Necessary and Sufficient Conditions for Mean Convergence of Lagrange Interpolation for Erdös Weights

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Abstract

We investigate mean convergence of Lagrange Interpolation at the zeros of orthogonal polynomials $p_n(W^2, x)$ for Erdös Weights $W^2 = \exp[-2Q]$. The archetypal example is $W_{k,\alpha} = \exp(-Q_{k,\alpha})$, where $Q_{k,\alpha}(x) := \exp(|x|^\alpha)\ \ \alpha > 1$, where $\exp_k = \exp(\exp(\exp(...)))$ is the kth iterated exponential. Following is our main result: Let $1 < p < \infty, \Delta \in \mathbb{R}\ \kappa > 0$. Let $L_n(f)$ denote the Lagrange interpolation polynomial to f at the zeros of $p_n(W^2, x) = p_n(\exp[-2Q], x)$. Then for,

$$\lim_{n\to\infty} \left\| f - L_n(f)\, W\, (1+Q)^{-\Delta} \right\|_{L_P(\mathbb{R})} = 0$$

to hold for every continuous function $f:\ \mathbb{R} \to \mathbb{R}$ satisfying,

$$\lim_{|x|\to\infty} (fW)(x)\, (\log|x|)^{1+\kappa} = 0$$

it is necessary and sufficient that,

$$\Delta > \max\left\{0, \frac{2}{3}\left(\frac{1}{4} - \frac{1}{p}\right)\right\}$$

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1 Introduction and Results.

In the past twenty years, there has begun to develop a general theory of orthogonal polynomials, and associated approximation theory, for weights on $\mathbb{R}$ [8,18]. In several aspects of the investigations, it has been helpful to distinguish between Erdős weights and Freud Weights.

Freud weights have the form $W^2 = \exp[-2Q]$, where $Q : \mathbb{R} \to \mathbb{R}$ is even and of polynomial growth at infinity. The archetypal example is,

$$W_\beta(x) := \exp\left(-Q_\beta(x)\right), Q_\beta(x) := \frac{1}{2}|x|^\beta,\ \beta > 0. \tag{1}$$

Erdös Weights have the form $W^2 = \exp[-2Q]$, where $Q : \mathbb{R} \to \mathbb{R}$ is even and of faster than polynomial growth at infinity. The archetypal example is,

$$W_{k,\alpha}(x) := \exp\left(-Q_{k,\alpha}(x)\right). \tag{2}$$

where

$$Q_{k,\alpha}(x) := \exp_k\left(|x|^\alpha\right),\ k \geq 1,\ \alpha > 0. \tag{3}$$

Here $\exp_k$ =$\exp_k = \exp\left(\exp\left(\exp\left(...\right)\right)\right)$ is the kth iterated exponential.

Given a weight $W : \mathbb{R} \to \mathbb{R}$ such as those above, we can define orthonormal polynomials,

$$p_n(x) = p_n\left(W^2, x\right) = \gamma_n x^n + ..., \gamma_n = \gamma_n\left(W^2\right) > 0,$$

satisfying,

$$\int_{\mathbb{R}} p_n\left(W^2, x\right) p_m\left(W^2, x\right) W^2\left(x\right) dx = \delta_{mn}$$

To those unfamiliar with the theory of weights on $\mathbb{R}$, writing W^2, rather than say w for a weight might seem strange. However the square reflects the L_2 norm, and facilitates formulation of theorems. We denote the zeros of p_n by,

$$-\infty < x_{nn} < x_{n-1,n} \cdots < x_{2,n} < x_{1,n} < \infty$$

The Lagrange Interpolation polynomial to a function $f : \mathbb{R} \to \mathbb{R}$ at the zeros $\{x_{j,n}\}_{j=1}^n$ is denoted by $L_n(f)$. Thus if P_m denotes the class of polynomials of degree $\leq m$ and $l_{jn} \in \mathcal{P}_{\setminus -\infty}$, $\infty \leq | \leq \setminus$, are the *fundemental polynomials* of Lagrange interpolation at $\{x_{j,n}\}_{j=1}^n$ satisfying,

$$l_{jn}\left(x_{j,n}\right) = \delta_{j,k}$$

then,

$$L_n(f)(x) = \sum_{j=1}^{n} f(x_{jn})\, l_{jn}(x) \tag{4}$$

For a large class of Freud Weights, mean convergence of Lagrange Interpolation was investigated by several authors [1], [4], [11], [17]. The possiblity of obtaining identical necassary and sufficient conditions for mean convergence of L_n arises from bounds obtained for $p_n(W^2, x)$ by A.L. Levin and the second author [6]. For notational simplicity, we recall the result of Matjila and the second author only for $W_\beta^2,\ \beta > 1$.

Theorem 1.1. Let $W_\beta(x) := \exp\left(-\frac{1}{2}|x|^\beta\right),\ \beta > 1$. Given $f : \mathbb{R} \to \mathbb{R}$, let $L_n[f]$ denote the Lagrange Interpolation polynomial to f at the zeros of $p_n(W^2, x)$. Let $1 < p < \infty,\ \Delta \in \mathbb{R},\ \alpha > 0$, and

$$\tau := \frac{1}{p} - \min\{1, \alpha\} + \max\left\{0, \frac{\beta}{6}\left(1 - \frac{4}{p}\right)\right\}.$$

Then for,

$$\lim_{n\to\infty} \left\| f - L_n(f)\, W (1 + |x|)^{-\Delta} \right\|_{L_P(\mathbb{R})} = 0$$

to hold for every continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying,

$$\lim_{|x|\to\infty} (fW)(x)(1 + |x|)^\alpha = 0$$

it is necessary and sufficient that,

$$\begin{aligned} &\Delta > \tau \text{ if } 1 < p \le 4 \\ &\Delta > \tau \text{ if } 4 < p \text{ and } \alpha = 1 \\ &\Delta > \tau \text{ if } 4 < p \text{ and } \alpha \ne 1. \end{aligned}$$

In describing analogous results for Erdős Weights, we need a class of weights W^2 for which suitable bounds are available for $p_n(W^2, .)$. These were found in [7] and L_p analoges were found in [10]. For our purposes, the folowing subclass of weights from [7] is suitable:

Definition 1.2. Let $W := \exp[-Q]$, where $Q : \mathbb{R} \to \mathbb{R}$ is even, continuous, $Q^{(2)}$exists in $(0, \infty)$ and the function,

$$T(x) := 1 + \frac{xQ^{(2)}(x)}{Q^{(1)}(x)} \tag{5}$$

is increasing in $(0, \infty)$ with,

$$\lim_{x \to \infty} T(x) = \infty, \ T(0+) := \lim_{x \to 0+} T(x) > 1. \tag{6}$$

Moreover, we assume for some $C_1, C_2, C_3 > 0$,

$$C_1 \leq \frac{T(x)}{\frac{xQ^{(1)}(x)}{Q(x)}} \leq C_2, \ x \geq C_3 \tag{7}$$

and for every $\varepsilon > 0$,

$$T(x) = O\left(Q(x)^{\varepsilon}\right), x \to \infty \tag{8}$$

Then we write $W \in \mathcal{E}_1$.

The new restrictions over those in [7] are (8) and $Q \geq 0$. The latter is easily achieved by replacing Q by $Q + |Q(0)|$. The former is needed in simplifing the formulation of our theorems. The principal example of $W = \exp[-Q] \in \mathcal{E}_1$ is $W_{k,\alpha} = \exp\left(-Q_{k,\alpha}\right)$ given by (3) with $\alpha > 1$. For this W,

$$T(x) = T_{k,\alpha}(x) = \alpha \left[1 + x^{\alpha} \sum_{l=1}^{k} \prod_{j=1}^{l-1} \exp_j\left(x^{\alpha}\right), \ x \geq 0.\right] \tag{9}$$

Here (7) holds in the stronger form,

$$\lim_{x \to \infty} \frac{T(x)}{\frac{xQ^{(1)}(x)}{Q(x)}} = 1 \tag{10}$$

and (8) holds in the stronger form,

$$\lim_{x \to \infty} \frac{T(x)}{\left[\prod_{j=1}^{k} \log_j Q(x)\right]} = \alpha \tag{11}$$

Here, and in the sequel, $\log(\log(...))$denotes the kth iterated logarithm. For $\alpha \leq 1$, the second part of (6) fails, but this can be circumvented by considering $W_{k,\frac{\alpha}{2}}\left(A + x^2\right)$, with A large enough to guarantee $T(0+) > 1$.

Another (more slowly decaying) example of $W = \exp[-Q] \in \mathcal{E}_1$ is given by,

$$Q(x) := \exp\left[\log\left(A + x^2\right)^{\beta}\right], \ \beta > 1, \ A \text{ large enough}, \tag{12}$$

for which,

$$T(x)=\frac{2x^2}{A+x^2}\left[\frac{\beta-1}{\log\left(A+x^2\right)}+\beta\left\{\log\left(A+x^2\right)\right\}^{\beta-1}\right]+\frac{2A}{A+x^2} \qquad (13)$$

Again (7) holds in the stronger form (10), while (8) holds in the stronger form,

$$\lim_{x\to\infty}\frac{T(x)\log x}{\log Q(x)}=\beta \qquad (14)$$

The first results for mean convergence of Lagrange interpolation for a class of Erdös weights appeared in [13]. However the restrictions there both on W and on the growth of f are more severe, because the correct bounds on $p_n\left(W^2,.\right)$ were not available.

