Energy estimates and the Weyl criterion on compact homogeneous manifolds

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Summary. The purpose of this paper is to demonstrate that a number of results concerning approximation, integration, and uniform distribution on spheres can be generalised to a much wider range of compact homogeneous manifolds. The essential ingredient is that a certain type of invariant kernels on the manifolds (the generalisation of zonal kernels on spheres or radial kernels in the euclidean spaces) have a spectral decomposition in terms of projection kernels onto invariant polynomial subspaces. In particular, we establish a Weyl's criterion on such manifolds and announce a discrepancy estimate that generalises some pertinent results of Damelin and Grabner.

Keywords and Phrases: Compact Homogeneous Manifold, Energy, Invariant Kernels, Invariant Polynomial Subspaces, Numerical Integration, Projection Kernels, Spherical Harmonic, Uniform Distribution, Weyl's Criterion.

1 Introduction

Let M be a $d \geq 1$ dimensional homogeneous space of a compact Lie group G embedded in $\mathbb{R}^{d+r}$ for some $r \geq 0$. Then (see [6]), we may assume that $G \subset O(d+r)$, the orthogonal group on $\mathbb{R}^{d+r}$. Thus $M = \{gp : g \in G\}$ where $p \in M$ is a non-zero vector in $\mathbb{R}^{d+r}$. For technical reasons, we will assume that M is *reflexive*. That is, for any given $x, y \in M$, there exists $g \in G$ such that $gx = y$ and $gy = x$.

Let $d(x,y)$ be the geodesic distance between $x, y, \in M$ induced by the embedding of M in $\mathbb{R}^{d+r}$ (see [5] for details). On the spheres, this corresponds to the usual geodesic distance. A real valued function $\kappa(x,y)$ defined on $M\times M$ is called a positive definite kernel on M, if for every nonempty finite subset $Y\subset M$, and arbitrary real numbers c_y, $y\in Y$, we have

$$\sum_{x\in Y}\sum_{y\in Y} c_x c_y \kappa(x,y)\geq 0.$$

If the above inequality becomes strict whenever the points y are distinct, and not all the c_y are zero, then the kernel κ is called strictly positive definite. A kernel κ is called G-invariant if $\kappa(gx,gy)=\kappa(x,y)$ for all $x,y\in M$ and $g\in G$. For example, if $M:=S^d$, the d dimensional sphere realized as a subset of $\mathbb{R}^{d+1}$ and $G:=O(d+1)$, then all the G–invariant kernels have the form $\phi(xy)$, where $\phi:[-1,1]\to\mathbb{R}$, and where xy denotes the usual inner product of x and y. A kernel of the form $\phi(xy)$ is often called a zonal kernel on the sphere in the literature.

Let μ be a G–invariant measure on M (which may be taken as an appropriately normalized 'surface' measure). Then, for two functions $f,g:M\to\mathbb{R}$, we define an inner product with respect to μ:

$$[f,g]=[f,g]_\mu:=\int_M fg\,d\mu$$

and let $L_2(M)_\mu$ denote the space of all square integrable functions from M into $\mathbb{R}$ with respect to the above inner product. In the usual way, we identify all functions as being equal in $L_2(M)_\mu$, if they are equal almost everywhere with respect to the measure μ.

Let $n\geq 0$ and P_n be the space of polynomials in $d+r$ variables of degree n restricted on M. Here, multiplication is taken pointwise on $\mathbb{R}^{d+r}$. The *harmonic polynomials* of degree n on M are $H_n:=P_n\bigcap P_{n-1}^{\perp}$. We may always (uniquely) decompose H_n into irreducible G-invariant subspaces $H_{n,k}$, $k=1,\ldots,\nu_n$. Indeed, the uniqueness of the decomposition follows from the minimality of the G–invariant space, since a different decomposition would give subspaces contained in minimal ones leading to a contradiction.

