1 \left(\frac{a_n}{n}\right)^r \left[ \sum_{k=1}^{l} (l - k + 1)^{\frac{ra}{2}} \left(\frac{a_n/n}{a_{2k}/2^k}\right)^{(r-a)q} \right]^{\frac{1}{q}} \\
    &\leq C_2 \left(\frac{a_n}{n}\right)^a \left[ \sum_{k=1}^{l} (l - k + 1)^{\frac{ra}{2}} \left(\frac{2^{k+1}}{2^k}\right)^{(r-a)q(1+a)} \right]^{\frac{1}{q}} \quad \text{(by (2.11))} \\
    &\leq C_3 \left(\frac{a_n}{n}\right)^a \left[ \sum_{j=0}^{\infty} j^{\frac{ra}{2}} a^j q^j \right]^{\frac{1}{q}} \quad \text{(for some } 0 < a < 1) \\
    &\leq C_4 \left(\frac{a_n}{n}\right)^a . \quad (6.3)
\end{align*}
\]

Now for \( t > 0 \) small enough, we may determine \( n \) by (1.18) and using Theorem 1.3, (2.25) and (6.2) deduce the Corollary for \( t \). \( \square \)

We now proceed to prove Corollary 1.7. We need first a lemma that will prove useful in other related contexts.

Lemma 6.1. Let \( W \in \mathcal{E}_1, \quad r \geq 1, \quad 0 < p \leq \infty \) and assume (1.20). Then for \( n \geq C \) and \( \forall P_n \in \mathcal{P}_n \) satisfying

\[ \|(f - P_n) W\|_{L_p(\mathbb{R})} \leq LE_n[f]_{W,p} \quad (6.4) \]

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for some \( L \geq 1 \), we have
\[
\|(f - P_n) W\|_{L_p(\mathbb{R})} + \left(\frac{a_n}{n}\right)^r \left\| P_n \Phi_{an}^r \right\|_{L_p(\mathbb{R})} \sim K \left(f, W, \left(\frac{a_n}{n}\right)^r\right), \tag{6.5}
\]
where the constants in the \( \sim \) relation depend on \( L \) but are independent of \( n \) and \( f \).

We remark that in particular, (6.4) holds for \( P_n^* \) the best approximant to \( f \).

**Proof.** Let \( P^\# \) satisfy the required hypotheses. Then by the definition of \( K_{r,p} \left(f, W, \left(\frac{a_n}{n}\right)^r\right) \), we have
\[
\left\{ \left\|(f - P_n^\#) W\right\|_{L_p(\mathbb{R})} + \left(\frac{a_n}{n}\right)^r \left\| P_n^\# \Phi_{an}^r \right\|_{L_p(\mathbb{R})} \right\} \\
\geq K_{r,p} \left(f, W, \left(\frac{a_n}{n}\right)^r\right). \tag{6.6}
\]

Next choose \( P_n \) such that
\[
\left\{ \left\|(f - P_n) W\right\|_{L_p(\mathbb{R})} + \left(\frac{a_n}{n}\right)^r \left\| P_n \Phi_{an}^r \right\|_{L_p(\mathbb{R})} \right\} \\
\leq 2K_{r,p} \left(f, W, \left(\frac{a_n}{n}\right)^r\right). \tag{6.7}
\]

Then
\[
\left\| (P_n - P_n^\#) W \right\|_{L_p(\mathbb{R})}^q \\
\leq \left\| (P_n - f) W \right\|_{L_p(\mathbb{R})}^q + \left\| (f - P_n^\#) W \right\|_{L_p(\mathbb{R})}^q \\
\leq C_1 K_{r,p} \left(f, W, \left(\frac{a_n}{n}\right)^r\right)^q. \tag{6.8}
\]

by (6.7).

Further using (1.20), we can write using (6.8)
\[
\left\| (P_n - P_n^\#) \Phi_{an}^r \right\|_{L_p(\mathbb{R})}^q \\
\leq C_2 \left(\frac{n}{a_n}\right)^{rq} \left\| (P_n - P_n^\#) W \right\|_{L_p(\mathbb{R})}^q \\
\leq C_3 \left(\frac{n}{a_n}\right)^{rq} K_{r,p} \left(f, W, \left(\frac{a_n}{n}\right)^r\right)^q. \tag{6.9}
\]

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Thus by (6.8) and (6.9)
\[
\left( \frac{a_n}{n} \right)^r \left\| P_n^\#(r) \Phi_r f \right\|_{L_p(\mathbb{R})}^q \\
\leq C_4 \left[ \left( \frac{a_n}{n} \right)^r \left\| P_n^\#(r) \Phi_r f \right\|_{L_p(\mathbb{R})}^q + \left( \frac{a_n}{n} \right)^r \left\| (P_n - P_n^\#(r)) \Phi_r f \right\|_{L_p(\mathbb{R})}^q \right] \\
\leq C_5 K_{r,p} \left( f, W, \left( \frac{a_n}{n} \right)^r \right)^q.
\]
so that (6.6) and (6.10) give the result. □