Following is our main result:

Theorem 1.3. Let $W:=\exp[-Q]\in\mathcal{E}_1$. Let $L_n[.]$ denote the Lagrange Interpolation to $f$at the zeros of $p_n\left(W^2,.\right)$. Let $1<p<\infty,\Delta\in\mathbb{R},\ \kappa>0$. Then for,

$$\lim_{n\to\infty}\left\|\left(f-L_n[f]\right)W\left(1+Q\right)^{-\Delta}\right\|_{L_P(\mathbb{R})}=0 \qquad (15)$$

to hold for every continuous function $f:\mathbb{R}\to\mathbb{R}$ satisfying,

$$\lim_{|x|\to\infty}|fW|(x)\left(\log|x|\right)^{1+\kappa}=0 \qquad (16)$$

it is necessary and sufficient that,

$$\Delta>\max\left\{0,\frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right)\right\} \qquad (17)$$

At first, the choice of the extra weighting factor $(1+Q)$ in (15) may seem rather severe. After all Q grows faster than any polynomial. However, even if f vanishes outside a fixed finite interval, we need such a factor if $p>4$:

Theorem 1.4. Let W, L_n be as above and $p>4$. Suppose that measurable $U:\mathbb{R}\to\mathbb{R}$ satisfies,

$$\liminf_{x\to\infty}U(x)x^{-\left(\frac{3}{2}-\frac{1}{p}\right)}Q(x)^{\frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right)}>0 \qquad (18)$$

Then ther exists continuous $f : \mathbb{R} \to \mathbb{R}$ vanishing outside $[-2, 2]$ such that,

$$\limsup_{n\to\infty} \|L_n[f]WU\|_{L_P[\mathbb{R}]} = \infty \tag{19}$$

So for $p > 4$, no growth restriction on f, however severe, allows us a weighting factor weaker than a power of $1 + Q$. One can formulate versions of Theorem 1.3 for $p > 4$ that involve $\Delta = \frac{2}{3}\left(\frac{1}{4} - \frac{1}{p}\right)$, and then one has to introduce extra factors in (15), such as negative powers of $1+|x|$ and negative powers of T or $\log(2 + Q)$. Unfortunately one then needs extra hypotheses on T to avoid very complicated formulations. One of the complicating features here is that T may grow faster than any power of $|x|$ (as in (9) for $k \geq 2$), like a power of x (as in (9) for $k = 1$), or slower than any power of $|x|$ (as in (13)). Moreover, one has to compare T to $\log Q$. We spare the reader the details.

For $p \leq 4$, the weighting factor $1 + Q$ is unnessesary strong. Let us recall the Erdős–Turan theorem, as extended by Shohat (see [3,Ch.2,p.97]). If $f : \mathbb{R} \to \mathbb{R}$ is Riemann integrable in each finite interval, and there exists an even entire function G with all non–negative Maclaurin series coefficients such that,

$$\lim_{|x|\to\infty} \frac{f^2(x)}{G(x)} = 0$$

and

$$\int_{\mathbb{R}} G(x)W^2(x)dx < \infty$$

then

$$\lim_{n\to\infty} \|(f - L_n[f])\, W\|_{L_2[\mathbb{R}]} = 0$$

For nice weights here, a result of Clunie and Kovari [2,Th4,pg19], allows us to choose G with

$$G(x) \sim W^{-2}(x)\,(1 + |x|)^{-1-\kappa} \quad x \in \mathbb{R},\ \kappa > 0.$$

Here and in the sequel, the notation involving $\sim$ means that the ratio of the two sides is bounded above and below by positive constants independent of x. (Later on, the dependence will be on n and possibly other parameters). Thus we can ensure that (20) holds provided,

$$\lim_{|x|\to\infty} (fW)(x)\,(1 + |x|)^{\frac{1}{2}+\frac{\kappa}{2}} = 0$$

Thus our result does not extend the classical result for $p = 2$.

Actually, the extension from continuous functions to Riemann integrable ones can be completed in the context of the present paper, but would substantially lengthen the proofs, so is ommited. Our main emphasis in any event, is the weighting factors required on L_n, or f.

By modifying the proofs presented below, we can prove results of the form,

$$\lim_{n\to\infty} \|(f - L_n[f])\, W\|_{L_P[\mathbb{R}]} = 0 \tag{20}$$

with $p < 4$, but under severe growth conditions on f. For example, we need to assume,

$$\lim_{|x|\to\infty} |fW|\,(x) T(x)^a |x|^b = 0$$

with suitable choices of $a, b > 0$. Of course T should not really be there. For $p = 4$, a weighting factor is needed in (21). We hope to improve these results to include the Erdős–Turan result in a future paper.

Thiis paper is organised as follows: In section 2, we gather technical estimates from other papers. In section 3, we present some quadrature sum estimates. In section 4, we prove the sufficiency part of Theorem 1.3, and in section 5, we prove the necessity part of Theorem 1.3 and also prove Theorem 1.4.

We close this section by introducing some more notation. Given Q as above, the *Mhaskar–Rhamanov–Saff* number a_u is the positive root of the equation,

$$u = \frac{2}{\pi} \int_0^1 a_u t Q'\left(a_u t\right) \frac{dt}{\sqrt{1-x^2}},\ u > 0 \tag{21}$$

For example, for $W_\beta,\ a_u = C(\beta) u^{\frac{1}{\beta}},\ u > 0$. It is instructive to see how a_u. $T\left(a_u\right),\ Q\left(a_u\right)$ grow for the example $Q = Q_{k,\alpha}$ of (3). Here,

$$a_u \sim \left(\log_k u\right)^{\frac{1}{\alpha}} \tag{22}$$

$$T\left(a_u\right) \sim \prod_{j=1}^{k} \log_j u \tag{23}$$

$$Q\left(a_u\right) \sim u \left\{ \prod_{j=1}^{k} \log_j u \right\}^{-\frac{1}{2}} \tag{24}$$

To the unfamiliar, one of the of a_u is in the identity [14],

$$\|PW\|_{L_\infty[\mathbb{R}]} = \|PW\|_{L_\infty[-a_n\ a_n]},\ P \in \mathcal{P}_\backslash \tag{25}$$

Here and the sequel, $\mathcal{P}_\backslash$ denotes the polynomials of degree $\leq n$. There are also several L_p analouges [15], [6], [7], for example, given $\varepsilon > 0$, there exists $C > 0$ such that for $n \geq 1$ and $P \in \mathcal{P}_\backslash$, [6], [7]

$$\|PW\|_{L_p[\mathbb{R}]} \leq C\|PW\|_{L_\infty[-a_n\ a_n]}, \tag{26}$$

In the sequel, $C, C_1, C_2...$ denote constants independent of n, x and $P \in \mathcal{P}_\backslash$. The same symbol does not necessarily denote the same constant in differences occurrences.

The nth *Christoffel function* for a weight W^2 is,

$$\begin{aligned} \lambda_n(x) \quad : &= \lambda_n\left(W^2, x\right) = \inf_{P \in \mathcal{P}_{n-1}} \int_{\mathbb{R}} \frac{(PW)^2 dt}{P^2(x)} \\ &= \frac{1}{\sum_{j=0}^{n-1} p_j^2(x)}. \end{aligned} \tag{27}$$

The *Christoffel numbers* are,

$$\lambda_{jn} = \lambda_n\left(W^2, x_{j,n}\right)\ \ 1 \leq j \leq n. \tag{28}$$

The fundamental polynomials l_{jn} of (4) admit the representation,

$$l_{jn}(x) = \lambda_{jn} \frac{\gamma_{n-1}}{\gamma_n} p_{n-1}\left(x_{j,n}\right) \frac{p_n(x)}{x - x_{j,n}} \tag{29}$$

The reproducing kernel for W^2 is,

$$\begin{aligned} K_n(x,t) \quad : &= K_n\left(W^2, x, t\right) = \sum_{j=0}^{n-1} p_j(x) p_j(t) \\ &= \frac{\gamma_{n-1} p_n(x) p_{n-1}(t) - p_n(t) p_{n-1}(x)}{\gamma_n (x - t)} \end{aligned} \tag{30}$$

(the Christoffel–Darboux formula).

Given measurable $f: \mathbb{R} \to \mathbb{R}$ with $f(x)x^jW^2 \in L_1(\mathbb{R})\ \forall j \geq 0$, the nth partial sum of its orthonormal expansion with respect to W^2 is denoted by $S_n[f](x)$, and admits the representation,

$$S_n[f](x) = \int_{\mathbb{R}} K_n(x,t)f(t)W^2(t)dt \tag{31}$$

If we introduce the **Hilbert Transform** of $g \in L_1(\mathbb{R})$ by,

$$H[g](x) := \lim_{\varepsilon\to 0+} \int_{|x-t|\geq\varepsilon} \frac{g(t)}{x-t}dt, \tag{32}$$

(this exists a.e. [20]), then we may use the Christoffel–Darboux formula for $K_n(x,t)$ to rewrite (32) as,

$$S_n[f] = \frac{\gamma_{n-1}}{\gamma_n}\left\{p_n H\left[fp_{n-1}W^2\right] - p_{n-1}H\left[fp_nW^2\right]\right\} \tag{33}$$

Finally, we define some auxillary quantities:

$$\delta_n := (nT(a_n))^{-\frac{2}{3}},\ n \geq 1. \tag{34}$$

This quantity is useful in describing the behaviour of $p_n(\exp[-2Q], t)$ near $x_{1,n}$. For example,

$$\left|\frac{x_{1,n}}{a_n(Q)} - 1\right| \leq \frac{L\delta_n}{2}. \tag{35}$$

Here L is independent of n. We often use the fact that δ_n is much smaller than any power of $\frac{1}{T(a_n)}$, see section 2. We also use the function,

$$\Psi_n(x) := \max\left\{\sqrt{1-\frac{|x|}{a_n}+L\delta_n},\ \left[T(a_n)\sqrt{1-\frac{|x|}{a_n}+L\delta_n}\right]^{-1}\right\},\ |x| \leq a_n \tag{36}$$

and set

$$\Psi_n(x) := \Psi_n(a_n),\ |x| > a_n \tag{37}$$

This function is used in describing spacing of zeros of p_n, behaviour of Christoffel functions and so on.