Any G–invariant kernel κ, has an associated integral operator which we define by

$$T_\kappa f(x)=\int_M \kappa(x,y)f(y)\,d\mu(y).$$

Now, for $n\geq 0, k\geq 1$, let $Y_{n,k}^1,\ldots,Y_{n,k}^{d_{n,k}}$ be any orthonormal basis for $H_{n,k}$, and set

$$Q_{n,k}(x,y):=\sum_{j=1}^{d_{n,k}} Y_{n,k}^j(x)Y_{n,k}^j(y).$$

Then $Q_{n,k}$ is the unique G-invariant kernel for the orthogonal projection $T_{Q_{n,k}}$ of $L_2(M)_\mu$ onto $H_{n,k}$ acting as

$$T_{Q_{n,k}} f(x) = \int_M Q_{n,k}(x,y) f(y)\, d\mu(y).$$

The symmetry of $Q_{n,k}$ in x and y implies that it is positive definite on M. In fact, for every nonempty finite subset $Y \subset M$, and arbitrary real numbers $c_y, y \in Y$, we have

$$\begin{aligned}\sum_{x\in Y}\sum_{y\in Y} c_x c_y Q_{n,k}(x,y) &= \sum_{j=1}^{d_{n,k}} \Big(\sum_{x\in Y} c_x Y^j_{n,k}(x)\Big)\Big(\sum_{y\in Y} c_y Y^j_{n,k}(y)\Big) \\ &= \sum_{j=1}^{d_{n,k}} \Big(\sum_{x\in Y} c_x Y^j_{n,k}(x)\Big)^2 \\ &\geq 0.\end{aligned}$$

We summarise a few basic facts about G-invariant kernels in the following lemma:

Lemma 1. *Let y, z be fixed points in M. Then*

a. $\int_M Q_{n,k}(y,x) Q_{n,k}(x,z) d\mu(x) = Q_{n,k}(y,z)$.
b. For all $x \in M$, we have $Q_{n,k}(x,x) = d_{n,k}$.
c. If κ is a G-invariant kernel, then for all pairs of $(x,y) \in M \times M$, we have $\kappa(x,y) = \kappa(y,x)$.
d. For all $(x,y) \in M \times M$, we have $|Q_{n,k}(x,y)| \leq Q_{n,k}(x,x)$.

Proof: Part (a) follows directly from the fact that $Q_{n,k}$ is the projection kernel from $L_2(M)_\mu$ onto $H_{n,k}$.

Part (b) is a consequence of the equation

$$Q_{n,k}(x,x) := \sum_{j=1}^{d_{n,k}} Y^j_{n,k}(x) Y^j_{n,k}(x).$$

Indeed, since $Q_{n,k}$ is G-invariant, $Q_{n,k}(x,x)$ is a constant function of x for all $x \in M$. Integrating the last equation over M and using the orthonormality of the $Y^j_{n,k}$, we then arrive at the desired result.

The proof of Part (c) needs the reflexivity of M. Indeed, pick a $g \in G$ so that $gx = y$ and $gy = x$. Then

$$\kappa(x,y) = \kappa(gy, gx) = \kappa(y,x)$$

using the G-invariance of κ.

Part (d) follows from a standard positive definiteness argument. Indeed, for each fixed pair $(x,y) \in M \times M$, the positive definiteness of the kernel $Q_{n,k}$ implies that the matrix

$$\begin{pmatrix} Q_{n,k}(x,x) & Q_{n,k}(x,y) \\ Q_{n,k}(y,x) & Q_{n,k}(y,y) \end{pmatrix}$$

is nonnegative definite, which further implies that

$$(Q_{n,k}(x,x))\,(Q_{n,k}(y,y)) - (Q_{n,k}(x,y))\,(Q_{n,k}(y,x)) \geq 0.$$

Since $Q_{n,k}(x,x) = Q_{n,k}(y,y)$, by Part (b), and $Q_{n,k}(x,y) = Q_{n,k}(y,x)$ by Part (c), we have the desired inequality. □

An important consequence of the development above is that each irreducible subspace is generated by the translates of a fixed element. For this result on the sphere S^d, see, for instance, [1].

