Necessary and Sufficient Conditions for Mean Convergence of Lagrange Interpolation For Erdős weights II

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Abstract

We continue our investigations of mean convergence of Lagrange Interpolation at the zeros of orthogonal polynomials $p_n(W^2, x)$ for Erdős Weights $W^2 = \exp[-2Q]$. The archetypal example is $W_{k,\alpha} = \exp\left[-Q_{k,\alpha}\right]$ where,

$$Q_{k,\alpha}(x) := \exp_k\left(|x|^\alpha\right)$$

$\alpha > 1$, $k \geq 1$, and $\exp_k = \exp\left(\exp(\exp \ldots)\right)$ is the kth iterated exponential. Following is our main result: Let $1 < p < 4$ and $\alpha \in \mathbb{R}$. Let $L_n[f]$ denote the Lagrange Interpolation polynomial to f at the zeros of $p_n\left(W^2, x\right) = p_n\left(\exp[-2Q], x\right)$. Then for,

$$\lim_{n \longrightarrow \infty} \|(f - L_n[f])\, W\|_{L_p[\mathbb{R}]} = 0$$

to hold for every continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying ,

$$\lim_{|x| \longrightarrow \infty} (fW)(x)\,(1 + |x|)^\alpha = 0,$$

it is necessary and sufficient that $\alpha > \frac{1}{p}$. This is, essentially, an extension of the Erdős–Turan theorem on L_p convergence. In an earlier paper, we analysed convergence for all $p > 1$, showing the necessity and sufficiency of using the weighting factor $1 + Q$ for all $p > 4$. Our proofs of convergence are based on converse quadrature sum estimates, that are established using ideas of König.

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1 Introduction and Results

In this paper, we continue our investigation from [2] of mean convergence of Lagrange interpolation at the zeros of orthogonal polynomials for Erdős weights. Recall that Erdős weights have the form $W^2 = \exp[-2Q]$, where $Q : \mathbb{R} \longrightarrow \mathbb{R}$ is even and faster than polynomial growth at infinity. The archetypal example is,

$$W_{k,\alpha}(x) := \exp\left(-Q_{k,\alpha}(x)\right), \tag{1}$$

where,

$$Q_{k,\alpha}(x) := \exp_k\left(|x|^\alpha\right),\ k \geq 1,\ \alpha > 0. \tag{2}$$

Here $\exp_k = \exp\left(\exp(\exp \ldots)\right)$ denotes the kth iterated exponential.

Given a weight $W : \mathbb{R} \longrightarrow \mathbb{R}$ such as those above, we can define orthonormal polynomials,

$$p_n(x) = p_n(W^2, x) = \gamma_n x^n + \ldots \gamma_n = \gamma_n(W^2) > 0.$$

satisfying,

$$\int_{\mathbb{R}} p_n(W^2, x) p_m(W^2, x) W^2(x) dx = \delta_{mn}$$

We denote the zeros of p_n by,

$$-\infty < x_{n,n} < x_{n-1,n} \ldots < x_{1,n} < \infty$$

The Lagrange interpolation polynomial to a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ at $\{x_{j,n}\}_{j=1}^n$ is denoted by $L_n[f]$. Thus if $\mathcal{P}_m$ denotes the class of polynomials of degree $\leq m$, and $l_{j,n} \in \mathcal{P}_{n-1},\ 1 \leq j \leq n$, are the *fundamental polynomials* of Lagrange interpolation at the $\{x_{j,n}\}_{j=1}^n$ satisfying,

$$l_{j,n}\left(x_{k,n}\right) = \delta_{j,k},$$

then,

$$L_n[f](x) = \sum_{j=1}^{n} f\left(x_{j,n}\right) l_{j,n}(x). \tag{3}$$

In [2], we investigated mean convergence of $L_n[\ .\ ]$ for the following class of Erdős weights:

Definition 1.1
Let $W := \exp[-Q]$, where $Q : \mathbb{R} \longrightarrow \mathbb{R}$ is even, continuous, $Q^{(2)}$ exists in $(0,\infty)$, $Q^{(j)} \geq 0$ in $(0,\infty)$, $j = 0, 1, 2$, and the function,

$$T(x) := 1 + \frac{xQ^{(2)}(x)}{Q^{(1)}(x)} \tag{4}$$

is increasing in $(0,\infty)$, with,

$$\lim_{x\longrightarrow\infty} T(x) = \infty;\ T(0+) := \lim_{x\longrightarrow 0+} T(x) > 1. \tag{5}$$

Moreover, we assume that for some C_1, C_2, $C_3 > 0$,

$$C_1 \leq \frac{T(x)}{\left(\frac{xQ'(x)}{Q(x)}\right)} \leq C_2,\ x \geq C_3 \tag{6}$$

and for every $\varepsilon > 0$,

$$T(x) = O\left(Q(x)^{\varepsilon}\right),\ x \longrightarrow \infty \tag{7}$$

Then we write $W \in \mathcal{E}_1$.

The principle example of $W = \exp[-Q] \in \mathcal{E}_1$ is $W_{k,\alpha} = \exp\left(-Q_{k,\alpha}\right)$ given by (2) with $\alpha > 1$. Another (more slowly decaying) example of $W = \exp[-Q] \in \mathcal{E}_1$ is given by,

$$Q(x) := \exp\left(\log\left(A + x^2\right)^{\beta}\right),\ \beta > 1,\ A \text{ large enough.}$$

The behaviour of $T(x)$, ect. for these weights is discussed in greater detail in [2], [7].

The first results for mean convergence of Lagrange interpolation for a class of Erdős weights appeared in [9], and the first "sharp" results appeared in [2]. Following is the main result of [2]:

Theorem 1.2

Let $W := \exp[-Q] \in \mathcal{E}_1$. Let $L_n[\ .\ ]$ denote the Lagrange interpolation polynomial to f at the zeros of $p_n(W^2,\ .\ )$. Let $1 < p < \infty,\ \Delta \in \mathbb{R},\ \kappa > 0$. Then for

$$\lim_{n \longrightarrow \infty} \left\| (f - L_n[f])\, W (1+Q)^{-\Delta} \right\|_{L_p[\mathbb{R}]} = 0 \tag{8}$$

to hold for every continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying,

$$\lim_{|x| \longrightarrow \infty} |fW|(x) \left(\log |x|\right)^{1+\kappa} = 0 \tag{9}$$

it is necessary and sufficient that,

$$\Delta > \max\left\{0, \frac{2}{3}\left(\frac{1}{4} - \frac{1}{p}\right)\right\}. \tag{10}$$

It was also shown in [2] that even if f vanishes outside a fixed finite interval, we need a factor like $(1+Q)^{-\Delta}$ with Δ large enough, if $p > 4$. We remarked there that for $p \leq 4$, the weighting factor $1 + Q$ is unnecessarily strong. After all, Q grows faster than any polynomial. Let us recall the Erdős–Turan theorem, as extended by Shohat (see [3, Ch.2, p.97]). If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is Riemann integrable in each finite interval, and there exists an even entire function G with all non–negative Maclaurin series coefficients such that,

$$\lim_{|x| \to \infty} \frac{f^2(x)}{G(x)} = 0$$

and

$$\int_{\mathbb{R}} G(x) W^2(x) dx < \infty$$

then

$$\lim_{n \longrightarrow \infty} \left\| (f - L_n[f])\, W \right\|_{L_p[\mathbb{R}]} = 0. \tag{11}$$

For nice weights here, a result of Clunie and Kovari [1, Th4, p.19], allows us to choose G with

$$G(x) \sim W^{-2}(x) \left(1 + |x|\right)^{-1-\kappa};\ x \in \mathbb{R},\ \kappa > 0.$$

Here and in the sequel, the notation involving $\sim$ means that the ratio of the two sides is bounded above and below by positive constants independent of

x. (Later on, the dependence will be on n and possibly other parameters.) Thus we can ensure that (20) holds provided,

$$\lim_{|x|\to\infty} (fW)(x)\,(1+|x|)^{\frac{1}{2}+\frac{\kappa}{2}} = 0$$

Thus our result does not extend the classical result for $p = 2$.

Following is our main result, which does essentially constitute an extension of the Erdős–Turan result.