2 Technical Lemmas

In this section, we gather technical estimates from various sources. We begin by recalling a number of estimates from [7]. Throughout, we assume that $W := \exp[-Q] \in \mathcal{E}_\infty$.

Lemma 2.1.

(a) Uniformly for $n \geq 1$ and $|x| \leq a_n$,

$$\lambda_n \left(W^2, x\right) \sim \frac{a_n}{n} W^2(x) \Psi_n(x) \tag{1}$$

(b) For $n \geq 1$,

$$\left|\frac{x_{1,n}}{a_n} - 1\right| \leq C\delta_n \tag{2}$$

Uniformly for $n \geq 2$ and $1 \leq j \leq n-1$,

$$x_{j,n} - x_{j+1,n} \sim \frac{a_n}{n} \Psi_n \left(x_{j,n}\right) \tag{3}$$

(c) For $n \geq 1$,

$$\sup_{x \in \mathbb{R}} |p_n W| (x) \left|1 - \frac{|x|}{a_n}\right|^{\frac{1}{4}} \sim a_n^{-\frac{1}{2}} \tag{4}$$

and

$$\sup_{x \in \mathbb{R}} |p_n W| (x) \sim a_n^{-\frac{1}{2}} \left(nT\left(a_n\right)\right)^{\frac{1}{6}} \tag{5}$$

(d) Let $0 < p \leq \infty$. There exists $C > 0$ such that for $n \geq 1$ and $P \in \mathcal{P}_\backslash$,

$$\|PW\|_{L_p[\mathbb{R}]} \leq C \|PW\|_{L_\infty[-a_n\ a_n]} \tag{6}$$

Moreover, given $r > 1$, there exists $C_1 > 0$such that,

$$\|PW\|_{L_p[\mathbb{R}]} \leq \exp\left[-C_1 n\right] \|PW\|_{L_\infty[-a_n\ a_n]} \tag{7}$$

(e) For $n \geq 1$,

$$\frac{\gamma_{n-1}}{\gamma_n} \sim a_n \tag{8}$$

(f) Uniformly for $n \geq 2$ and $1 \leq j \leq n-1$,

$$1 - \frac{|x_{j,n}|}{a_n} + L\delta_n \sim 1 - \frac{|x_{j+1,n}|}{a_n} + L\delta_n \tag{9}$$

and

$$\Psi_n\left(x_{j,n}\right) \sim \Psi_n\left(x_{j+1,n}\right) \tag{10}$$

Here, $L$is chosen so large enough that (36) is true.

(g) Uniformly for $n \geq 2$ and $2 \leq j \leq n-1$,

$$\frac{a_n^{\frac{3}{2}}}{n} \Psi_n\left(x_{j,n}\right) \left(1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right)^{\frac{1}{2}} |p_n' W|\left(x_{j,n}\right) \tag{11}$$

$$\sim \quad a_n^{\frac{1}{2}} |p_{n-1} W|\left(x_{j,n}\right) \sim \left(1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right)^{\frac{1}{4}}$$

Proof.

(a) This is part of Theorem 1.2 in [7].

(b) (40) is part of Corrolary 1.3 in [7] and (41) is Corrolary 1.3 in [10] (A weaker form of (41) appears in Corollary 1.3 in [7]).

(c) This is Corolary 1.4 (a) in [7].

(d) (44) is a special case of Theorem 1.5 in [7] and (45) is a special case of Lemma 2.5 in [7]. This is (10.33) in [7]. (47) is (9.9) in [7] and (48) follows immediately from (9.9). This is Corollary 1.4 (b) in [7]. □

Next, we recall some results from [9], [10], involving mostly the fundamental polynomials of Lagrange interpolation:

Lemma 2.2.

(a) Let $0 < p < \infty$. Then for $n \geq 2$,

$$\|p_n W\|_{L_P(\mathbb{R})} \sim a_n^{\frac{1}{p}-\frac{1}{2}} \times \begin{bmatrix} \\ 1 & , p < 4 \\ (\log n)^{\frac{1}{4}} & , p = 4 \\ nT(a_n)^{\frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right)} & , p > 4 \end{bmatrix} \tag{12}$$

(b) Uniformly for $n \geq 1,\ 1 \leq j \leq n, x \in \mathbb{R}$,

$$|l_{j,n}(x)| \sim \frac{a_n^{\frac{3}{2}}}{n} (\Psi_n W)(x_{j,n}) \left(\left(1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right)^{\frac{1}{4}} \left|\frac{P_n(x)}{x - x_{j,n}}\right| . \right) \tag{13}$$

(c) Uniformly for $n \geq 1,\ 1 \leq j \leq n,\ x \in \not{\mathbb{R}}$,

$$|l_{j,n}(x)| W(x) W(x_{j,n}) \leq C \tag{14}$$

(d) For $n \geq 2,\ 1 \leq j \leq n-1,\ x \in [x_{j,n}\ x_{j+1,n}]$,

$$l_{j,n}(x) W(x) W^{-1}(x_{j,n}) + l_{j+1,n}(x) W(x) W^{-1}(x_{j+1,n}) \geq 1 \tag{15}$$

Proof.

(a) This is Theorem 1.1 in [10].

(b),(c) These are Theorem 1.2 in [10].

(d) is a special case of the main result in [9]. □

Next, some technical estimates on the growth of $a_u, Q(a_u), T(a_u)$ect:

Lemma 2.3.

(a) Given $r > 0$, there exists x_0 such that for $x \geq x_0$ and $j = 0, 1, 2$, $\frac{Q^{(j)}(x)}{x^r}$ is increasing in (x_o, ∞).

(b) Uniformly for $u \geq C$ and $j = 0, 1, 2$,

$$a_u^j Q^{(j)}(a_u) \sim uT(a_u)^{j-\frac{1}{2}} \tag{16}$$

(c) Let $0 < \alpha < \beta$. Then uniformly for $u \geq C,\ j = 0, 1, 2$,

$$T(a_{\alpha u}) \sim T(a_{\beta u})\,; Q^{(j)}(a_{\alpha u}) \sim Q^{(j)}(a_{\beta u}) \tag{17}$$

(d) Given fixed $r > 1$,

$$\frac{a_{ru}}{a_u} \geq 1 + \frac{\log r}{T(a_{ru})},\ u \in (0, \infty). \tag{18}$$

Moreover,

$$a_{ru} \sim a_u \quad u \in (1, \infty). \tag{19}$$

(e) Uniformly for $t \in (C, \infty)$,

$$\frac{a'_t}{a_t} \sim \frac{1}{tT(a_t)} \tag{20}$$

(f) Uniformly for $u \in (C, \infty$, and $v \in \left[\frac{u}{2}, 2u\right]$, we have,

$$\left|\frac{a_u}{a_v} - 1\right| \sim \left|\frac{u}{v} - 1\right| \frac{1}{T(a_u)} \tag{21}$$

Proof.

(a) This is Lemma 2.1 (3) in [7].

(b)–(f) are part of Lemma 2.2 in [7]. □

Our final lemma in this section concerns estimates that specifically follow from (1.8):

Recall that δ_n was defined by (35).

Lemma 2.4.

(a) Let $\varepsilon > 0$. Then,

$$a_n \leq Cn^{\varepsilon}, T\left(a_n\right) \leq Cn^{\varepsilon},\ n \geq 1. \tag{22}$$

(b) Given $A > 0$, we have,

$$\delta_n \leq CT\left(a_n\right)^{-A},\ n \geq 1. \tag{23}$$

(c) Let $0 < \eta < 1$. Uniformly for $n \geq 1,\ 0 < |x| \leq a_{\eta n},\ |x| = a_s$, we have,

$$C_1 \leq T(x)\left(1 - \frac{|x|}{a_n}\right) \leq C_2 \log \frac{n}{s} \tag{24}$$

Proof.

(a) From (54) for $j = 0$, we have,

$$Q\left(a_n\right) \sim nT\left(a_n\right)^{-\frac{1}{2}} \leq nT\left(a_1\right)^{-\frac{1}{2}}$$

Since Q grows faster than any power of x (Lemma 2.3 (a)), we deduce,

$$a_n \leq n^{\varepsilon},$$

for n large enough. Also (8) then shows that,

$$T\left(a_n\right) = O\left(Q\left(a_n\right)^{\varepsilon}\right) \leq Cn^{\varepsilon}.$$

(b) This follows as,

$$\delta_n \leq n^{-\frac{2}{3}} T\left(a_1\right)^{-\frac{2}{3}},$$

that is δ_n decays faster than a power of n, while $T\left(a_n\right)$ grows slower than any power of n.

(c) Firstly if $\frac{|x|}{a_n} \leq \frac{1}{2}$, then,

$$T(x)\left(1 - \frac{|x|}{a_n}\right) \geq T\left(0+\right)\frac{1}{2} > \frac{1}{2}.$$

If $\frac{|x|}{a_n} \geq \frac{1}{2}$, write $|x| = a_s$, so that $s \leq \eta n$,

$$T(x)\left(1-\frac{|x|}{a_n}\right) \geq T\left(a_s\right)\left(1-\frac{a_s}{a_{\frac{s}{\eta}}}\right) \geq C_1,$$

by Lemma 2.3 (d). So we have the lower bound in (2.24). We proceed to the upper bound. We can assume that $x = a_s,\ s \geq 1$, and $n \geq n_0$. Then using the inequality

$$1-u \leq |\log u|\,,\ u \in (0,1).$$

we obtain,

$$\begin{aligned}\left(1-\frac{|x|}{a_n}\right) &\leq \left|\log \frac{a_s}{a_n}\right| = \int_s^n \frac{a_t'}{a_t} dt \\ &\leq C\int_s^n \frac{dt}{tT\left(a_t\right)} \leq \frac{C}{T\left(a_s\right)} \log \frac{n}{s} = \frac{C}{T(x)} \log \frac{n}{s}.\Box\end{aligned}$$

3 Quadrature Sum Estimates

We present two quadrature sum estimates, the first of which is really part of a Lebesgue function type estimate. The second involves quadrature sums for polynomials.