Proposition 1. *Let $Y \in H_{n,k}$, $Y \neq 0$. Then $H_{n,k} = \mathrm{span}\{Y(g\cdot); g \in G\}$.*

Proof: It is clear that $V = \mathrm{span}\{Y(g\cdot); g \in G\}$ is a G-invariant subspace of $H_{n,k}$, and since Y is not zero this is a non-trivial subspace. But $H_{n,k}$ is irreducible, so that V cannot be a proper subspace of $H_{n,k}$. Thus $V = H_{n,k}$. □

Lemma 2. *Let κ_1 and κ_2 be continuous G-invariant kernels. If M is a reflexive space, $T_{\kappa_1}T_{\kappa_2} = T_{\kappa_2}T_{\kappa_1}$.*

Proof: Let $f \in L_2(M)_\mu$. Then

$$\begin{aligned}[T_{\kappa_1}T_{\kappa_2}f](x) &= \int_M \kappa_1(x,y)\left\{\int_M \kappa_2(y,z)f(z)d\mu(z)\right\}d\mu(y)\\ &= \int_M f(z)\left\{\int_M \kappa_1(x,y)\kappa_2(y,z)d\mu(y)\right\}d\mu(z).\end{aligned}$$

Since the manifold is reflexive there is a $g \in G$ which interchanges x and z. Thus,

$$\int_M \kappa_1(x,y)\kappa_2(y,z)d\mu(y) = \int_M \kappa_1(z,y)\kappa_2(y,x)d\mu(y),$$

so that

$$\begin{aligned}[T_{\kappa_1}T_{\kappa_2}f](x) &= \int_M f(z)\left\{\int_M \kappa_1(z,y)\kappa_2(y,x)d\mu(y)\right\}d\mu(z)\\ &= \int_M \kappa_2(x,y)\left\{\int_M \kappa_1(y,z)f(z)d\mu(z)\right\}d\mu(y)\\ &= [T_{\kappa_2}T_{\kappa_1}f](x),\end{aligned}$$

where the penultimate step uses Lemma 1 (c). The changes of order of integration are easy to justify since the kernels are continuous and $f \in L_2(M)_\mu$. □

We are now able to show that a G–invariant kernel has a spectral decomposition in terms of projection kernels onto invariant polynomial subspaces. This is contained in the following theorem.

Theorem 1. *If M is a reflexive manifold, then any G–invariant kernel κ has the spectral decomposition*

$$\kappa(x,y) = \sum_{n=0}^{\infty} \sum_{k=1}^{\nu_n} a_{n,k}(\kappa) Q_{n,k}(x,y),$$

where

$$a_{n,k}(\kappa) = \frac{1}{d_{n,k}} \int_M \kappa(x,y) Q_{n,k}(x,y)\, d\mu(y),\ n \geq 0,\ k \geq 1.$$

Here the convergence is in the topology of $L_2(M)_\mu$.

Proof: If $Y \in H_{n,k}$ then $T_{Q_{n,k}} Y = Y$. Thus

$$\begin{aligned} T_\kappa Y &= T_\kappa (T_{Q_{n,k}} Y) \\ &= T_{Q_{n,k}} (T_\kappa Y) \in H_{n,k}, \end{aligned}$$

since $T_{Q_{n,k}}$ is the orthogonal projection onto $H_{n,k}$. Here we have used Lemma 2.

Since T_κ is a symmetric operator, it can be represented on the finite dimensional subspace by a symmetric matrix. Either this matrix is the zero matrix, in which case all the pertinent $a_{n,k}(\kappa)$ are zero, or T_κ has a non-trivial range. Since the matrix is symmetric, it must have a non-zero real eigenvalue. Let γ be a nonzero eigenvalue of the matrix, and let Y be an associated eigenvector, i.e., $T_\kappa Y = \gamma Y$. This implies that, for any fixed $g \in G$, $Y(g\cdot)$ is also an eigenvector. In fact, we have

$$\begin{aligned} [T_\kappa Y(g\cdot)](x) &= \int_M \kappa(x,y) Y(gy) d\mu(y) \\ &= \int_M \kappa(x, g^{-1}y) Y(y) d\mu(g^{-1}y) \\ &= \int_M \kappa(gx, y) Y(y) d\mu(y), \end{aligned}$$