Theorem 1.3

Let $W := \exp[-Q] \in \mathcal{E}_1$. Let $1 < p < 4$, and $\alpha \in \mathbb{R}$. Let $L_n[f]$ denote the Lagrange interpolating polynomial to f at the zeros of $p_n(W^2, \,.\,)$. Then the following are equivalent.

(a) For every continuous $f : \mathbb{R} \longrightarrow \mathbb{R}$ with,

$$\lim_{|x|\longrightarrow\infty} |f(x)|\,W(x)\,(1+|x|)^{\alpha} = 0 \tag{12}$$

we have,

$$\lim_{n\longrightarrow\infty} \|(f - L_n[f])\,W\|_{L_p[\mathbb{R}]} = 0 \tag{13}$$

(b) $\alpha > \frac{1}{p}$.

We next show that we cannot insert any positive power of $1+|x|$ inside the L_p norm in (13) at least when $\alpha > \frac{1}{p}$:

Theorem 1.4

Let $W := \exp[-Q] \in \mathcal{E}_1$. Let $1 < p < 4$ and $\Delta \in \mathbb{R}$. Then the following are equivalent:

(a) For every $\alpha > \frac{1}{p}$ and every continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying (12), we have,

$$\lim_{n\longrightarrow\infty} \left\|(f - L_n[f])\,(x)W(x)\,(1+|x|)^{\Delta}\right\|_{L_p[\mathbb{R}]} = 0 \tag{14}$$

(b)

$$\Delta \leq 0. \tag{15}$$

We note that with more work, we can replace continuity of f in the above two theorems by Riemann integrability, and we can replace $(1+|x|)^{\alpha}$, $\alpha > \frac{1}{p}$, by $(1+|x|)^{\frac{1}{p}} (\log(2+|x|))^{\frac{1}{p}+\varepsilon}$, some $\varepsilon > 0$, (and so on).

In [2], it was shown that even for f vanishing outside $[-2,2]$, and $p > 4$, we needed $(1+Q)^{-\Delta}$ in (8), with $\Delta \geq \frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right)$. Following is an analagous result for $p = 4$:

Theorem 1.5

Let $W := \exp[-Q] \in \mathcal{E}_1$. Suppose that a measurable function $U : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies,

$$\lim_{x \longrightarrow \infty} U(x) x^{\frac{-3}{4}} (\log Q(x))^{\frac{1}{4}} = \infty \tag{16}$$

Then there exists a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ vanishing outside $[-2,2]$ such that,

$$\lim \sup_{n \longrightarrow \infty} \|L_n[f] W U\|_{L_p[\mathbb{R}]} = \infty \tag{17}$$

If for example, $Q(x)$ grows faster than $\exp(x^{3+\varepsilon})$, some $\varepsilon > 0$, then Theorem 1.4 shows that we cannot choose $U \equiv 1$ and hope for convergence. So there is no analogue of Theorem 1.3 for $p = 4$. However, it seems that a negative power of $\log Q$, rather than the $1+Q$ required for $p > 4$, will allow some analgue of Theorem 1.2 for $p = 4$.

While the methods of this paper use many techniques and tools of H. König [4], [5], we also use estimates and results from [7], [8]. However the reader need only have a copy of [2] available for reading this paper.

This paper is organised as follows: In Section 2, we gather technical estimates from other papers. In Section 3, we prove a converse quadrature sum inequality using the same methods as H. König in [4], [5]. In Section 4, we prove the sufficiency conditions of Theorem 1.3 and 1.4, and in Section 5, we prove the necessity conditions of Theorems 1.3, 1.4, and also prove Theorem 1.5. At a first reading, it is best to skip the technical Section 2, and concentrate on Section 3. Then read Sections 4 and 5, and finally return to Section 2.

We close this section by introducing some more notation. Given Q as above, the *Mhaskar–Rhamanov–Saff* number a_u is the positive root of the equation,

$$u = \frac{2}{\pi} \int_0^1 a_u t Q'(a_u t) \frac{dt}{\sqrt{1-x^2}}, \quad u > 0 \tag{18}$$

For example, for $Q = Q_{k,\alpha}$ of (2),

$$a_u \sim (\log_k u)^{\frac{1}{\alpha}}$$

(see [2], [7]). To the unfamiliar, one of the uses of a_u is in the identity [10],

$$\|PW\|_{L_\infty[\mathbb{R}]} = \|PW\|_{L_\infty[-a_n\ a_n]}, \quad P \in \mathcal{P}_n \tag{19}$$

Here and the sequel, $\mathcal{P}_n$ denotes the polynomials of degree $\leq n$.

In the sequel, $C, C_1, C_2...$ denote constants independent of n, x and $P \in \mathcal{P}_n$.The same symbol does not necessarily denote the same constant in different occurrences.

The nth *Christoffel function* for a weight W^2 is,

$$\begin{aligned} \lambda_n(x) : &= \lambda_n\left(W^2, x\right) = \inf_{P \in \mathcal{P}_{n-1}} \int_{\mathbb{R}} \frac{(PW)^2 dt}{P^2(x)} \\ &= \frac{1}{\sum_{j=0}^{n-1} p_j^2(x)}. \end{aligned} \tag{20}$$

The *Christoffel numbers* are,

$$\lambda_{jn} = \lambda_n\left(W^2, x_{j,n}\right), \quad 1 \leq j \leq n. \tag{21}$$

The fundamental polynomials l_{jn} of (4) admit the representation,

$$l_{jn}(x) = \lambda_{jn} \frac{\gamma_{n-1}}{\gamma_n} p_{n-1}\left(x_{j,n}\right) \frac{p_n(x)}{x - x_{j,n}} \tag{22}$$

We define the **Hilbert transform** of $g \in L_1(\mathbb{R})$ by,

$$H[g](x) := \lim_{\varepsilon \longrightarrow 0+} \int_{|x-t| \geq \varepsilon} \frac{g(t)}{x-t} dt, \tag{23}$$

(this exists ae. [12]).

Finally, we define some auxiliary quantities:

$$\delta_n := \left(nT\left(a_n\right)\right)^{-\frac{2}{3}}, \ n \geq 1. \tag{24}$$

This quantity is useful in describing the behaviour of $p_n\left(\exp[-2Q], \ . \ \right)$ near $x_{1,n}$. For example,

$$\left|\frac{x_{1,n}}{a_n(Q)} - 1\right| \leq \frac{L\delta_n}{2}. \tag{25}$$

Here L is independent of n. We often use the fact that δ_n is much smaller than any power of $\frac{1}{T(a_n)}$, see Section 2. We also use the function,

$$\Psi_n(x) := \max\left\{\sqrt{1-\frac{|x|}{a_n}+L\delta_n},\ \left[T\left(a_n\right)\sqrt{1-\frac{|x|}{a_n}+L\delta_n}\right]^{-1}\right\},\ |x|\leq a_n \tag{26}$$

and set

$$\Psi_n(x) := \Psi_n\left(a_n\right),\ |x|>a_n \tag{27}$$

This function is used in describing spacing of zeros of p_n, behaviour of Christoffel functions and so on. Finally, we set

$$x_{0,n} := x_{1,n}\left(1+L\delta_n\right);\ x_{n+1,n} := x_{n,n}\left(1+L\delta_n\right); \tag{28}$$

and

$$I_{j,n} := \left(x_{j,n}, x_{j-1,n}\right);\ |I_{j,n}| := x_{j-1,n}-x_{j,n},\ 1\leq j\leq n \tag{29}$$

Also, in proving our quadrature estimates, we use

$$f_{j,n}(x) := \min\left\{\frac{1}{|I_{j,n}|}, \frac{|I_{j,n}|}{\left(x-x_{j,n}\right)^2}\right\}\left[\left|1-\frac{|x|}{a_n}\right|+L\delta_n\right]^{-\frac{1}{4}} \tag{30}$$

Define the characteristic function of $I_{j,n}$,

$$\chi_{j,n}(x) := \chi_{I_{j,n}}(x) := \left\{\begin{array}{ll} 1, & x\in I_{j,n} \\ 0, & x\notin I_{j,n}\end{array}\right\}. \tag{31}$$

2 Technical Estimates

In this section, we gather technical estimates from various sources. We begin by recalling some lemmas from [7]. Throughout, we assume that $W := \exp[-Q]\in\mathcal{E}_1$.