Lemma 3.1. Let $\beta \in \left(0, \frac{1}{4}\right)$and,

$$\Sigma_n(x) := \sum_{\left|x_{k,n}\right| \geq a_{\beta n}} \left|l_{k,n}(x)\right| W^{-1}\left(x_{k,n}\right). \tag{1}$$

We have for $|x| \leq a_{\frac{\beta n}{2}}$ and $|x| \geq a_{2n}$,

$$\left(\Sigma_n W\right)(x) \leq C \tag{2}$$

Moreover, for $a_{\frac{\beta n}{2}} \leq |x| \leq a_{2n}$,

$$\left(\Sigma_n W\right)(x) \leq C\left(\log n \ + a_n^{\frac{1}{2}}\left|P_n W\right|(x) T\left(a_n\right)^{\frac{-1}{4}}\right) \tag{3}$$

Proof.

Let $\Sigma_{n,1}(x)$ denote the sum $\Sigma_n(x)$ omitting those terms $x_{k,n}$ for which $x \in [x_{k+2,n}\ , x_{k-2,n}]$, (if there are any such k). Here and the sequel, we set for $l \geq 1$,

$$x_{1-l,n} := x_{1,n} + l\delta_n ; x_{n+l,n} := x_{n,n} - l\delta_n \tag{4}$$

Of course the sum $\Sigma_n - \Sigma_{n,1}$ consists of at most 8 terms. Each of these 8 terms admits the bound in Lemma 2.2 (c). So,

$$|(\Sigma_n - \Sigma_{n,1}) W(x)| \leq C_1. \tag{5}$$

Next, by Lemma 2.2 (b) and Lemma 2.1 (b),

$$(\Sigma_{n,1}W)(x) \sim a_n^{\frac{1}{2}} |P_n W|(x) \sum_{|x_{k,n}| \geq a_{\beta n}}^{(1)} \frac{(x_{k,n} - x_{k+1,n})}{|x - x_{k,n}|} \left(1 - \frac{|x_{k,n}|}{a_n} + L\delta_n\right)^{\frac{1}{4}} \tag{6}$$

Here the (1) indicates that the sum omits those k for which $x \in [x_{k+2,n}\ x_{k-2,n}]$.

Now (cf. (47)),

$$1 - \frac{|t|}{a_n} + L\delta_n \sim 1 - \frac{|x_{k,n}|}{a_n} + L\delta_n,\ t \in [x_{k+1}, x_{k,n}]\,, \tag{7}$$

uniformly in k and n. Next, if $x \in [x_{k+2,n}\ x_{k-2,n}]$, and $t \in [x_{k+1,n}\ x_{k,n}]$,

$$\left|\frac{x-t}{x - x_{k,n}} - 1\right| = \left|\frac{t - x_{k,n}}{x - x_{k,n}}\right| \leq \frac{x_{k,n} - x_{k+1,n}}{|x_{k\pm2,n} - x_{k,n}|} \leq C.$$

Similarly we bound $\frac{(x - x_{k,n})}{x-t}$. So,

$$|x - t| \sim |x - x_{k,n}|\,,\ t \in [x_{k+1,n}\, x_{k,n}]\,,\ x \notin [x_{k+2,n}\ x_{k-2,n}]\,. \tag{8}$$

In view of the spacing of the zeros (Lemma 2.1 (b)), we deduce that,

$$\begin{aligned} (\Sigma_{n,1}W)(x) &\sim a_n^{\frac{1}{2}} |P_n W|(x) \int_{\substack{a_{\beta n} \leq |t| \leq a_n \\ |t-x| \geq C\frac{a_n}{n}\Psi_n(x)}} \frac{\left(1 - \frac{|t|}{a_n} + L\delta_n\right)^{\frac{1}{4}}}{|t-x|} dt \qquad (9) \\ &= a_n^{\frac{1}{2}} |P_n W|(x) \int_{\substack{\frac{a_{\beta n}}{a_n} \leq |s| \leq 1 \\ \left|s - \frac{x}{a_n}\right| \geq C\frac{1}{n}\Psi_n(x)}} \frac{(1 - |s| + L\delta_n)^{\frac{1}{4}}}{\left|s - \frac{|x|}{a_n}\right|} ds \end{aligned}$$

Note that since δ_n is much smaller than $\frac{1}{T(a_n)}$,

$$1+s+L\delta_n \leq C_2\left(1-\frac{a_{\beta n}}{a_n}\right) \leq C_3\frac{1}{T\left(a_n\right)}.$$

(See Lemma 2.3 (f)). Then we obtain the bound,

$$\left(\Sigma_{n,1}W\right)(x) \leq C a_n^{\frac{1}{2}}\left|P_nW\right|(x)T\left(a_n\right)^{-\frac{1}{4}}\int_{\substack{\frac{a_{\beta n}}{a_n}\leq|s|\leq 1 \\ \left|s-\frac{x}{a_n}\right|\geq C\frac{1}{n}\Psi_n(x)}}\frac{ds}{\left|s-\frac{x}{a_n}\right|}$$

Now if $0 \leq x \leq a_{\frac{\beta n}{2}}$ or $x \geq a_{2n}$, then for $n \geq n_o$???? we can bound the integral by,

$$\begin{aligned}
&\int_{\frac{a_{\beta n}}{a_n}}\frac{ds}{\left|s-\frac{x}{a_n}\right|} \\
\lesssim\ &\left(1-\frac{a_{\beta n}}{a_n}\right)\max\left(\left|1-\frac{a_{2n}}{a_n}\right|^{-1},\left|\frac{a_{\beta n}}{a_n}-\frac{a_{\frac{\beta n}{2}}}{a_n}\right|^{-1}\right) \leq C_4,
\end{aligned}$$

by Lemma 2.3 (f). In this case the bound (42) gives,

$$\left(\Sigma_nW\right)(x) \leq C_5\left(\left|1+\frac{|x|}{a_n}\right|^{\frac{-1}{4}}T\left(a_n\right)^{-\frac{1}{4}}\right) \leq C_6.$$

so we have (64). Now let us turn to to the more difficult case where $a_{\frac{\beta n}{2}} \leq a_{2n}$. We bound the integral in (71) as follows:

$$\begin{aligned}
&\int_{\substack{\frac{a_{\beta n}}{a_n}\leq|s|\leq 1 \\ \left|s-\frac{x}{a_n}\right|\geq C\frac{1}{n}\Psi_n(x)}}\frac{\left(1-|s|+L\delta_n\right)^{\frac{1}{4}}}{\left|s-\frac{|x|}{a_n}\right|}ds \\
\leq\ & C_7[\int_{\substack{\frac{a_{\beta n}}{a_n}\leq|s|\leq 1 \\ \left|s-\frac{x}{a_n}\right|\geq C\frac{1}{n}\Psi_n(x)}}\frac{(1-s)^{\frac{1}{4}}}{\left|s-\frac{|x|}{a_n}\right|}ds+\int_{\substack{\frac{a_{\beta n}}{a_n}\leq|s|\leq 1 \\ \left|s-\frac{x}{a_n}\right|\geq C\frac{1}{n}\Psi_n(x)}}\frac{\left(\delta_n\right)^{\frac{1}{4}}}{\left|s-\frac{|x|}{a_n}\right|}]ds \\
=\ & :\left[I_1+I_2\right].
\end{aligned}$$

Now since $\frac{1}{n}\Psi_n(x)$ is bounded below by a power of n, we see that,

$$I_2 \leq C_8\delta_n^{\frac{1}{4}}\log n.$$

If $x \geq a_n$, we estimate,

$$I_1 \leq \int_{\frac{a_{\beta n}}{a_n}} \frac{(1-s)^{\frac{1}{4}}}{|s-1|} ds \leq C_9 T\left(a_n\right)^{-\frac{1}{4}}.$$

If $x < a_n$, we make the substitution $1-s = \left(1-\frac{x}{a_n}\right) v$to get,

$$\begin{aligned}
I_1 &= \left(1-\frac{x}{a_n}\right)^{\frac{1}{4}} \int_{\substack{v \in \left[0, \frac{\left(1-\frac{a_{\beta n}}{a_n}\right)}{\left(1-\frac{x}{a_n}\right)}\right] \\ |v-1| \geq C \frac{\Psi_n(x)}{n\left(1-\frac{x}{a_n}\right)}}} \frac{v^{\frac{1}{4}}}{|v-1|} dv \\
&\leq C_{10} \left(1-\frac{x}{a_n}\right)^{\frac{1}{4}} \{ \int_{\substack{v \in [0,2] \\ |v-1| \geq C \frac{\Psi_n(x)}{n\left(1-\frac{x}{a_n}\right)}}} \frac{1}{|v-1|} dv + \\
&\quad \int_2^{\left(1-\frac{a_{\beta n}}{a_n}\right)\left(1-\frac{x}{a_n}\right)} v^{-\frac{3}{4}} dv \} \\
&\leq C_{11} \left(1-\frac{x}{a_n}\right)^{\frac{1}{4}} \log n + T\left(a_n\right)^{-\frac{1}{4}} \}
\end{aligned}$$

Combining our estimates for I_1, I_2 and using the bound,

$$a_n^{\frac{1}{2}} \left|p_n W(x)\right| \delta_n^{\frac{1}{4}} \leq C,$$

which follows from (43), we deduce (65) from (79). $\square$

In our second quadrature sum estimate, we need the kernel function for the Chebyshev weight,

$$v(t) := \left(1-t^2\right)^{-\frac{1}{2}}, \ t \in (-1,1) \tag{10}$$

If $p_j(v,x) = \sqrt{\frac{2}{\pi}} T_j(x)$is the jth orthonormal polynomial for v (at least for $j \geq 1$), then,

$$K_n(v,x,t) := \sum_{j=0}^{n-1} p_j(v,x) p_j(v,t) \tag{11}$$

admits the following results [19,p36], [16,p108]:

$$K_n(v,xx) \sim n, \ |x| \leq 1. \tag{12}$$

Also,

$$|K_n(v,x,t)| \le C \min\left\{n, \frac{\sqrt{1-x^2}+\sqrt{1-t^2}}{|x-t|}\right\},\ x,t \in [-1,1]. \tag{13}$$

Lemma 3.2. Let $0 < \eta < 1$. Let $\phi : \mathbb{R} \to (0, \infty)$ be a continuous function with the following property: For $n \ge 1$, there exist polynomials R_n of degree $\le n$ such that,

$$C_1 \le \frac{\phi(t)}{R_n(t)} \le C_2,\ |t| \le a_{4n}. \tag{14}$$

Then for $n \ge n_0$ and $P \in \mathcal{P}_n$,

$$\sum_{|x_{j,n}| \le a_{\eta n}} \lambda_{j,n} PW^{-1}(x_{j,n})\, \phi(x_{j,n}) \le C \int_{-a_{4n}}^{a_{4n}} |PW|\phi. \tag{15}$$

Proof.