using the G-invariance of both κ and μ. But Y is an eigenvector of T_κ, so that

$$[T_\kappa Y(g\cdot)](x) = \gamma Y(gx).$$

Now, using Proposition 1 we see that $H_{n,k}$ is an eigenspace for T_κ with single eigenvalue γ. We can compute γ by evaluating T_κ on $Q_{n,k}(\cdot, y)$ for a fixed y:

$$\int_M \kappa(z,x) Q_{n,k}(x,y) d\mu(x) = \gamma Q_{n,k}(z,y).$$

Setting $z = y$ and using Lemma 1 (b) we have

$$\gamma = \frac{1}{d_{n,k}} \int_M \kappa(y,x) Q_{n,k}(x,y) d\mu(x),$$

and the appropriate form for γ follows using the symmetry of G-invariant kernels (Lemma 1 (c)). $\square$

2 Weyl's criterion

Weyl's criterion concerns uniformly distributed sequences $\{x_l : l \in \mathbf{N}\} \subset M$. These are sequences for which

$$\lim_{N\to\infty} \frac{1}{N}\sum_{l=1}^{N} \delta_{x_l}$$

converge weakly to the measure μ. In this section we provide alternative characterisations for uniformly distributed sequences. The equivalence of the above definition to that of Part (a) of the following theorem follows from standard arguments (see Kuipers and Niederreiter [4]).

In this section, we assume that $a_{n,k}(\kappa) > 0$ for all n, k, and

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\nu_n} d_{n,k} a_{n,k}(\kappa) < \infty. \tag{1}$$

Thus κ is bounded and continuous on $M \times M$. More importantly for our purpose in this section, κ is strictly positive definite on M. We will prove the equivalence of two characterisations of uniform distribution of points on M. Our main result of this section is as follows.

Theorem 2. *The following two criteria of a uniformly distributed sequence on M are equivalent.*

a. *A sequence $\{x_l : l \in \mathbf{N}\}$ is uniformly distributed on M if and only if*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{l=1}^{N} Y_{n,k}^{j}(x_l) = 0$$

for all $n \geq 0$ and $1 \leq k \leq \nu_n$, $1 \leq j \leq d_{n,k}$.

b. *Let κ be a strictly positive definite G-invariant kernel on M. A sequence $\{x_l : l \in \mathbf{N}\}$ is uniformly distributed on M if and only if*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{l=1}^{N} \kappa(x_l, y) = a_{0,0}(\kappa),$$

holds true uniformly for $y \in M$.

Proof: Using the series expansion for κ we have for any $y \in M$,

$$\frac{1}{N}\sum_{l=1}^{N} \kappa(x_l, y) = \sum_{n=0}^{\infty}\sum_{k=1}^{\nu_n} a_{n,k}(\kappa) \sum_{j=1}^{d_{n,k}} Y_{n,k}^{j}(y) \left(\frac{1}{N}\sum_{l=1}^{N} Y_{n,k}^{j}(x_l)\right). \tag{2}$$

Suppose $\{x_l : l \in \mathbf{N}\}$ is uniformly distributed by Criterion (a). Using Lemma 1, Part (d), we can dominate the right hand side of the last equation by

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\nu_n} a_{n,k}(\kappa)\frac{1}{N}\sum_{l=1}^{N}|Q_{n,k}(x_l,y)| \leq \sum_{n=0}^{\infty}\sum_{k=1}^{\nu_n} d_{n,k}a_{n,k}(\kappa).$$

The right hand side of the inequality is bounded from Equation (1). This allows us to use the dominated convergence theorem to pass the limit in N through the sum to get

$$\begin{aligned}
&\lim_{N\to\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\nu_n} a_{n,k}(\kappa)\sum_{j=1}^{d_{n,k}} Y_{n,k}^j(y)\left(\frac{1}{N}\sum_{l=1}^{N} Y_{n,k}^j(x_l)\right)\\
&=\sum_{n=1}^{\infty}\sum_{k=1}^{\nu_n} a_{n,k}(\kappa)\sum_{j=1}^{d_{n,k}} Y_{n,k}^j(y)\lim_{N\to\infty}\left(\frac{1}{N}\sum_{l=1}^{N} Y_{n,k}^j(x_l)\right)\\
&=0,
\end{aligned}$$

by assumption. Thus

$$\lim_{N\to\infty}\frac{1}{N}\sum_{l=1}^{m}\kappa(x_l,y)=a_{0,0}(\kappa)$$

uniformly for each y by (1), and the sequence $\{x_l : l\in\mathbf{N}\}$ is thus uniformly distributed by Criterion (b).