Lemma 2.1

(a) Uniformly for $n\geq 1$ and $|x|\leq a_n$,

$$\lambda_n\left(W^2,x\right)\sim\frac{a_n}{n}W^2(x)\Psi_n(x) \tag{32}$$

(b) For $n \geq 1$,

$$\left|\frac{x_{1,n}}{a_n} - 1\right| \leq C\delta_n \tag{33}$$

Uniformly for $n \geq 2$ and $0 \leq j \leq n-1$,

$$x_{j,n} - x_{j+1,n} \sim \frac{a_n}{n}\Psi_n\left(x_{j,n}\right) \tag{34}$$

(c) For $n \geq 1$,

$$\sup_{x\in\mathbb{R}} \left|p_n W\right|(x)\left|1 - \frac{|x|}{a_n}\right|^{\frac{1}{4}} \sim a_n^{-\frac{1}{2}} \tag{35}$$

and

$$\sup_{x\in\mathbb{R}} \left|p_n W\right|(x) \sim a_n^{-\frac{1}{2}}\left(nT\left(a_n\right)\right)^{\frac{1}{6}} \tag{36}$$

(d) Let $0 < p \leq \infty,\ K > 0$. There exists $C > 0$ such that for $n \geq n_0$ and $P \in \mathcal{P}_n$,

$$\|PW\|_{L_p[\mathbb{R}]} \leq C\|PW\|_{L_\infty[-a_n(1-K\delta_n)\ a_n(1-K\delta_n)]} \tag{37}$$

Moreover, given $r > 1$, there exists $C_1 > 0$ such that,

$$\|PW\|_{L_p[|x|\geq a_{rn}]} \leq \exp\left[-C_1 nT\left(a_n\right)^{-\frac{1}{2}}\right]\|PW\|_{L_\infty[-a_n\ a_n]} \tag{38}$$

(e) For $n \geq 1$,

$$\frac{\gamma_{n-1}}{\gamma_n} \sim a_n \tag{39}$$

(f) Uniformly for $n \geq 2$ and $0 \leq j \leq n-1$,

$$1 - \frac{|x_{j,n}|}{a_n} + L\delta_n \sim 1 - \frac{|x_{j+1,n}|}{a_n} + L\delta_n \tag{40}$$

and

$$\Psi_n\left(x_{j,n}\right) \sim \Psi_n\left(x_{j+1,n}\right) \tag{41}$$

Here, L is chosen so large enough that (25) is true.

(g) Uniformly for $n \geq 2$ and $1 \leq j \leq n-1$,

$$\frac{a_n^{\frac{3}{2}}}{n} \Psi_n\left(x_{j,n}\right)\left(1-\frac{\left|x_{j,n}\right|}{a_n}+L\delta_n\right)^{\frac{1}{2}}\left|p_n' W\right|\left(x_{j,n}\right)$$

$$\sim a_n^{\frac{1}{2}}\left|p_{n-1} W\right|\left(x_{j,n}\right) \sim\left(1-\frac{\left|x_{j,n}\right|}{a_n}+L\delta_n\right)^{\frac{1}{4}} \tag{42}$$

Proof.

This is Lemma 2.1 in [2], except for (2.3), (2.9) and (2.10) for $j=0$, which follow from the definition of $x_{0,n}$ and Ψ_n. $\square$

Lemma 2.2

(a) Let $0<p<\infty$. Then for $n \geq 2$,

$$\left\|p_n W\right\|_{L_p(\mathbb{R})} \sim a_n^{\frac{1}{p}-\frac{1}{2}}\left[\begin{array}{ll} 1, & p<4 \\ (\log n)^{\frac{1}{4}}, & p=4 \\ nT\left(a_n\right)^{\frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right)}, & p>4 \end{array}\right]. \tag{43}$$

(b) Uniformly for $n \geq 1,\ 1 \leq j \leq n,\ x \in \mathbb{R}$,

$$\left|l_{j,n}(x)\right| \sim \frac{a_n^{\frac{3}{2}}}{n}\left(\Psi_n W\right)\left(x_{j,n}\right)\left(\left(1-\frac{\left|x_{j,n}\right|}{a_n}+L\delta_n\right)^{\frac{1}{4}}\left|\frac{P_n(x)}{x-x_{j,n}}\right|\right). \tag{44}$$

(c) Uniformly for $n \geq 1,\ 1 \leq j \leq n,\ x \in \mathbb{R}$,

$$\left|l_{j,n}(x)\right| W(x) W\left(x_{j,n}\right) \leq C. \tag{45}$$

(d) For $n \geq 2,\ 1 \leq j \leq n-1,\ x \in\left[x_{j,n}\ x_{j+1,n}\right]$,

$$l_{j,n}(x) W(x) W^{-1}\left(x_{j,n}\right)+l_{j+1,n}(x) W(x) W^{-1}\left(x_{j+1,n}\right) \geq 1. \tag{46}$$

Proof.

This is Lemma 2.2 in [2]. $\square$

Lemma 2.3

(a) Given $r>0$, there exists x_0 such that for $x \geq x_0$ and $j=0,1,2$, $\frac{Q^{(j)}(x)}{x^r}$ is increasing in (x_0,∞).

(b) Uniformly for $u \geq C$ and $j=0,1,2$,

$$a_u^j Q^{(j)}\left(a_u\right) \sim uT\left(a_u\right)^{j-\frac{1}{2}} \tag{47}$$

(c) Let $0<\alpha<\beta$. Then uniformly for $u \geq C,\ j=0,1,2$,

$$T\left(a_{\alpha u}\right) \sim T\left(a_{\beta u}\right); \quad Q^{(j)}\left(a_{\alpha u}\right) \sim Q^{(j)}\left(a_{\beta u}\right) \tag{48}$$

(d) Given fixed $r>1$,

$$\frac{a_{ru}}{a_u} \geq 1+\frac{\log r}{T\left(a_{ru}\right)},\ u \in (0,\infty). \tag{49}$$

Moreover,

$$a_{ru} \sim a_u, \quad u \in (1,\infty). \tag{50}$$

(e) Uniformly for $t \in (C,\infty)$,

$$\frac{a_t'}{a_t} \sim \frac{1}{tT\left(a_t\right)} \tag{51}$$

(f) Uniformly for $u \in (C,\infty)$, and $v \in \left[\frac{u}{2},2u\right]$, we have,

$$\left|\frac{a_u}{a_v}-1\right| \sim \left|\frac{u}{v}-1\right| \frac{1}{T\left(a_u\right)} \tag{52}$$

Proof.
This is Lemma 2.3 in [2]. $\square$

Lemma 2.4

(a) Let $\varepsilon>0$. Then,

$$a_n \leq Cn^{\varepsilon},\ T\left(a_n\right) \leq Cn^{\varepsilon},\ n \geq 1. \tag{53}$$

(b) Given $A > 0$, we have,

$$\delta_n \leq CT\left(a_n\right)^{-A}, \ n \geq 1. \tag{54}$$

(c) Let $0 < \eta < 1$. Uniformly for $n \geq 1, \ 0 < |x| \leq a_{\eta n}, |x| = a_s$, we have,

$$C_1 \leq T(x)\left(1 - \frac{|x|}{a_n}\right) \leq C_2 \log \frac{n}{s} \tag{55}$$

Proof.
This is Lemma 2.4 in [2]. $\square$
Next, we present a lemma from König [5]: Recall the notation,

$$\|g\|_{L_p[d\mu]} := \left(\int_\Omega |g|^p d\mu\right)^{\frac{1}{p}},$$

for μ measurable functions g on a measure space (Ω, μ).

Lemma 2.5
Let $1 < p < \infty$ and $q := \frac{p}{(p-1)}$. Let (Ω, μ) be a measure space, $k, r : \Omega^2 \longrightarrow \mathbb{R}$ and,

$$T_k[f](u) := \int_\Omega k(u,v) f(v) d\mu(v) \tag{56}$$

for μ measurable $f : \Omega \longrightarrow \mathbb{R}$. Assume that,

$$\sup_u \int_\Omega |k(u,v)|\, |r(u,v)|^q \, d\mu(v) \leq M. \tag{57}$$

$$\sup_v \int_\Omega |k(u,v)|\, |r(u,v)|^{-p} \, d\mu(u) \leq M. \tag{58}$$

Then T_k is a bounded operator from $L_p(d\mu)$ to $L_p(d\mu)$. More precisely,

$$\|T_k\|_{L_p(d\mu) \longrightarrow L_p(d\mu)} \leq M. \tag{59}$$

Proof.