Essentially the proof is the same as in [13], and the ideas appeared much earlier [16], [17] but we include the details.

Step 1: Christoffel function type estimate

We first note that for $P_1 \in \mathcal{P}_{4n}$,

$$\begin{aligned}(P_1W)^2(x) &\le \lambda_{4n}^{-1}\left(W^2, x\right) \int_{\mathbb{R}} (P_1W)^2(t)dt \\ &\le C_1 \frac{n}{a_n} (\Psi_{4n}(x))^{-1} \int_{-a_{4n}}^{a_{4n}} (P_1W)^2(t)dt,\end{aligned}$$

by Lemma 2.1 (a), (d). We deduce that,

$$\left\|P_1 W \Psi_{4n}^{\frac{1}{2}}\right\|^2_{L_\infty[-a_{4n}\ a_{4n}]} \le C_1 \frac{n}{a_n} \int_{-a_{4n}}^{a_{4n}} \left|P_1 \Psi_{4n}^{-\frac{1}{2}} W(t)dt\right| \left\|P_1 W \Psi_{4n}^{\frac{1}{2}}\right\|_{L_\infty[-a_{4n}\ a_{4n}]}$$

and hence that for $|x| \le a_{4n}$,

$$\left|P_1 \Psi_{4n}^{\frac{1}{2}} W(x)\right| \le C_1 \frac{n}{a_n} \int_{-a_{4n}}^{a_{4n}} \left|P_1 \Psi_{4n}^{-\frac{1}{2}} W(t)dt\right|$$

Now we apply this for fixed $|x| \leq a_{4n}$ to

$$P_1(t) := P_2(t)K_n^2\left(v, \frac{x}{a_{4n}}, \frac{t}{a_{4n}}\right)$$

where $P_2 \in \mathcal{P}_{2n}$. We obtain, using (74) that,

$$\left|P_2\Psi_{4n}^{\frac{1}{2}}W(x)\right| \leq C_2\frac{1}{na_n}\int_{-a_{4n}}^{a_{4n}}\left|P_2\Psi_{4n}^{-\frac{1}{2}}W(t)\right|K_n^2\left(v, \frac{x}{a_{4n}}, \frac{t}{a_{4n}}\right)dt$$

In particular, applying this to $P_2 := PR_n$, where $P \in \mathcal{P}_n$, and using (76), we obtain,

$$\left|P\Psi_{4n}^{\frac{1}{2}}W\phi(x)\right| \leq C_3\frac{1}{na_n}\int_{-a_{4n}}^{a_{4n}}\left|P\phi\Psi_{4n}^{-\frac{1}{2}}W(t)\right|K_n^2\left(v, \frac{x}{a_{4n}}, \frac{t}{a_{4n}}\right)dt \qquad (16)$$

Step 2: The general quadrature sum bounded in terms of a special quadrature sum

We take (78) for $x = x_{j,n}$, multiply by $\lambda_{j,n}W^{-2}(x_{j,n})\Psi_{4n}^{-\frac{1}{2}}(x_{j,n})$, and sum over all $|x_{j,n}| \leq a_{\eta n}$. Using our estimate for Christoffel function $\lambda_n(W^2, .)$ in Lemma 2.1 (a), we obtain,

$$\begin{aligned} &\sum_{|x_{j,n}|\leq a_{\eta n}}\lambda_{j,n}\left|PW^{-1}\right|(x_{j,n})\,\Phi(x_{j,n}) && (17)\\ \leq\ & C_4\int_{-a_{4n}}^{a_{4n}}|PW\Phi|(t)\Sigma_n(t)dt, \end{aligned}$$

where,

$$\Sigma_n(t) := n^{-2}\sum_{|x_{j,n}|\leq a_{\eta n}}\Psi_n(x_{j,n})\,\Psi_{4n}^{-\frac{1}{2}}(x_{j,n})\,K_n^2\left(v, \frac{x_{j,n}}{a_{4n}}, \frac{t}{a_{4n}}\right)\Psi_{4n}^{-\frac{1}{2}}(t) \qquad (18)$$

Then the result will follow if we can show

$$\Sigma_n(t) \leq C_5,\ |t| \leq a_{4n} \qquad (19)$$

Step 3: Estimation of (80)

First note that for $|x| \leq a_{\eta n}$,

$$\Psi_n(x) \sim \Psi_{4n}(x) \sim \left(1 - \frac{|x|}{a_n}\right)^{\frac{1}{2}}$$

Now since $\frac{1}{n}\Psi_n(x)$ is bounded below by a power of n, we see that,

$$I_2 \leq C_8 \delta_n^{\frac{1}{4}} \log n.$$

If $x \geq a_n$, we estimate,

$$I_1 \leq \int_{\frac{a_{\beta n}}{a_n}} \frac{(1-s)^{\frac{1}{4}}}{|s-1|} ds \leq C_9 T\left(a_n\right)^{-\frac{1}{4}}.$$

If $x < a_n$, we make the substitution $1 - s = \left(1 - \frac{x}{a_n}\right) v$to get,

$$\begin{aligned}
I_1 &= \left(1 - \frac{x}{a_n}\right)^{\frac{1}{4}} \int_{\substack{v \in \left[0, \frac{\left(1 - \frac{a_{\beta n}}{a_n}\right)}{\left(1 - \frac{x}{a_n}\right)}\right] \\ |v-1| \geq C \frac{\Psi_n(x)}{n\left(1 - \frac{x}{a_n}\right)}}} \frac{v^{\frac{1}{4}}}{|v-1|} dv \\
&\leq C_{10} \left(1 - \frac{x}{a_n}\right)^{\frac{1}{4}} \{ \int_{\substack{v \in [0,2] \\ |v-1| \geq C \frac{\Psi_n(x)}{n\left(1 - \frac{x}{a_n}\right)}}} \frac{1}{|v-1|} dv + \\
&\quad \int_2^{\left(1 - \frac{a_{\beta n}}{a_n}\right)\left(1 - \frac{x}{a_n}\right)} v^{-\frac{3}{4}} dv \} \\
&\leq C_{11} \left(1 - \frac{x}{a_n}\right)^{\frac{1}{4}} \log n + T\left(a_n\right)^{-\frac{1}{4}} \}
\end{aligned}$$

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$$a_n^{\frac{1}{2}} \left| p_n W(x) \right| \delta_n^{\frac{1}{4}} \leq C,$$

which follows from (43), we deduce (65) from (79). □

In our second quadrature sum estimate, we need the kernel function for the Chebyshev weight,

$$v(t) := \left(1 - t^2\right)^{-\frac{1}{2}}, \ t \in (-1, 1) \tag{20}$$

If $p_j(v,x)=\sqrt{\frac{2}{\pi}}T_j(x)$ is the jth orthonormal polynomial for v (at least for $j\geq 1$), then,

$$K_n(v,x,t):=\sum_{j=0}^{n-1}p_j(v,x)p_j(v,t) \tag{21}$$

admits the following results [19,p36], [16,p108]:

$$K_n(v,xx)\sim n,\ |x|\leq 1. \tag{22}$$

Also,

$$|K_n(v,x,t)|\leq C\min\left\{n,\frac{\sqrt{1-x^2}+\sqrt{1-t^2}}{|x-t|}\right\},\ x,t\in[-1,1]. \tag{23}$$

Lemma 3.2. Let $0<\eta<1$. Let $\phi:\mathbb{R}\to(0,\infty)$ be a continuous function with the following property: For $n\geq 1$, there exist polynomials R_n of degree $\leq n$ such that,

$$C_1\leq\frac{\phi(t)}{R_n(t)}\leq C_2,\ |t|\leq a_{4n}. \tag{24}$$

Then for $n\geq n_0$ and $P\in\mathcal{P}_n$,

$$\sum_{|x_{j,n}|\leq a_{\eta n}}\lambda_{j,n}PW^{-1}\left(x_{j,n}\right)\phi\left(x_{j,n}\right)\leq C\int_{-a_{4n}}^{a_{4n}}|PW|\phi. \tag{25}$$

Proof.

Essentially the proof is the same as in [13], and the ideas appeared much earlier [16], [17] but we include the details.