We can now give:

**The Proof of Corollary 1.7 (a).** We shall show that
\[
\left\| W \Delta_h^r \Phi_t(f, x, \mathbb{R}) \right\|_{L_p[|x| \leq \sigma(2t)]} \leq C_1 t^r \left\| f^r \Phi_t W \right\|_{L_p(\mathbb{R})}
\]
and
\[
\inf_{P \in \mathcal{P}_{r-1}} \left\| W(f - P) \right\|_{L_p[|x| \geq \sigma(4t)]} \leq C_2 t^r \left\| f^r \Phi_t W \right\|_{L_p(\mathbb{R})}.
\]

We begin with:

**The Proof of (6.11).** We begin with an observation.

If \( h > 0 \) we may write
\[
|\Delta_h^r (f, x, \mathbb{R})| = \left| \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \cdots \int_{-\frac{h}{2}}^{\frac{h}{2}} f^{(r)}(x + t_1 + \cdots + t_r) dt_1 \cdots dt_r \right| \\
\leq h^{r-1} \int_{-\frac{h}{2}}^{\frac{h}{2}} |f^{(r)}(x + s)| ds.
\]
Now note that for \( s \in \left[ -\frac{rh\Phi_t(x)}{2}, \frac{rh\Phi_t(x)}{2} \right] \) and \( x \in [-\sigma(2t), \sigma(2t)] \) we have by (2.26)
\[
\Phi_t(x) \sim \Phi_t(x + s).
\]
Thus we may deduce from (6.13) that for \(|x| \leq \sigma(2t)| as
\[
\left| W \Delta_h^r \Phi_t(f, x, \mathbb{R}) \right| \leq C_3 h^r \frac{1}{\frac{rh\Phi_t(x)}{2}} \int_{-\frac{rh\Phi_t(x)}{2}}^{\frac{rh\Phi_t(x)}{2}} |W f^{(r)}(x + s)| ds.
\]
Case 1. \( p > 1 \). We recall the definition of the maximal function operator

\[
M[g](x) := \sup_{u>0} \frac{1}{2u} \int_{-u}^{u} |g(x+s)| \, ds
\]

which is bounded from \( L_p \) to \( L_p \), \( 1 < p \leq \infty \). It follows that (6.14) can be rewritten as

\[
\left\| W \Delta_h \Phi_t(x) \left( f, x, \mathbb{R} \right) \right\|_{L_p[|x| \leq \sigma(2\zeta)]} \leq C_4 h^r \left\| M \left[ W \Phi_t^r f(r) \right] \right\|_{L_p(\mathbb{R})}
\]

\[
\leq C_5 t^r \left\| f(r) \Phi_t^r W \right\|_{L_p(\mathbb{R})}.
\]

Case 2. \( p = 1 \). Integrating (6.14) and noting that if \( u = x+s \), then for the range of \( x \) and \( s \) above,

\[
\Phi_t(x) \sim \Phi_t(x+s),
\]

we obtain

\[
\int_{|x| \leq \sigma(2\zeta)} \left| W \Delta_h \Phi_t(x) \left( f, x, \mathbb{R} \right) \right| \, dx
\]

\[
\leq C_6 h^{r-1} \int_{|x| \leq \sigma(2\zeta)} \frac{1}{\Phi_t(x)} \int_{|s| \leq \frac{h}{2} \Phi_t(x)} \left| W f(r) \Phi_t^r \right| (x+s) \, ds \, dx
\]

\[
\leq C_7 h^{r-1} \int_{u=x+s \in \mathbb{R}} \frac{1}{\Phi_t(u)} \int_{|s| \leq \frac{h}{2} \Phi_t(u)} ds \, du
\]

\[
\leq C_8 h^{r} \int_{\mathbb{R}} \left| f(r) W \Phi_t^r \right| (u) \, du.
\]

Next we give:

The Proof of (6.12). We mimic the proof of (4.2) for \( p > 1 \). For the given \( t > 0 \), write \( 4t = \frac{a}{u} \). Determine \( n = n(t) \) by (1.18) and recall \( u \sim n \) (see (2.26)) so that

(a) \[ \sigma(4t) \leq a_u \leq a_{\alpha n} \] (6.15)
(b)\[
\sigma(4t) \geq a_{\frac{\alpha}{2}} \geq a_{\beta n}
\]
for some \( \alpha > 1 \) and \( \beta > 0 \).

As in the proof of (5.9), we may without loss of generality suppose that \( x > 0 \). Suppose first that \( r = 1 \). We have
\[
\inf_{P \in \mathcal{P}_{r-1}} \|W(f - P)\|_{L_p[x \geq \sigma(4t)]} \leq \|W(f - f(a_{\beta n}))\|_{L_p[x \geq a_{\beta n}]} = \left\|W(x) \int_{a_{\beta n}}^{x} f'(u) du\right\|_{L_p[x \geq a_{\beta n}]}
\]
\[
\leq C_4 \frac{a_n}{nT(a_n)^\frac{1}{2}} \|Wf'\|_{L_p[x \geq a_{\beta n}]}
\]
\[
\leq C_5 \frac{a_n}{T(a_n)^\frac{1}{2}} \|Wf'\|_{L_p[x \geq a_{\beta n}]}
\]
\[
\leq C_6 \frac{a_n}{T(\sigma(t))^\frac{1}{2}} \|Wf'\|_{L_p[x \geq a_{\beta n}]}
\]
\[
\leq C_7 \frac{a_n}{n} \|Wf'\|_{L_p[x \geq a_{\beta n}]}
\]
by Lemma 4.2, (2.2) and (2.16).