We sketch this, as no proof is given in [5], though such lemmas are standard. First use the dual expression for the L_p norm of $T_k[f]$, and then Fubini's theorem, and then Hölder's inequality, to show that,

$$\|T_k[f]\|_{L_p(d\mu)} \leq \|f\|_{L_p(d\mu)} \sup_g \left[\int_\Omega \left|\int_\Omega k(u,v)g(u)d\mu(u)\right|^q d\mu(v)\right]^{\frac{1}{q}},$$

where the sup is taken over all g with $\|g\|_{L_q(d\mu)} = 1$. Let us call the sup J. So we must show that J is bounded by M. Using Hölder's inequality on the inner integral in J gives,

$$\begin{aligned}
&\left|\int_\Omega k(u,v)g(u)d\mu(u)\right|^q \\
&\quad \leq \left[\int_\Omega |k(u,v)|\,|r(u,v)|^{-p}\,d\mu(u)\right]^{\frac{q}{p}} \int_\Omega |k(u,v)|\,|r(u,v)|^q\,|g(u)|^q\,d\mu(v) \\
&\quad \leq M^{\frac{q}{p}} \sup_g \left[\int_\Omega |k(u,v)|\,|r(u,v)|^q\,|g(u)|^q\,d\mu(v)d\mu(u)\right]^{\frac{1}{q}} \\
&\quad \leq M^{\frac{1}{p}} M^{\frac{1}{q}} = M. \quad \square
\end{aligned}$$

The next lemma essentially already appears in 1970 papers of Muckenhoupt [11, pp.449–451], and later in H. Konig's paper [5] and is of course implied by results on the weighted L_p boundedness of Hilbert tranforms (Muckenhoupt's A_p condition):

Lemma 2.6

Let $1 < p < 4$. Then,

$$\left\|H[g](x)\left|1 - \frac{|x|}{a_n}\right|^{-\frac{1}{4}}\right\|_{L_p[\mathbb{R}]} \leq \left\|g(x)\left|1 - \frac{|x|}{a_n}\right|^{-\frac{1}{4}}\right\| \tag{60}$$

with C independent of n and $g \in L_p[\mathbb{R}]$.

Proof.

The proof appears with $a_n = \sqrt{2n}$ in [5], but we sketch the proof from [5]: Consider the operator T_k of Lemma 2.5, with

$$k(u,v) := \frac{\left|\left|\frac{u}{v}\right|^{\frac{1}{4}} - 1\right|}{[u-v]}$$

Using $r(u,v) := \left|\frac{u}{v}\right|^{\frac{1}{pq}}$,where $q := \frac{p}{p-1}$. Lemma 2.5 can be used to show that T_k is bounded from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$. Comparison of T_k and the bounded operator H show that,

$$H_1[g](v) := \lim_{\varepsilon \longrightarrow 0+} \int_{|u-v|\geq \varepsilon} \frac{g(u)}{v-u} \left|\frac{u}{v}\right|^{\frac{1}{4}} du$$

is bounded from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$. Replacing u by $a_n \pm u$, and v by $a_n \pm v$, easily gives the result. $\square$

Our final lemma in this section concerns bounds on the difference between $\frac{1}{(x-x_{j,n})}$ and the Hilbert transform of a weighted characteristic function. Recall the notation (29–31) for $I_{j,n}$, $f_{j,n}$ and $\chi_{j,n}$. In particular, recall that,

$$f_{j,n}(x) := \min\left\{\frac{1}{|I_{j,n}|}, \frac{|I_{j,n}|}{(x-x_{j,n})^2}\right\}\left[1-\frac{|x|}{a_n}+L\delta_n\right]^{\frac{-1}{4}}.$$

Lemma 2.7

Uniformly for $n \geq 1$ and $1 \leq j \leq n$ and $x \in [x_{n,n}\ x_{1,n}]$,

$$\begin{aligned} \tau_{j,n}(x) : &= a_n^{\frac{1}{2}}\left[p_n\left(W^2,x\right)W(x)\right]\left|\frac{1}{x-x_{j,n}} - \frac{1}{|I_{j,n}|}H\left[\chi_{j,n}\right](x)\right| \\ &\leq Cf_{j,n}(x). \end{aligned} \tag{61}$$

Proof

The idea already appears in [5]. Note first that,

$$H\left[\chi_{j,n}\right](x) = \log\left|\frac{x-x_{j,n}}{x_{j-1,n}-x}\right| = -\log\left|1-\frac{|I_{j,n}|}{x-x_{j,n}}\right|. \tag{62}$$

We consider two ranges:

Case 1: $|x-x_{j,n}| \geq 2\,|I_{j,n}|$

Using the inequality $|t+\log(1-t)| \leq t^2$, $|t| \leq \frac{1}{2}$, we see that,

$$\begin{aligned} \left|\frac{1}{x-x_{j,n}} - \frac{1}{|I_{j,n}|}H\left[\chi_{j,n}\right](x)\right| &= \frac{1}{|I_{j,n}|}\left|\frac{|I_{j,n}|}{x-x_{j,n}} + \log\left[1-\frac{|I_{j,n}|}{x-x_{j,n}}\right]\right| \\ &\leq \frac{|I_{j,n}|}{(x-x_{j,n})^2}. \end{aligned}$$

Next, the bounds (35), (36) show that uniformly in n and x,

$$a_n^{\frac{1}{2}} \left| p_n W \right| (x) \leq C \left[\left| 1 - \frac{x}{a_n} \right| + L\delta_n \right]^{\frac{-1}{4}} . \tag{63}$$

So we obtain the result for this range of x.

Case 2: $|x - x_{j,n}| \leq 2 \left| I_{j,n} \right|$

From the identity,

$$a_n^{\frac{1}{2}} \left(p_n W \right)(x) = \left(l_{j,n} W \right)(x) W^{-1} \left(x_{j,n} \right) \left(x - x_{j,n} \right) a_n^{\frac{1}{2}} \left(p_n' W \right) \left(x_{j,n} \right),$$

(for both j and $j-1$) and from (34), (40), (42), (45), we obtain for $|x_{j,n}| \leq 2 \left| I_{j,n} \right|, 2 \leq j \leq n$,

$$a_n^{\frac{1}{2}} \left| p_n W \right| (x) \leq C_1 f_{j,n}(x) \min \left\{ \left| x - x_{j,n} \right|, \left| x - x_{j-1,n} \right| \right\}. \tag{64}$$

For $j = 1$, this holds with the minimum replaced by $|x - x_{j,n}|$. Then for $2 \leq j \leq n$,

$$\tau_{j,n}(x) \leq C_2 f_{j,n}(x) \left[1 + \min \left\{ \left| x - x_{j,n} \right|, \left| x - x_{j-1,n} \right| \right\} \frac{1}{|I_{j,n}|} \log \left| \frac{x - x_{j,n}}{x_{j-1,n} - x} \right| \right]. \tag{65}$$

Since $|I_{j,n}| \geq C_3 \max \left\{ \left| x - x_{j,n} \right|, \left| x - x_{j-1,n} \right| \right\}$, we see that with,

$$u := \left| \frac{x - x_{j,n}}{x_{j-1,n} - x} \right|$$

we obtain for both signs of the exponent,

$$\tau_{j,n}(x) \leq C_4 f_{j,n}(x) \left[1 + 2u^{\pm 1} \left| \log u^{\pm 1} \right| \right]$$

As either u or u^{-1} lies in $[0, 1]$ and $t \left| \log t \right|$ is bounded for $t \in [0, 1]$, we have (61). It remains to handle the case $j = 1$. Note that for $x \in [x_{n,n}, x_{1,n}]$ (it is only here that we need this resriction) with $|x - x_{1,n}| \leq 2 \left| I_{1,n} \right|$, we have

$$|x - x_{0,n}| \sim a_n \delta_n.$$

(See (33), (34), (28), (29)). Then instead of (65), we obtain,

$$\tau_{j,n}(x) \leq C f_{1,n}(x) \left[1 + C_1 \frac{|x - x_{1,n}|}{a_n \delta_n} \left| \log \sigma \frac{|x - x_{1,n}|}{a_n \delta_n} \right| \right]$$

where $\sigma \sim 1$ independently of x, j, n. As $|x - x_{1,n}| \leq C_2 a_n \delta_n$, the boundedness of $u |\log u|$ in any finite interval in $[0, \infty)$ again gives our result. $\square$