Step 1: Christoffel function type estimate

We first note that for $P_1\in\mathcal{P}_{4n}$,

$$\begin{aligned}\left(P_1W\right)^2(x) &\leq \lambda_{4n}^{-1}\left(W^2,x\right)\int_{\mathbb{R}}\left(P_1W\right)^2(t)dt\\ &\leq C_1\frac{n}{a_n}\left(\Psi_{4n}(x)\right)^{-1}\int_{-a_{4n}}^{a_{4n}}\left(P_1W\right)^2(t)dt,\end{aligned}$$

by Lemma 2.1 (a), (d). We deduce that,

$$\left\| P_1 W \Psi_{4n}^{\frac{1}{2}} \right\|_{L_\infty[-a_{4n}\ a_{4n}]}^2 \leq C_1 \frac{n}{a_n} \int_{-a_{4n}}^{a_{4n}} \left| P_1 \Psi_{4n}^{-\frac{1}{2}} W(t) dt \right| \left\| P_1 W \Psi_{4n}^{\frac{1}{2}} \right\|_{L_\infty[-a_{4n}\ a_{4n}]}$$

and hence that for $|x| \leq a_{4n}$,

$$\left| P_1 \Psi_{4n}^{\frac{1}{2}} W(x) \right| \leq C_1 \frac{n}{a_n} \int_{-a_{4n}}^{a_{4n}} \left| P_1 \Psi_{4n}^{-\frac{1}{2}} W(t) dt \right|$$

Now we apply this for fixed $|x| \leq a_{4n}$ to

$$P_1(t) := P_2(t) K_n^2 \left(v, \frac{x}{a_{4n}}, \frac{t}{a_{4n}} \right)$$

where $P_2 \in \mathcal{P}_{2n}$. We obtain, using (74) that,

$$\left| P_2 \Psi_{4n}^{\frac{1}{2}} W(x) \right| \leq C_2 \frac{1}{n a_n} \int_{-a_{4n}}^{a_{4n}} \left| P_2 \Psi_{4n}^{-\frac{1}{2}} W(t) \right| K_n^2 \left(v, \frac{x}{a_{4n}}, \frac{t}{a_{4n}} \right) dt$$

In particular, applying this to $P_2 := P R_n$, where $P \in \mathcal{P}_n$, and using (76), we obtain,

$$\left| P \Psi_{4n}^{\frac{1}{2}} W \phi(x) \right| \leq C_3 \frac{1}{n a_n} \int_{-a_{4n}}^{a_{4n}} \left| P \phi \Psi_{4n}^{-\frac{1}{2}} W(t) \right| K_n^2 \left(v, \frac{x}{a_{4n}}, \frac{t}{a_{4n}} \right) dt \qquad (26)$$

Step 2: The general quadrature sum bounded in terms of a special quadrature sum

We take (78) for $x = x_{j,n}$, multiply by $\lambda_{j,n} W^{-2}(x_{j,n}) \Psi_{4n}^{-\frac{1}{2}}(x_{j,n})$, and sum over all $|x_{j,n}| \leq a_{\eta n}$. Using our estimate for Christoffel function $\lambda_n(W^2, .)$ in Lemma 2.1 (a), we obtain,

$$\begin{aligned} & \sum_{|x_{j,n}| \leq a_{\eta n}} \lambda_{j,n} \left| P W^{-1} \right| (x_{j,n}) \Phi(x_{j,n}) & (27) \\ \leq \quad & C_4 \int_{-a_{4n}}^{a_{4n}} |P W \Phi|(t) \Sigma_n(t) dt, \end{aligned}$$

where,

$$\Sigma_n(t) := n^{-2} \sum_{|x_{j,n}| \leq a_{\eta n}} \Psi_n(x_{j,n}) \Psi_{4n}^{-\frac{1}{2}}(x_{j,n}) K_n^2 \left(v, \frac{x_{j,n}}{a_{4n}}, \frac{t}{a_{4n}} \right) \Psi_{4n}^{-\frac{1}{2}}(t) \qquad (28)$$

Then the result will follow if we can show

$$\Sigma_n(t) \le C_5,\ |t| \le a_{4n} \tag{29}$$

Step 3: Estimation of (80)

First note that for $|x| \le a_{\eta n}$,

$$\Psi_n(x) \sim \Psi_{4n}(x) \sim \left(1 - \frac{|x|}{a_n}\right)^{\frac{1}{2}}$$

Now since $\frac{1}{n}\Psi_n(x)$ is bounded below by a power of n, we see that,

$$I_2 \le C_8 \delta_n^{\frac{1}{4}} \log n.$$

If $x \ge a_n$, we estimate,

$$I_1 \le \int_{\frac{a_{\beta n}}{a_n}} \frac{(1-s)^{\frac{1}{4}}}{|s-1|} ds \le C_9 T\left(a_n\right)^{-\frac{1}{4}}.$$

If $x < a_n$, we make the substitution $1 - s = \left(1 - \frac{x}{a_n}\right) v$to get,

$$\begin{aligned} I_1 &= \left(1 - \frac{x}{a_n}\right)^{\frac{1}{4}} \int_{\substack{v \in \left[0, \frac{\left(1 - \frac{a_{\beta n}}{a_n}\right)}{\left(1 - \frac{x}{a_n}\right)}\right] \\ |v-1| \ge C \frac{\Psi_n(x)}{n\left(1 - \frac{x}{a_n}\right)}}} \frac{v^{\frac{1}{4}}}{|v-1|} dv \\ &\le C_{10} \left(1 - \frac{x}{a_n}\right)^{\frac{1}{4}} \{ \int_{\substack{v \in [0,2] \\ |v-1| \ge C \frac{\Psi_n(x)}{n\left(1 - \frac{x}{a_n}\right)}}} \frac{1}{|v-1|} dv + \\ &\quad \int_2^{\left(1 - \frac{a_{\beta n}}{a_n}\right)\left(1 - \frac{x}{a_n}\right)} v^{-\frac{3}{4} dv} \} \\ &\le C_{11} \left(1 - \frac{x}{a_n}\right)^{\frac{1}{4}} \log n + T\left(a_n\right)^{-\frac{1}{4}} \} \end{aligned}$$

Combining our estimates for I_1, I_2 and using the bound,

$$a_n^{\frac{1}{2}} \left|p_n W(x)\right| \delta_n^{\frac{1}{4}} \le C,$$

which follows from (43), we deduce (65) from (79). $\square$

In our second quadrature sum estimate, we need the kernel function for the Chebyshev weight,

$$v(t) := \left(1-t^2\right)^{-\frac{1}{2}}, \ t \in (-1,1) \tag{30}$$

If $p_j(v,x) = \sqrt{\frac{2}{\pi}}T_j(x)$ is the jth orthonormal polynomial for v (at least for $j \geq 1$), then,

$$K_n(v,x,t) := \sum_{j=0}^{n-1} p_j(v,x)p_j(v,t) \tag{31}$$

admits the following results [19,p36], [16,p108]:

$$K_n(v,xx) \sim n, \ |x| \leq 1. \tag{32}$$

Also,

$$|K_n(v,x,t)| \leq C \min\left\{n, \frac{\sqrt{1-x^2}+\sqrt{1-t^2}}{|x-t|}\right\}, \ x,t \in [-1,1]. \tag{33}$$

Lemma 3.2. Let $0 < \eta < 1$. Let $\phi : \mathbb{R} \to (0,\infty)$ be a continuous function with the following property: For $n \geq 1$, there exist polynomials R_n of degree $\leq n$ such that,

$$C_1 \leq \frac{\phi(t)}{R_n(t)} \leq C_2, \ |t| \leq a_{4n}. \tag{34}$$

Then for $n \geq n_0$ and $P \in \mathcal{P}_n$,

$$\sum_{|x_{j,n}| \leq a_{\eta n}} \lambda_{j,n} PW^{-1}\left(x_{j,n}\right)\phi\left(x_{j,n}\right) \leq C \int_{-a_{4n}}^{a_{4n}} |PW|\phi. \tag{35}$$

Proof.

Essentially the proof is the same as in [13], and the ideas appeared much earlier [16], [17] but we include the details.

Step 1: Christoffel function type estimate

We first note that for $P_1 \in \mathcal{P}_{4n}$,

$$\begin{aligned}(P_1W)^2(x) &\leq \lambda_{4n}^{-1}\left(W^2, x\right)\int_{\mathbb{R}}(P_1W)^2(t)dt \\ &\leq C_1\frac{n}{a_n}\left(\Psi_{4n}(x)\right)^{-1}\int_{-a_{4n}}^{a_{4n}}(P_1W)^2(t)dt,\end{aligned}$$

by Lemma 2.1 (a), (d). We deduce that,

$$\left\|P_1W\Psi_{4n}^{\frac{1}{2}}\right\|_{L_\infty[-a_{4n}\ a_{4n}]}^2 \leq C_1\frac{n}{a_n}\int_{-a_{4n}}^{a_{4n}}\left|P_1\Psi_{4n}^{-\frac{1}{2}}W(t)dt\right|\left\|P_1W\Psi_{4n}^{\frac{1}{2}}\right\|_{L_\infty[-a_{4n}\ a_{4n}]}$$

and hence that for $|x| \leq a_{4n}$,

$$\left|P_1\Psi_{4n}^{\frac{1}{2}}W(x)\right| \leq C_1\frac{n}{a_n}\int_{-a_{4n}}^{a_{4n}}\left|P_1\Psi_{4n}^{-\frac{1}{2}}W(t)dt\right|$$

Now we apply this for fixed $|x| \leq a_{4n}$ to

$$P_1(t) := P_2(t)K_n^2\left(v, \frac{x}{a_{4n}}, \frac{t}{a_{4n}}\right)$$

where $P_2 \in \mathcal{P}_{2n}$. We obtain, using (74) that,

$$\left|P_2\Psi_{4n}^{\frac{1}{2}}W(x)\right| \leq C_2\frac{1}{na_n}\int_{-a_{4n}}^{a_{4n}}\left|P_2\Psi_{4n}^{-\frac{1}{2}}W(t)\right|K_n^2\left(v, \frac{x}{a_{4n}}, \frac{t}{a_{4n}}\right)dt$$