Conversely suppose that $\{x_l : l\in\mathbf{N}\}$ is uniformly distributed by Criterion (b). Then, as in Equation (2), we have

$$\frac{1}{N^2}\sum_{m=1}^{N}\sum_{l=1}^{N}\kappa(x_m,x_l)=\sum_{n=0}^{\infty}\sum_{k=1}^{\nu_n} a_{n,k}(\kappa)\sum_{j=1}^{d_{n,k}}\left(\frac{1}{N}\sum_{l=1}^{N} Y_{n,k}^j(x_l)\right)^2.$$

Now, for each x_m, by hypothesis

$$\lim_{N\to\infty}\frac{1}{N}\sum_{l=1}^{N}\phi(x_m,x_l)=\int_M \phi(x_m,x)d\mu(x)=a_{0,0}(\kappa).$$

Thus,

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{m=1}^{N}\sum_{l=1}^{N}\phi(x_l,x_j)=\int_M \phi(x,x_j)d\mu(x)=a_{0,0}(\kappa).$$

Therefore

$$\lim_{N\to\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\nu_n} a_{n,k}(\kappa)\sum_{j=1}^{d_{n,k}}\left(\frac{1}{N}\sum_{l=1}^{N} Y_{n,k}^j(x_l)\right)^2=0,$$

and since $a_{n,k}(\kappa)>0$, $n\in\mathbf{N}$ and $1\leq k\leq\nu_n$, it must be that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{l=1}^{N} Y_{n,k}^{j}(x_l) = 0,$$

so that $\{x_l : l \in \mathbf{N}\}$ is uniformly distributed by (a). $\square$

We note that Criterion (a) is called Weyl's criterion in the literature.

3 Energy on manifolds

In this section, we work with kernels κ that satisfy the following two conditions:

1. There exists a positive constant C, independent of x, such that
$$\int_M |\kappa(x, y)| d\mu(y) \le C.$$
2. For each non-trivial continuous function ϕ on M, we have
$$\int_M \int_M \kappa(x, y)\phi(x)\phi(y) d\mu(x) d\mu(y) > 0.$$

We will call a kernel κ satisfying the above two conditions *admissible*. The archetype for admissible kernels is the *Riesz kernel*

$$\kappa(x, y) = \|x - y\|^{-s}, \quad 0 < s < d + r, \quad x, y \in M,$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^{d+r}$.

We are interested in studying errors of numerical integration of continuous functions $f : M \to \mathbb{R}$ over a set $Z \subset M$ of cardinality $N \ge 1$. In particular, we seek a generalization of results of Damelin and Grabner in [2]. More precisely, given an admissible kernel κ and such a point set Z, we define the discrete energy

$$E_\kappa(Z) = \frac{1}{N^2} \sum_{\substack{y, z \in Z \\ y \ne z}} \kappa(y, z)$$

and for the normalised G–invariant measure μ on M, denote by

$$R(f, Z, \mu) := \left| \int_M f d\mu - \frac{1}{N} \sum_{y \in Z} f(y) \right|$$

the error of numerical integration of f with respect to μ over M.

For an admissible kernel κ and probability measure ν on M, we define the energy integral

$$\mathcal{E}_\kappa(\nu) = \int_M \int_M \kappa(x, y) d\nu(x) d\nu(y).$$

We have

Lemma 3. *The energy integral $\mathcal{E}_\kappa(\nu)$ is uniquely minimised by the normalized G–invariant measure μ.*

Proof: Since κ satisfies Condition 2 we have $a_{n,k} > 0$, $\mathcal{E}_\kappa(\nu) \geq 0$ for every Borel probability measure ν. Also, a simple computation shows that $\mathcal{E}_\kappa(\mu) = a_{0,0}(\kappa)$.