3 A Converse Quadrature Sum Estimate

The main result of this section is,

Theorem 3.1
Let $W := \exp[-Q] \in \mathcal{E}_1$ and $1 < p < 4$. There exists $C > 0$ such that for $n \geq 1$ and $P \in \mathcal{P}_n$,

$$\|PW\|_{L_p[\mathbb{R}]} \leq C \left\{ \sum_{j=1}^{n} \lambda_{j,n} W^{-2}(x_{j,n}) |PW|^p (x_{j,n}) \right\}^{\frac{1}{p}} \tag{66}$$

Our proof of Theorem 3.1 follows that of H. König. We shall divide the proof into several steps: In the sequel, we shall use the abbreviation,

$$\mu_{j,n} := \lambda_{j,n} W^{-2}(x_{j,n}) \sim |I_{j,n}| = x_{j-1,n} - x_{j,n}. \tag{67}$$

(See (32) and (34)).

Step 1: Express PW as a sum of two terms.
Let $P \in \mathcal{P}_n$. We write,

$$\begin{aligned} (PW)(x) &= (L_n[P]W)(x) = \sum_{j=1}^{n} P(x_{j,n})(l_{j,n}W)(x) \\ &= a_n^{\frac{1}{2}} (p_n W)(x) \sum_{j=1}^{n} y_{j,n} \left\{ \frac{1}{x - x_{j,n}} - \frac{1}{|I_{j,n}|} H[\chi_{j,n}](x) \right\} \\ &\quad + a_n^{\frac{1}{2}} (p_n W)(x) H\left[\sum_{j=1}^{n} y_{j,n} \frac{\chi_{j,n}}{|I_{j,n}|} \right](x) \\ &=: J_1(x) + J_2(x). \end{aligned} \tag{68}$$

Here,

$$y_{j,n} := a_n^{\frac{-1}{2}} \frac{(PW)(x_{j,n})}{(p_n' W)(x_{j,n})}. \tag{69}$$

Note that in view of the behaviour of the smallest and largest zeros (see (33)) and in view of the infinite–finite range inequality (37), it suffices to estimate $\|PW\|_{L_p[x_{n,n}\ x_{1,n}]}$ in terms of the right–hand side of (61).

Step 2: Estimate $\|J_2\|$

(We begin J_2 as it is easier to handle). Using our bound (35) for p_n, and then the weighted boundedness of the Hilbert transform in Lemma 2.6 gives,

$$\begin{aligned}
\|J_2\|_{L_p[x_{n,n}\ x_{1,n}]} &\leq C\left\|\sum_{j=1}^{n} y_{j,n}\frac{\chi_{j,n}(x)}{|I_{j,n}|}\left|1-\frac{|x|}{a_n}\right|^{\frac{-1}{4}}\right\|_{L_p[\mathbb{R}]} \\
&= C_1\left[\sum_{j=1}^{n}\left\{\frac{|y_{j,n}|}{|I_{j,n}|}\right\}^p\int_{I_{j,n}}\left|1-\frac{|x|}{a_n}\right|^{\frac{-p}{4}}\right]^{\frac{1}{p}} dx.
\end{aligned}$$

Using the spacing (34) and also (40), one deduces that,

$$\int_{I_{j,n}}\left|1-\frac{|x|}{a_n}\right|^{\frac{-p}{4}} dx \sim |I_{j,n}|\left|1-\frac{|x_{j,n}|}{a_n}+\delta_n\right|^{\frac{-p}{4}}.$$

Next, from (69) and (42), we see that,

$$|y_{j,n}| \sim |PW|(x_{j,n})\,|I_{j,n}|\left|1-\frac{|x_{j,n}|}{a_n}+\delta_n\right|^{\frac{+1}{4}}. \tag{70}$$

Hence,

$$\begin{aligned}
\|J_2\|_{L_p[x_{n,n}\ x_{1,n}]} &\leq C_2\left[\sum_{j=1}^{n}|I_{j,n}|\,|PW|^p(x_{j,n})\right]^{\frac{1}{p}} \\
&\leq C_3\left[\sum_{j=1}^{n}\lambda_{j,n}W^{-2}(x_{j,n})\,|PW|^p(x_{j,n})\right]^{\frac{1}{p}}
\end{aligned}$$

by (62).

Step 3: Estimate J_1.

By Lemma 2.7,

$$|J_1(x)| \leq C_4\sum_{j=1}^{n}|y_{j,n}|\,f_{j,n}(x),\ x\in[x_{n,n},\ x_{1,n}].$$

Then,

$$\|J_1\|_{L_p[x_{nn}\ ,x_{1,n}]} \leq C_4\left\{\sum_{k=2}^{n}\int_{I_{k,n}}\left[\sum_{j=1}^{n}|y_{j,n}|\,f_{j,n}(x)\right]^p dx\right\}^{\frac{1}{p}}.$$

Using the spacing (34), (40) and the definition (30) of $f_{j,n}$, we see that,

$$f_{j,n}(x) \sim \frac{|I_{j,n}|}{(x_{k,n}-x_{j,n})^2}\left[\left|1-\frac{|x_{k,n}|}{a_n}\right|+\delta_n\right]^{\frac{-1}{4}}, x \in I_{k,n},$$

uniformly in n and $j \neq k$. We deduce that,

$$\|J_1\|_{L_p[x_{nn}\ ,x_{1,n}]} \leq C_5\left(S_1+S_2\right) \tag{71}$$

where,

$$S_1 := C_4\left\{\sum_{k=2}^{n}|I_{k,n}|\left[\sum_{\substack{j=1\\ j\neq k}}^{n}|y_{j,n}|\frac{|I_{j,n}|}{(x_{k,n}-x_{j,n})^2}\left[\left|1-\frac{|x_{k,n}|}{a_n}\right|+\delta_n\right]^{\frac{-1}{4}}\right]^p\right\}^{\frac{1}{p}}, \tag{72}$$

and by (30),

$$S_2 := \left\{\sum_{k=2}^{n}|y_{k,n}|^p\,|I_{k,n}|^{1-p}\left[\left|1-\frac{|x_{k,n}|}{a_n}\right|+\delta_n\right]^{\frac{-1}{4}p}dx\right\}^{\frac{1}{p}}.$$

Exactly as in the last part of Step 2, we see that (70) gives,

$$S_2 \leq C_6\left[\sum_{j=1}^{n}\lambda_{j,n}W^{-2}\left(x_{j,n}\right)|PW|^p\left(x_{j,n}\right)\right]^{\frac{1}{p}}.$$

To deal with S_1, we use Lemma 2.5 with a discrete measure space. Using (62) and (70), we see that,

$$S_1 \leq C_7\left\{\sum_{k=1}^{n}\left[\sum_{j=1}^{n}b_{k,j}\left\{\mu_{j,n}^{\frac{1}{p}}PW\left(x_{j,n}\right)\right\}\right]^p\right\}^{\frac{1}{p}},$$

where,

$$\begin{aligned}
b_{k,k} :&= 0 = b_{1,k}\ \forall\ k \text{ and for } j\neq k,\\
b_{k,j} :&= |I_{j,n}|^{2-\frac{1}{p}}\,|I_{k,n}|^{\frac{1}{p}}\left(x_{j,n}-x_{k,n}\right)^{-2}\left[\left|1-\frac{|x_{j,n}|}{a_n}\right|+\delta_n\right]^{\frac{1}{4}}\left[\left|1-\frac{|x_{k,n}|}{a_n}\right|+\delta_n\right]^{\frac{-1}{4}}.
\end{aligned}$$

Note the order: $b_{k,j}$ rather than $b_{j,k}$. Defining $B := (b_{k,j})_{k,j=1}^{n}$, we see that if L_p^n denotes the usual (little) L_p space on $\mathbb{R}^n$, then,

$$S_1 \leq C_8 \|B\|_{L_p^n \longrightarrow L_p^n} \left[\sum_{j=1}^{n} \mu_{j,n} |PW|^p (x_{j,n}) \right]^{\frac{1}{p}}$$

so the result follows if we can show that independently of n,

$$\|B\|_{L_p^n \longrightarrow L_p^n} \leq C_9. \tag{73}$$

Step 4: We prove (73).