In particular, applying this to $P_2 := PR_n$, where $P \in \mathcal{P}_n$, and using (76), we obtain,

$$\left|P\Psi_{4n}^{\frac{1}{2}}W\phi(x)\right| \leq C_3\frac{1}{na_n}\int_{-a_{4n}}^{a_{4n}}\left|P\phi\Psi_{4n}^{-\frac{1}{2}}W(t)\right|K_n^2\left(v, \frac{x}{a_{4n}}, \frac{t}{a_{4n}}\right)dt \qquad (36)$$

Step 2: The general quadrature sum bounded in terms of a special quadrature sum

We take (78) for $x = x_{j,n}$, multiply by $\lambda_{j,n}W^{-2}(x_{j,n})\Psi_{4n}^{-\frac{1}{2}}(x_{j,n})$, and sum over all $|x_{j,n}| \leq a_{\eta n}$. Using our estimate for Christoffel function $\lambda_n(W^2, .)$ in Lemma 2.1 (a), we obtain,

$$\begin{aligned}&\sum_{|x_{j,n}|\leq a_{\eta n}}\lambda_{j,n}\left|PW^{-1}\right|(x_{j,n})\,\Phi(x_{j,n}) \qquad (37)\\ \leq\ & C_4\int_{-a_{4n}}^{a_{4n}}|PW\Phi|(t)\Sigma_n(t)dt,\end{aligned}$$

where,

$$\Sigma_n(t) := n^{-2} \sum_{|x_{j,n}| \le a_{\eta n}} \Psi_n\left(x_{j,n}\right) \Psi_{4n}^{-\frac{1}{2}}\left(x_{j,n}\right) K_n^2\left(v, \frac{x_{j,n}}{a_{4n}}, \frac{t}{a_{4n}}\right) \Psi_{4n}^{-\frac{1}{2}}(t) \tag{38}$$

Then the result will follow if we can show

$$\Sigma_n(t) \le C_5,\ |t| \le a_{4n} \tag{39}$$

Step 3: Estimation of (80)

First note that for $|x| \le a_{\eta n}$,

$$\Psi_n(x) \sim \Psi_{4n}(x) \sim \left(1 - \frac{|x|}{a_n}\right)^{\frac{1}{2}}$$

This follows easily from the fact that $1 - \frac{|x|}{a_n} \ge C_6 T\left(a_n\right)$ for this range. Moreover,

$$\Psi_{4n}(t) \ge \left(1 - \frac{|t|}{a_{4n}} + L\delta_n\right)^{\frac{1}{2}}$$

for $|t| \le a_{4n}$. Let us set,

$$y_{j,n} := \frac{x_{j,n}}{a_{4n}};\ T := \frac{t}{a_{4n}}.$$

Then we have, using also (75) and the spacing in Lemma 2.1 (b), that,

$$\begin{aligned}
\Sigma_n(t)\left(1 - \frac{|t|}{a_n} + L\delta_n\right)^{\frac{1}{4}} &\le \frac{C_7}{n a_n} \sum_{|x_{j,n}| \le a_{\eta n}} \left(x_{j,n} - x_{j+1,n}\right)\left(1 - \frac{|x_{j,n}|}{a_{4n}}\right)^{-\frac{1}{4}} K_n^2\left(v, \frac{x_{j,n}}{a_{4n}}, \frac{t}{a_{4n}}\right) \quad (40)\\
&\le C_8 n^{-1} \sum_{|y_{j,n}| \le a_{\eta n}} \left(y_{j,n} - y_{j+1,n}\right)\left(1 - |y_{j,n}|\right)^{-\frac{1}{4}}\\
&\quad \times \min\left\{n, \frac{\sqrt{1 - {y_{j,n}}^2} + \sqrt{1 - T^2}}{|y_{j,n} - T|}\right\}^2\\
&\le C_9 n^{-1} \int_{-1}^{1} \left(1 - |y|\right)^{-\frac{1}{4}} \min\left\{n, \frac{\sqrt{1 - y^2} + \sqrt{1 - T^2}}{|y - T|}\right\}^2 dy
\end{aligned}$$

In bounding the sum in terms of the integral, we have used (47). Let us assume that $1 - n^{-2} \geq T \geq 0$. Then we can continue the above as,

$$\begin{aligned}
\Sigma_n(t)(1-T)^{\frac{1}{4}} &\leq C_{10} n^{-1} \left\{ \begin{array}{c} n^2 \int_{y \in [0,1]: |y-T| \leq \frac{1}{n}(1-T)^{\frac{1}{2}}} (1-y)^{-\frac{1}{4}} dy \\ + \int_{y \in [0,1]: |y-T| \geq \frac{1}{n}(1-T)^{\frac{1}{2}}} (1-y)^{-\frac{1}{4}} \frac{1-y+1-T}{|y-T|^2} dy \end{array} \right\} \\
&= C_{10} n^{-1} \left\{ \begin{array}{c} n^2 (1-T)^{\frac{3}{4}} \int_{w: |1-w| \leq \frac{1}{n}(1-T)^{-\frac{1}{2}}} w^{-\frac{1}{4}} dw \\ +(1-T)^{-\frac{1}{4}} \int_{w: |1-w| \geq \frac{1}{n}(1-T)^{-\frac{1}{2}}} w^{-\frac{1}{4}} \frac{|1+w|}{|1-w|^2} dw \end{array} \right\} \\
&\quad \text{(substitution } 1-y = (1-T)w\text{)} \\
&\leq C_{11} (1-T)^{\frac{1}{4}}
\end{aligned}$$

Here we have used the fact that,

$$\frac{1}{n}(1-T)^{-\frac{1}{2}} \leq 1.$$

So in this case, we have (81). In the remaining case where $1 - n^{-2} \leq T < 1$, we continue (82) as,

$$\begin{aligned}
\Sigma_n(t) \left(L\delta_n\right)^{\frac{1}{4}} &\leq C_{12} n^{-1} \left\{ \begin{array}{c} n^2 \int_{y \in [0,1]: |y-T| \leq 4n^{-2}} (1-y)^{-\frac{1}{4}} dy \\ + \int_0^{1-2n^{-2}} (1-y)^{-\frac{1}{4}} \frac{1-y+n^{-2}}{|y-T|^2} dy \end{array} \right\} \\
&\leq C_{13} n^{-\frac{1}{2}}
\end{aligned}$$

Since $\delta_n^{\frac{1}{4}}$ decays scarcely faster than $n^{-\frac{1}{6}}$ we again have (81). $\square$

4 Proof of the sufficiency conditions

In proving the sufficiency conditions, we split our functions into pieces that vanish inside or outside $\left[-a_{\frac{n}{9}}, a_{\frac{n}{9}}\right]$. Throughout, we let χ_S denote the characteristic function of a set S. Also, we set for some fixed $\kappa > 0$,

$$\phi(x) := \left(\log\left(2 + x^2\right)\right)^{-1-\kappa}. \tag{1}$$

Throughout, we assume that $W = \exp[-Q] \in \mathcal{E}_1$, that $1 < p < \infty$, and,

$$\Delta > \max\left\{0, \frac{2}{3}\left(\frac{1}{4} - \frac{1}{p}\right)\right\}. \tag{2}$$

Lemma 4.1.

Let $\{f\}_{n=1}^{\infty}$ be a sequence of measurable functions from $\mathbb{R} \to \mathbb{R}$ such that for $n \geq 1$,

$$f_n(x) = 0,\ |x| < a_{\frac{n}{9}}; \tag{3}$$

$$|f_n W|(x) \leq \Phi(x),\ x \in \mathbb{R} \tag{4}$$

Then,

$$\lim_{n\to\infty} \left\| L_n [f_n] W (1+Q)^{-\Delta} \right\|_{L_P(\mathbb{R})} = 0 \tag{5}$$

Proof.

Firstly for $|x| \leq a_{\frac{n}{18}}$ or $|x| \geq a_{2n}$, Lemma 3.1 (with $\beta = \frac{1}{9}$) and (86), (87) show that,

$$\begin{aligned} |L_n [f_n] W|(x) &\leq \Phi\left(a_{\frac{n}{9}}\right) \sum_{\left|x_{k,n}\right| \geq a_{\frac{n}{9}}} |l_{k,n}(x)| W^{-1}(x_{k,n}) W(x) \\ &\leq C_1 \phi\left(a_{\frac{n}{9}}\right). \end{aligned}$$

So,

$$\begin{aligned} &\left\| L_n [f_n] W (1+Q)^{-\Delta} \right\|_{L_P\left(\left(|x| \leq a_{\frac{n}{18}}\right) \cup (|x| \geq a_{2n})\right)} \\ \leq\ & C_1 \phi\left(a_{\frac{n}{9}}\right) \left\| (1+Q)^{-\Delta} \right\|_{L_P[\mathbb{R}]} \leq C_2 \phi\left(a_{\frac{n}{9}}\right) \end{aligned}$$

Here we have used the fact that Q grows faster than any power of x (Lemma 2.3 (a)). Next, for $a_{\frac{n}{18}} \leq |x| \leq a_{2n}$, Lemma 3.1 gives,

$$|L_n [f_n] W|(x) \leq C_3 \phi\left(a_{\frac{n}{9}}\right) \left\{ \log n + a_n^{\frac{1}{2}} |P_n W|(x)\, T(a_n)^{-\frac{1}{4}} \right\}$$