Next, for an arbitrary probability measure σ on M, we use Lemma 1, Part (d) to write down

$$\begin{aligned}
\mathcal{E}_\kappa(\sigma) &= \int_M \int_M \left\{ \sum_{n=0}^{\infty} \sum_{k=1}^{\nu_n} a_{n,k}(\kappa) Q_{n,k}(x,z) \right\} d\sigma(x) d\sigma(z) \\
&= a_{0,0}(\kappa) + \sum_{n=1}^{\infty} \sum_{k=1}^{\nu_n} a_{n,k}(\kappa) \int_M \int_M Q_{n,k}(x,z) d\sigma(x) d\sigma(z) \\
&= a_{0,0}(\kappa) + \sum_{n=1}^{\infty} \sum_{k=1}^{\nu_n} a_{n,k}(\kappa) \int_M \int_M \int_M Q_{n,k}(x,y) Q_{n,k}(y,z) d\mu(y) d\sigma(x) d\sigma(z) \\
&= a_{0,0}(\kappa) + \sum_{j=1}^{\infty} \sum_{k=1}^{\nu_n} a_{n,k}(\kappa) \int_M \left\{ \int_M Q_{n,k}(x,y) d\sigma(x) \right\}^2 d\mu(y).
\end{aligned}$$

If ν is a probability measure on M that minimises $\mathcal{E}_\kappa(\sigma)$, i.e.,

$$\mathcal{E}_\kappa(\nu) = \min_\sigma \mathcal{E}_\kappa(\sigma),$$

where the minimum is taken over all the probability measures on M, then ν must satisfy

$$\int_M Q_{n,k}(x,y) d\nu(x) = 0, \quad k = 1, \ldots, \nu_n, \quad n = 1, \ldots .$$

Hence, since μ also annihilates all polynomials of degree ≥ 0, $\nu - \mu$ annihilates all polynomials. Because the polynomials are dense in the continuous functions, we see that $\nu - \mu$ is the zero measure and the result is proved. $\square$

Heuristically, one expects that a point distribution Z of minimal energy gives a discrete approximation to the measure μ, in the sense that the integral with respect to the measure is approximated by a discrete sum over the points of Z. For the sphere, this was shown by Damelin and Grabner in [2] for Riesz kernels. The essence of our main result below is that we are able to formulate a general analogous result which works on M and for a subclass of admissible kernels κ. To describe this result, we need some more notations.

Let σ_α be a sequence of kernels converging to the δ distribution (the distribution for which all Fourier coefficients are unity) as $\alpha \to 0$. Let κ be admissible and for $\alpha < \alpha_0$ for some fixed α_0, we wish the convolution $\kappa_\alpha = \kappa * \sigma_\alpha$ to have the following properties:

a. κ_α is positive definite

b. $\kappa_\alpha(x,y) \le \kappa(x,y)$ for all $x, y \in M$.

If the above construction is possible, we say that κ is *strongly admissible*. Besides Riesz kernels on d dimensional spheres see [2, 3], we have as a futher natural example on the 2-torus embedded in $\mathbb{R}^4$, strongly admissible kernels defined as products of univariate kernels:

$$\kappa(x,y) = \rho(x_1,y_1)\rho(x_2,y_2), \quad x_1, y_1, x_2, y_2 \in S^1,$$

where

$$\rho(s,t) = |1 - st|^{-1/2}, \quad s,t \in S^1$$

and S^1 is the one dimensional circle (realized as a subset of $\mathbb{R}^2$). See [3] for further details.

We now give an interesting result which demonstrates the way in which results on the sphere can be transplanted onto more general manifolds. The reader is directed to [3] for the proof and further results.

Theorem 3. *Let κ be strongly admissible on M and $Z \subset M$ be a point subset of cardinality $N \ge 1$. Fix $x \in M$. If q is a polynomial of degree at most $n \ge 0$ on M then, for $\alpha < \alpha_0$,*

$$|R(f,Z,\mu)| \le \max_{j \le n,\, l \le h_j} \frac{1}{(a_{j,l}(\kappa_\alpha))^{1/2}} \|q\|_2 \left(E_\kappa(Z) + \frac{1}{N}\kappa_\alpha(x,x) - a_{0,0}(\kappa_\alpha) \right)^{1/2}.$$

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