This is far more complicated than the analogous proof for the Hermite weight [5] because of the more complicated behaviour of the spacing of the zeros of the orthogonal polynomials. We apply Lemma 2.5 with the discreet measure space $\Omega := \{1, 2, \ldots n\}$ and $\mu(\{j\}) = 1,\ j = 1, 2, \ldots n$. Moreover, we set there,

$$k(k,j) := b_{k,j};\ r_{k,j} := \left(\frac{|I_{j,n}|}{|I_{k,n}|} \right)^{\frac{1}{pq}}$$

Note that because of the way we order the variables ($b_{k,j}$ rather than $b_{j,k}$), the variable u in (57) – (58) is k, and the variable v in (57) – (58) is j. So (57–58) become,

$$\sup_{k} \sum_{\substack{j=1 \\ j \neq k}}^{n} |I_{j,n}|^2 (x_{j,n} - x_{k,n})^{-2} \left[\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + \delta_n \right]^{\frac{1}{4}} \left[\left| 1 - \frac{|x_{k,n}|}{a_n} \right| + \delta_n \right]^{\frac{-1}{4}} \leq M \tag{74}$$

$$\sup_{j} \sum_{\substack{k=1 \\ j \neq k}}^{n} |I_{j,n}| |I_{k,n}| (x_{j,n} - x_{k,n})^{-2} \left[\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + \delta_n \right]^{\frac{1}{4}} \left[\left| 1 - \frac{|x_{k,n}|}{a_n} \right| + \delta_n \right]^{\frac{-1}{4}} \leq M \tag{75}$$

Recall that given fixed $\beta \in (0,1)$, we have uniformly in l and n,

$$|I_{l,n}| \sim \frac{a_n}{n} \left(1 - \frac{|x_{l,n}|}{a_n} \right)^{\frac{1}{2}},\ |x_{l,n}| \leq a_{\beta n} \tag{76}$$

$$|I_{l,n}| \sim \frac{a_n}{n} T(a_n)^{-1} \left(1 - \frac{|x_{l,n}|}{a_n} + \delta_n\right)^{-\frac{1}{2}}, \ |x_{l,n}| \geq a_{\beta n} \tag{77}$$

(See (26) and (34)). To take account of this dual behaviour of $|I_{l,n}|$, we consider three ranges of $x_{j,n}, x_{k,n}$. It is not difficult to see that we may consider only $x_{j,n}, x_{k,n} \geq 0$.

Range 1 : $0 < x_{j,n}, x_{k,n} < a_{\frac{3n}{4}}$.

Using (76), we see that if we restrict summation in the sum in (74) to $j : |x_{j,n}| \leq a_{\frac{3n}{4}}$, then the resulting sum is bounded by a constant times,

$$I_{11} := \frac{a_n}{n}\left(1 - \frac{x_{k,n}}{a_n}\right)^{\frac{-1}{4}} \int_{\substack{0 \leq t \leq a_{\frac{4n}{5}} \\ |t - x_{k,n}| \geq C_{10} |I_{k,n}|}} \frac{\left|1 - \frac{t}{a_n}\right|^{\frac{3}{4}}}{(t - x_{k,n})^2} dt$$

We make the substitution,

$$1 - \frac{t}{a_n} = \left(\left(1 - \frac{x_{k,n}}{a_n}\right) u\right)$$

in this integral, and use (76) again to give,

$$\begin{aligned} I_{11} &\leq \frac{1}{n}\left(1 - \frac{x_{k,n}}{a_n}\right)^{\frac{-1}{2}} \int_{\substack{0 \leq u \leq \left(1 - \frac{x_{k,n}}{a_n}\right)^{-1} \\ |1-u| \geq C_{1,1} n^{-1}\left(1 - \frac{x_{k,n}}{a_n}\right)^{\frac{-1}{2}}}} \frac{|u|}{(1-u)^2} du \\ &\leq C_{12} \frac{1}{n}\left(1 - \frac{x_{k,n}}{a_n}\right)^{\frac{-1}{2}} \left[n\left(1 - \frac{x_{k,n}}{a_n}\right)^{\frac{1}{2}} + 1\right] \\ &\leq C_{13}\left[1 + \frac{1}{n} T(a_n)^{\frac{1}{2}}\right] \leq C_{14} \end{aligned}$$

by (52) and (53). Next, if we restrict summation in (75) to $k : |x_{k,n}| \leq a_{\frac{3n}{4}}$, and we use (76), we see that the resulting sum is bounded above by a constant times,

$$I_{12} := \frac{a_n}{n}\left(1 - \frac{x_{j,n}}{a_n}\right)^{\frac{3}{4}} \int_{\substack{0 \leq t \leq a_{\frac{4n}{5}} \\ |t - x_{j,n}| \geq C_{15} |I_{j,n}|}} \frac{\left|1 - \frac{t}{a_n}\right|^{\frac{3}{4}}}{(t - x_{j,n})^2} dt$$

The same substitution as before shows that $I_{1,2}$ has a similar upper bound to that for $I_{1,1}$, and hence is bounded independently of j, n.

Range 2 : $x_{j,n}, x_{k,n} \geq a_{\frac{n}{2}}$

Using (77), we see that after restricting summation in the sum in (74) to $j : |x_{j,n}| \geq a_{\frac{n}{2}}$, then the resulting sum is bounded by a constant times,

$$\begin{aligned}
&\sum_{\substack{j:\left|x_{j,n}\right|\geq a_{\frac{n}{4}} \\ j\neq k}} \frac{\left|I_{j,n}\right|^{\frac{3}{2}}\left|I_{k,n}\right|^{\frac{1}{2}}}{\left(x_{j,n}-x_{k,n}\right)^{2}} \\
&\leq C_{16} \sum_{\substack{j:\left|x_{j,n}\right|\geq a_{\frac{n}{4}} \\ j\neq k}} \frac{\left|I_{j,n}\right|^{\frac{3}{2}}\left|I_{k,n}\right|^{\frac{1}{2}}}{\left(x_{j,n}-x_{k,n}\right)^{2}} \\
&\leq C_{17} \int_{t:\left|t-x_{k,n}\right|\geq C_{18}\left|I_{k,n}\right|} \frac{dt}{\left(t-x_{k,n}\right)^{2}} \leq C_{18}.
\end{aligned}$$

Similarly, after restricting summation in the sum in (75) to $k : |x_{k,n}| \geq a_{\frac{n}{2}}$, then the resulting sum is bounded by a constant times,

$$\sum_{\substack{j:\left|x_{k,n}\right|\geq a_{\frac{n}{4}} \\ k\neq j}} \frac{\left|I_{k,n}\right|^{\frac{3}{2}}\left|I_{j,n}\right|^{\frac{1}{2}}}{\left(x_{j,n}-x_{k,n}\right)^{2}}$$

After swopping the indicies j and k, we see that this is the same as the sum just estimated.