Assume (6.16) holds for 1, 2, \ldots, \( r - 1 \). Choose \( S \in \mathcal{P}_{r-2} \) such that
\[
\|W(f' - S)\|_{L_p[x \geq \sigma(t)]} \leq C_6 \left(\frac{a_n}{n}\right)^{r-1} \|f^{(r)}\Phi_{\sigma(t)}^{-1}W\|_{L_p(\mathbb{R})}.
\]
Set
\[
P(x) := f(a_{\beta n}) + \int_{a_{\beta n}}^{x} S(u) du
\]
Then we can bound the left hand side of (6.12) by
\[
\inf_{P \in \mathcal{P}_{r-1}} \|W(f - P)\|_{L_p[x \geq a_{\beta n}]}
\]
\[
\leq C_7 \left\|W(x) \int_{a_{\beta n}}^{x} (f' - S)(u) du\right\|_{L_p[x \geq a_{\beta n}]}
\]
\[
\leq C_8 \frac{a_n}{n^rT(a_n)^\frac{1}{2}} \|f^{(r)}W\Phi_{\sigma(t)}^{-1}\|_{L_p[x \geq a_{\beta n}]}
\]
\[
\leq C_9 \|f^{(r)}\Phi_{\sigma(t)}^{-1}W\|_{L_p(\mathbb{R})}
\]
(6.17)
and we have our result. □

Finally we give:

**Proof of Corollary 1.7 (b).** Write $t = \frac{a_n}{u}$ and let $n = n(t)$ be determined by (1.18).

Firstly

\[
K_{r,p} (f, W, t^*) = \inf_{P \in \mathcal{P}_n} \left\{ \|(f - P)W\|_{L_p(\mathbb{R})} + t^* \left\| WP_n^{(r)} \Phi_t^r \right\|_{L_p(\mathbb{R})} \right\}
\]

\[
\geq \inf_g \left\{ \|(f - g)W\|_{L_p(\mathbb{R})} + t^* \left\| Wg^{(r)} \Phi_t^r \right\|_{L_p(\mathbb{R})} \right\}
\]

\[
= K_{r,p}^* (f, W, t^*) . \tag{6.18}
\]

Next, we may choose $g$ such that

\[
\|(f - g)W\|_{L_p(\mathbb{R})} + t^* \left\| Wg^{(r)} \Phi_t^r \right\|_{L_p(\mathbb{R})} \leq 2K_{r,p}^* (f, W, t^*) \tag{6.19}
\]

Also by Lemma 6.1, Theorem 1.3 and Corollary 1.4 we may choose $P_n$ such that

\[
\|(P_n - g)W\|_{L_p(\mathbb{R})} \leq C_2 w_{r,p} \left( g, W, \frac{a_n}{n} \right) \tag{6.20}
\]

and

\[
\left( \frac{a_n}{n} \right)^r \left\| WP_n^{(r)} \Phi_t^r \right\|_{L_p(\mathbb{R})} \leq C_3 w_{r,p} \left( g, W, \frac{a_n}{n} \right) . \tag{6.21}
\]

Thus by (6.19–6.21) we have

\[
K_{r,p} (f, W, t^*) \leq \left\{ \|(f - P_n)W\|_{L_p(\mathbb{R})} + t^* \left\| WP_n^{(r)} \Phi_t^r \right\|_{L_p(\mathbb{R})} \right\}
\]

\[
\leq C_4 \left[ \|(f - g)W\|_{L_p(\mathbb{R})} + \|(g - P_n)W\|_{L_p(\mathbb{R})} + t^* \left\| WP_n^{(r)} \Phi_t^r \right\|_{L_p(\mathbb{R})} \right]
\]

\[
\leq C_5 \left[ \|(f - g)W\|_{L_p(\mathbb{R})} + w_{r,p} \left( g, W, \frac{a_n}{n} \right) \right] \tag{by (6.2) }
\]

\[
\leq C_6 \left[ \|(f - g)W\|_{L_p(\mathbb{R})} + w_{r,p} (g, W, t) \right] \tag{by Corollary 1.7 (a)}
\]

\[
\leq C_7 \left[ \|(f - g)W\|_{L_p(\mathbb{R})} + t^* \left\| g^{(r)} \Phi_t^r W \right\|_{L_p(\mathbb{R})} \right] \tag{by Corollary 1.7 (a)}
\]

\[
\leq C_8 K_{r,p}^* (f, W, t^*) . \tag{6.22}
\]

Then (6.18) and (6.22) give the result. □
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