Range 3 : $x_{j,n} < a_{\frac{n}{2}}$ **and** $x_{k,n} > a_{\frac{3n}{4}}$**; or** $x_{j,n} > a_{\frac{3n}{4}}$ **and** $x_{k,n} < a_{\frac{n}{2}}$**.**

Here,

$$|x_{j,n} - x_{k,n}| \geq a_{\frac{3n}{2}} - a_{\frac{n}{2}} \geq C_{19} \frac{a_n}{T(a_n)}$$

(See (52)). Also, given fixed small $\varepsilon > 0$, we see that,

$$|I_{l,n}| \leq C_{20} n^{-\frac{2}{3}+\varepsilon}, \text{ uniformly in } l \text{ and } n$$

(See (24), (53), (76), (77)). Finally,

$$\left[\left|1 - \frac{|x_{k,n}|}{a_n}\right| + \delta_n\right]^{-\frac{1}{4}} \leq C_{21} n^{\frac{1}{6}+\varepsilon}.$$

Then we see after suitably restricting the range of summation in (74), we obtain a sum bounded by,

$$C_{22} n^{\frac{-1}{2}+2\varepsilon} T\left(a_n\right)^2 \sum_j \left|I_{j,n}\right| \leq C_{23} n^{\frac{-1}{2}+2\varepsilon} T\left(a_n\right)^2 a_n = o(1).$$

Similarly the sum arising from (75) is $o(1)$. So we have completed the proof of (73). □

4 Proof of the Sufficiency Conditions

We begin with the,

Proof of the Sufficiency part of Theorem 1.3

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous and satisfy (12) with $\alpha > \frac{1}{p}$. We must show (13). Let $\varepsilon \in (0,1)$. We can choose a polynomial P such that,

$$\left\|(f-P)(x) W(x)\left(1+|x|\right)^{\alpha}\right\|_{L_\infty[\mathbb{R}]} \leq \varepsilon.$$

(Compare [6]). Then for n large enough,

$$\begin{aligned}
&\left\|\left(f-L_n[f]\right) W\right\|_{L_p[\mathbb{R}]} \\
&\quad \leq \|(f-P)W\|_{L_p[\mathbb{R}]} + \left\|L_n[P-f]W\right\|_{L_p[\mathbb{R}]} \\
&\quad \leq \varepsilon \left\|(1+|x|)^{-\alpha}\right\|_{L_p[\mathbb{R}]} + \left\|L_n[P-f]W\right\|_{L_p[\mathbb{R}]} \qquad (78)
\end{aligned}$$

The first norm in the right–hand side of (78) is of course finite as $\alpha p > 1$. Next, Theorem 3.1 shows that for large enough n,

$$\begin{aligned}
\left\|L_n[P-f]W\right\|_{L_p[\mathbb{R}]} &\leq C_1 \left\{\sum_{j=1}^{n} \lambda_{j,n} W^{-2}\left(x_{j,n}\right) |(P-f)W|^p \left(x_{j,n}\right)\right\}^{\frac{1}{p}} \\
&\leq C_2 \varepsilon \left\{\sum_{j=1}^{n} \left|I_{j,n}\right| \left(1+\left|x_{j,n}\right|\right)^{-\alpha p}\right\}^{\frac{1}{p}} \\
&\leq C_3 \varepsilon \left\|(1+|x|)^{-\alpha}\right\|_{L_p[\mathbb{R}]}
\end{aligned}$$

Substituting into (78), and noting that the various constants are independent of ε, gives the result. □

Proof of the Sufficiency Part of Theorem 1.4

As $(1+|x|)^{\Delta} \leq 1$ if $\Delta \leq 0$, the limit (14) follows from (13). □

5 Proof of the Necessary Conditions

We begin with,

Lemma 5.1
Let $0 < p < \infty$. Let $0 < A < B < \infty$ and $\xi : \mathbb{R} \longrightarrow (0, \infty)$ be a continuous function such that for $1 \leq s,\, t < \infty$ with $\frac{1}{2} \leq \frac{s}{t} \leq 2$, we have,

$$A \leq \frac{\xi\left(a_s\right)}{\xi\left(a_t\right)} \leq B. \tag{79}$$

For $n \geq 1$, let $\Im_n \subset [-a_n\ a_n]$ be an interval containing at least two zeros of $p_n\left(W^2,\ .\ \right)$. Then for $n \geq 1$,

$$\left\|p_n W \xi\right\|_{L_p[\Im_n]} \geq C_1 a_n^{\frac{-1}{2}} \left\| \xi(t) \left(\left| 1 - \frac{|t|}{a_n} \right| + \delta_n \right)^{\frac{-1}{4}} \right\|_{L_p[\Im_n]} \tag{80}$$

Here C_1 depends only on A, B (and not on ξ or n or $\Im_n$).

Proof.
From (46), for $x \in [x_{j+1,} x_{j,n}]$,

$$\max\left\{ l_{j,n}(x) W^{-1}\left(x_{j,n}\right) W(x), l_{j+1,n}(x) W^{-1}\left(x_{j+1,n}\right) W(x) \right\} \geq \frac{1}{2}$$

and hence for such x,

$$\begin{aligned} \left|p_n W\right|(x) &\geq \frac{1}{2} \min\left\{ \left|x - x_{j,n}\right| \left|p_n' W\right|\left(x_{j,n}\right), \left|x - x_{j+1,n}\right| \left|p_n' W\right|\left(x_{j+1,n}\right) \right\} \\ &\geq C_2 \frac{n}{a_n^{\frac{3}{2}}} \Psi_n^{-1}\left(x_{j,n}\right) \left(\left| 1 - \frac{\left|x_{j,n}\right|}{a_n} \right| + \delta_n \right)^{\frac{-1}{4}} \min\left\{ \left|x - x_{j,n}\right|, \left|x - x_{j+1,n}\right| \right\} \end{aligned}$$

by (40), (41) and (42). Let,

$$\Im_{j,n} := \left[x_{j+1,n} + \frac{1}{4}\left(x_{j,n} - x_{j+1,n}\right), x_{j,n} + \frac{1}{4}\left(x_{j,n} - x_{j+1,n}\right) \right],$$

so that $\Im_{j,n}$ has length $\frac{1}{2}\left(x_{j,n} - x_{j+1,n}\right)$. By (34),

$$\left|p_n W\right|(x) \geq C_3 a_n^{\frac{-1}{2}} \left(\left| 1 - \frac{\left|x_{j,n}\right|}{a_n} \right| + \delta_n \right)^{\frac{-1}{4}},\ x \in \Im_{j,n}.$$

Then using also (40),

$$\begin{aligned}
&\int_{x_{j+1,n}}^{x_{j,n}} |p_n W|^p (t) \xi^p(t) dt \\
\geq\ & C_4 a_n^{-\frac{p}{2}} \left(\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + \delta_n \right)^{\frac{-1}{4}} \int_{\Im_{j,n}} \xi^p(t)\ dt.
\end{aligned}$$

The result follows if we can show that,

$$\int_{\Im_{j,n}} \xi^p(t)\ dt \geq C_5 \int_{x_{j+1,n}}^{x_{j,n}} \xi^p(t) dt.$$

(The L_p norm of $\xi(t) \left(\left| 1 - \frac{|t|}{a_n} \right| + \delta_n \right)^{\frac{-1}{4}}$ over that part of $\Im_{j,n}$ near the end-points of this interval is easily estimated in terms of the rest). To do this if suffices to show that,

$$\xi(t) \sim \xi\left(x_{j,n}\right),\ t \in \left[x_{j+1,n}\ x_{j,n}\right].$$

Now in view of (79), it suffices to show that if $x_{j+1,n} = a_s$ and $x_{j,n} = a_t$, where $s \geq s_o > 0$. (Here we use the continuity of the map $u :\longrightarrow a_u$), then,

$$1 \leq \frac{s}{t} \leq 2. \tag{81}$$

But if $t \geq 2s$, then (48) and (49) give,

$$\frac{x_{j,n}}{x_{j+1,n}} - 1 \geq \frac{a_{2s}}{a_s} - 1 \geq C_6 \frac{1}{T\left(a_s\right)} \geq C_7 \frac{1}{T\left(a_n\right)}$$

while our spacing (2.3) gives,

$$\frac{x_{j,n}}{x_{j+1,n}} - 1 \leq C_8 \frac{a_n}{n} \frac{\Psi_n\left(x_{j,n}\right)}{x_{j+1,n}} \leq C_9 \frac{a_n}{n} \Psi_n\left(a_n\right) \leq C_{10} a_n \left(nT\left(a_n\right)\right)^{-\frac{2}{3}}.$$

Our hypothesis shows that $T\left(a_n\right)$ is much larger than any negative power of n, for n large, and we have a contradicton. So (81) and the result follow. $\square$

We can now proceed with the,

Proof of the necessity parts of Theorem 1.3 and 1.4.

Fix $\alpha, \Delta \in \mathbb{R}$ and $1 < p < 4$. Assume moreover that we have the convergence (14) for every continuous f satisfying (12). Let $\eta : \mathbb{R} \longrightarrow (0, \infty)$ be a positive even continuous function, decreasing in $(0, \infty)$, with limit 0 at ∞. We shall assume it decays very slowly later on. Let

$$X := \left\{ f : \mathbb{R} \longrightarrow \mathbb{R} \text{ continuous with } \|f\|_X := \sup_{x \in \mathbb{R}} |fW|(x) \left(1 + |x|\right)^{\alpha} \eta(x)^{-1} < \infty \right\}.$$

Moreover, let Y be the space of all measurable functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ with

$$\|f\|_Y := \left\| (fW)(x) \left(1 + |x|\right)^{\Delta} \right\|_{L_p[\mathbb{R}]} < \infty.$$

Each $f \in X$ satisfies (12), so the conclusion of Thm 1.4 ensures that,

$$\lim_{n \longrightarrow \infty} \|(f - L_n[f])\|_Y = 0$$

Since X is a Banach space, the uniform boundedness principle gives,

$$\|(f - L_n[f])\|_Y \leq C \|f\|_X,$$

with C independent of n and f. In particular as $L_1[f] = f(0)$ (recall that $p_1(x) = \gamma_1(x)$), we deduce that for $f \in X$ with $f(0) = 0$,

$$\|f\|_Y \leq C \|f\|_X.$$

So for such f,

$$\|L_n[f]\|_Y \leq 2C \|f\|_X \tag{82}$$

Choose g_n continuous in $\mathbb{R}$, with $g_n = 0$ in $[0, \infty) \cup (-\infty, \frac{-1}{2} a_n]$, with,

$$\|g_n\|_X = \sup_{x \in \mathbb{R}} |g_n W|(x) \left(\log\left(2 + |x|\right)^{\delta}\right) = 1,$$

and for $x_{j,n} \in \left(-\frac{1}{2} a_n, 0\right)$,

$$g_n W\left(x_{j,n}\right) \left(\log\left(2 + |x_{j,n}|\right)^{\delta} sign\left(p_n'\left(x_{j,n}\right)\right)\right) = 1.$$

For example, $\left(g_n W(x)\left(\log\left(2+|x|\right)\right)^{\delta}\right)$ can be chosen to be piecewise linear. Then for $x \in \left[1.\frac{a_n}{4}\right]$,

$$\begin{aligned}
|L_n[g_n](x)| &= \left|\sum_{x_{j,n}\in[-\frac{1}{2}a_n,0)} g_n(x_{j,n})\frac{p_n(x)}{p_n'(x_{j,n})(x-x_{j,n})}\right| \\
&= |p_n(x)|\sum_{x_{j,n}\in[-\frac{1}{2}a_n,0)}\frac{(1+|x_{j,n}|)^{-\alpha}\eta(x_{j,n})}{|p_n'W|(x_{j,n})(x+|x_{j,n}|)} \\
&\geq C_1 a_n^{\frac{1}{2}}|p_n(x)|\eta(a_n)\sum_{x_{j,n}\in[-2x,-x)}|I_{j,n}|\frac{(1+|x_{j,n}|)^{-\alpha}}{(x+|x_{j,n}|)} \\
&\quad \text{(by (42))} \\
&\geq C_2 a_n^{\frac{1}{2}}|p_n(x)|\eta(a_n)\int_x^{2x} t^{-1-\alpha}dt \\
&\quad \text{(by (34))} \\
&\geq C_3 a_n^{\frac{1}{2}}|p_n(x)|\eta(a_n)x^{-\alpha}
\end{aligned}$$

Then by (83),

$$\begin{aligned}
2C &= 2C\|g_n\|_X \geq \|L_n[g_n]\|_Y \\
&\geq C_4 a_n^{\frac{1}{2}}\eta(a_n)\left\|p_n W(x)x^{\Delta-\alpha}\right\|_{L_p\left[1,\frac{a_n}{4}\right]} \\
&\geq C_5\eta(a_n)\left\|x^{\Delta-\alpha}\right\|_{L_p\left[1,\frac{a_n}{4}\right]}
\end{aligned}$$

by Lemma 5.1. We may assume that η decays so slowly to 0 that,

$$\eta(a_n) \geq (\log\log a_n)^{-1}.$$

(Note that we could have imposed this condition on η at the start, but delayed this for clarity).

Suppose now that $\Delta - \alpha \geq \frac{-1}{p}$. Then we obtain,

$$2C \geq C_6(\log\log(a_n))^{-1}\log a_n.$$

Then for large n, we obtain a contradiction. So we deduce $\Delta - \alpha < \frac{-1}{p}$ is necessary. Consequently if for a given $\Delta \in \mathbb{R}$, we have the convergence (14) for every continuous f satisfying (12) and for every $\alpha > \frac{1}{p}$ then we must have $\Delta \leq 0$. The necessity part of Theorem 1.4 is proved.

Finally, for the necessity part of Theorem 1.3, we take $\Delta = 0$ in the above and deduce that $\alpha > \frac{1}{p}$. $\square$

Proof of Theorem 1.5

This is similar to the previous proof. We let X be the Banach space of continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ vanishing outside $[-2, 2]$, with norm,

$$\|f\|_X := \|f\|_{[-2,2]}.$$

We let Y be the space of all measurable $f : \mathbb{R} \longrightarrow \mathbb{R}$ with,

$$\|f\|_Y := \|fWU\|_{L_p[\mathbb{R}]} < \infty.$$

Assume that we cannot find f satisfying (17). Then the uniform boundedness principle gives (82) for all $f \in X$. Again, when $f(0) = 0$, we obtain (83). We now choose $g_n \in X$, with

$$\|g_n\|_X = 1$$

$$(g_n W)(x_{j,n})\, sign\left(p_n'(x_{j,n})\right) = 1$$

in $\left[-1, -\frac{1}{2}\right]$, $g_n = 0$ in $(-\infty, -2] \cup [0, \infty)$ and

$$g_n W(x_{j,n})\, sign\left(p_n'(x_{j,n})\right) \geq 0$$

in $[-2, 2]$. Much as before, we deduce that for $x \geq 1$,

$$|L_n[g_n](x)| \geq C a_n^{\frac{1}{2}} \frac{|p_n(x)|}{x}$$

Also by hypothesis, given $A > 0$, there exists C_2 such that,

$$U(x) \geq A x^{\frac{3}{4}} [\log Q(x)]^{-\frac{1}{4}},\ x \geq C_2.$$

Hence by (83),

$$\begin{aligned}
2C &= 2C \|g_n\|_X \geq \|L_n[g_n]\|_Y \\
&\geq C_1 A a_n^{\frac{1}{2}} \left\| p_n(x) W(x) x^{\frac{-1}{4}} [\log Q(x)]^{\frac{-1}{4}} \right\|_{L_4[C_2, a_n]} \\
&\geq C_3 A a_n^{\frac{1}{4}} [\log n]^{\frac{-1}{4}} \|p_n W\|_{L_4\left[a_{\frac{n}{2}},\, a_n\right]}
\end{aligned}$$

by (47) and (53). Now by Lemma 5.1,

$$\begin{aligned}
\|p_nW\|_{L_4\left[a_{\frac{n}{2}},\, a_n\right]} &\geq C_4 a_n^{\frac{-1}{2}} \left\| \left(1+\frac{t}{a_n}+\delta_n\right)^{\frac{-1}{4}} \right\|_{L_4\left[a_{\frac{n}{2}},\, a_n\right]} \\
&= C_4 a_n^{\frac{-1}{4}} \left[\int_{0\leq s\leq \left(1-\frac{a_{\frac{n}{2}}}{a_n}\right)/\delta_n} (1+s)^{-1}\, ds \right]^{\frac{1}{4}} \\
&\geq C_5 a_n^{\frac{-1}{4}} \left[\log\left\{1+C_6\delta_n^{-1}\left(T\left(a_n\right)^{-1}\right)\right\}\right]^{\frac{1}{4}} \\
&\geq C_6 a_n^{\frac{-1}{4}} \left(\log n\right)^{\frac{1}{4}}.
\end{aligned}$$

Here we make the substitution $1-\frac{t}{a_n}=\delta_n s$, and also used (52) and (53). Finally, using (84), we obtain

$$2C \geq C_7 A.$$

It is clear that C_7 is independent of A. Of course, this is impossible for large A. So there must exist continuous f vanishing outside $[-2,2]$ satisfying (17). □

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