Also for this range of x,

$$Q(x) \sim Q(a_n) \sim nT(a_n)^{-\frac{1}{2}}.$$

So,

$$\begin{aligned}
&\left\|L_n\left[f_n\right]W(1+Q)^{-\Delta}\right\|_{L_P\left[a_{\frac{n}{18}}\leq|x|\leq a_{2n}\right]} \\
\leq\ & C_4\phi\left(a_{\frac{n}{9}}\right)\left(nT\left(a_n\right)^{-\frac{1}{2}}\right)^{-\Delta}\left\{\log n\left(a_{2n}-a_{\frac{n}{18}}\right)^{\frac{1}{p}}+a_n^{\frac{1}{2}}T\left(a_n\right)^{-\frac{1}{4}}\left\|P_nW\right\|_{L_P[\mathbb{R}]}\right\} \\
\leq\ & C_5\phi\left(a_{\frac{n}{9}}\right)\left(nT\left(a_n\right)^{-\frac{1}{2}}\right)^{-\Delta}(\log n)\left(\frac{a_n}{T\left(a_n\right)}\right)^{\frac{1}{p}} \\
& +C_5\phi\left(a_{\frac{n}{9}}\right)\left(nT\left(a_n\right)^{-\frac{1}{2}}\right)^{-\Delta}T\left(a_n\right)^{-\frac{1}{4}}a_n^{\frac{1}{p}}\left\{\begin{array}{ll}1 & ,p<4\\ (\log n)^{\frac{1}{4}} & ,p=4\\ nT\left(a_n\right)^{\frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right)} & ,p>4\end{array}\right\}
\end{aligned}$$

by Lemma 2.2 (a) and Lemma 2.3 (f). Since $T\left(a_n\right)$ and a_n grow slower than any positive power of n (Lemma 2.4 (a)), we see that the right hand side is $o\left(\phi\left(a_{\frac{n}{9}}\right)\right)=o(1)$, because of (84). $\square$

Next, we deal with functions that vanish outside $\left[-a_{\frac{n}{9}},a_{\frac{n}{9}}\right]$. We separately estimate the weighted L_p norms of their Lagrange interpolants over $\left[-a_{\frac{n}{8}},a_{\frac{n}{8}}\right]$ and $\mathbb{R}\backslash\left[-a_{\frac{n}{8}},a_{\frac{n}{8}}\right]$.

Lemma 4.2.

Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of measurable functions from $\mathbb{R}\to\mathbb{R}$ such that for $n\geq 1$,

$$g_n(x)=0,\ |x|\geq a_{\frac{n}{9}};\tag{6}$$

$$\left|g_nW\right|(x)\leq\phi(x),\ x\in\mathbb{R}\tag{7}$$

Then,

$$\lim_{n\to\infty}\left\|L_n\left[g_n\right]W(1+Q)^{-\Delta}\right\|_{L_P\left[|x|\geq a_{\frac{n}{8}}\right]}=0\tag{8}$$

Proof.

For $x\geq a_{\frac{n}{8}}$,

$$\left|L_n\left[g_n\right](x)\right|\ \leq\ \sum_{\left|x_{k,n}\right|\leq a_{\frac{n}{9}}}l_{k,n(x)}W^{-1}\left(x_{k,n}\right)\phi\left(x_{k,n}\right)$$

$$
\begin{aligned}
&\leq C_1 a_n^{\frac{1}{2}} |p_n(x)| \sum_{|x_{k,n}| \leq a_{\frac{n}{9}}} (x_{k,n} - x_{k+1,n}) \frac{\left(1 - \frac{|x_{k,n}|}{a_n} + L\delta_n\right)^{\frac{1}{4}}}{|x - x_{k,n}|} \phi(x_{k,n}) \\
&\quad \text{(by Lemma 2.2 (b) and (2.3))} \\
&\leq C_2 a_n^{\frac{1}{2}} |p_n(x)| \int_{-a_{\frac{n}{9}}}^{a_{\frac{n}{9}}} \frac{\left(1 - \frac{|t|}{a_n} + L\delta_n\right)^{\frac{1}{4}}}{|x - t|} \phi(t) dt.
\end{aligned}
$$

Here we have used the monotonicity of ϕ and (69). Next, for $t \in \left[0, a_{\frac{n}{9}}\right]$, $x \geq a_{\frac{n}{8}}$,

$$
0 \leq \frac{a_n - t}{x - t} = 1 + \frac{\frac{a_n}{x} - 1}{1 - \frac{t}{x}} \leq 1 + \frac{\frac{a_n}{a_{\frac{n}{8}}} - 1}{1 - \frac{a_{\frac{n}{9}}}{a_{\frac{n}{8}}}} \leq C_3,
$$

by Lemma 2.3 (f). Moreover,

$$
1 - \frac{|t|}{a_n} \geq C_4 \frac{1}{T(a_n)} >> \delta_n
$$

So,

$$
\begin{aligned}
|L_n[g_n](x)| &\leq C_5 a_n^{\frac{1}{4}} |p_n(x)| \int_0^{a_{\frac{n}{9}}} \frac{(a_n - t)^{\frac{1}{4}}}{x - t} \phi(t) dt \\
&\leq C_6 a_n^{\frac{1}{4}} |p_n(x)| \int_0^{a_{\frac{n}{9}}} (x - t)^{-\frac{3}{4}} \phi(t) dt
\end{aligned}
$$

Here if $t = a_s$, $\frac{n}{9} \geq s \geq 1$, we have for $x \geq a_{\frac{n}{8}}$,

$$
x - t = x\left(1 - \frac{t}{x}\right) \geq a_{\frac{n}{8}} \left(1 - \frac{a_s}{a_{\frac{9s}{8}}}\right) \geq C_7 \frac{a_n}{T(a_s)}.
$$

So,

$$
|L_n[g_n](x)| \leq C_5 a_n^{-\frac{1}{2}} |p_n(x)| \int_0^{a_{\frac{n}{9}}} T(t)^{\frac{3}{4}} \phi(t) dt
$$

Thus,

$$
\begin{aligned}
&\left\| L_n[g_n] W(1+Q)^{-\Delta} \right\|_{L_P\left[|x| \geq a_{\frac{n}{8}}\right]} \\
&\leq C_9 a_n^{-\frac{1}{2}} \left[\int_0^{a_{\frac{n}{9}}} T(t)^{\frac{3}{4}} \phi(t) dt\right] Q\left(a_{\frac{n}{8}}\right)^{-\Delta} \|p_n W\|_{L_P[\mathbb{R}]}.
\end{aligned}
$$

It is easy to see that the integral involving ϕ in the last right hand side grows slower than any power of n. Then using (84) and the estimate on $\|p_n W\|_{L_P[\mathbb{R}]}$ provided by Lemma 2.2 (a), we obtain (90). $\square$

We now turn to the most difficult part of the sufficiency proof, namely the estimation of $\left\|L_n\left[g_n\right]W(1+Q)^{-\Delta}\right\|_{L_P\left[|x|\leq a_{\frac{n}{8}}\right]}$. We present the most technical part of this as a seperate lemma. Recall the notation (31–34) for partial sums $S_n[\,.\,]$ of orthonormal expansions with respect to W^2.

Lemma 4.3.

Let $\sigma:\mathbb{R}\rightarrow\mathbb{R}$ be a bounded measurable function. Then,

$$\left\|S_n\left[\sigma\phi W^{-1}\right]W(1+Q)^{-\Delta}\right\|_{L_P\left[|x|\leq a_{\frac{n}{8}}\right]}\leq C\|\sigma\|_{L_\infty(\mathbb{R})}, \tag{9}$$

for $n\geq 1$. Here C is independent of σ and n.

Proof.

We split this into several steps. Part of the difficulty lies in that we cannot simply estimate Hilbert Transforms in L_p with the weight $(1+Q)^{-\Delta}$, as it does not satisfy Muckenhoupts $\mathrm{A}_p$condition [20]. We may assume that $\|\sigma\|_{L_\infty[\mathbb{R}]}=1$.

Step 1: Split $S_n[.](x)$ into several terms depending on the location of x

First note that by (34) and by our estimates for $\frac{\gamma_{n-1}}{\gamma_n}$ and p_n (see Lemma 2.1 (c), (e)),

$$S_n\left[\sigma\phi W^{-1}\right]W(x)\leq C_1 a_n^{\frac{1}{2}}\left(1-\frac{|x|}{a_n}\right)^{-\frac{1}{4}}\sum_{j=n-1}^{n}H\left[\sigma\phi p_j W\right](x) \tag{10}$$

Now let us choose $l:=l(n)$ such that,

$$2^l\leq\frac{n}{8}\leq 2^{l+1} \tag{11}$$

Note that us choose $l=l(n)$ guarantees that,

$$2^{l+3}\leq n. \tag{12}$$

Define,

$$\Im_k := [a_{2^k},\ a_{2^{k+1}}],\ k \geq 1 \tag{13}$$

The reason for this choice of intervals is that,

$$Q(x) \sim Q(a_{2^k}) \sim 2^k T(a_{2^k}),\ x \in \Im_k \tag{14}$$

uniformly in k. For $j = n-1, n$ and $x \in \Im_k$, we split,

$$\begin{aligned} H[\sigma\phi p_j W](x) &= [\int_{-\infty}^{0} + \int_{0}^{a_{2k-1}} + P.V.\int_{a_{2k-1}}^{a_{2k+2}} + \int_{a_{2k+2}}^{\infty}]\frac{\sigma\phi p_j W(t)}{x-t}dt \qquad (15) \\ &= : I_1(x) + I_2(x) + I_3(x) + I_4(x). \end{aligned}$$

Here $P.V$ stands for principle value.

Step 2: Estimation of I_1 and I_2 for $x \in \Im_k$

We see that (recall $x \geq a_2$),

$$\begin{aligned} |I_1(x)| &\leq \int_0^\infty \frac{|p_j W\phi|(-t)}{t+x}dt \\ &\leq C_2 a_n^{-\frac{1}{2}} \int_0^{\frac{a_n}{2}} \frac{\phi(t)}{t+a_2}dt + C_2 a_n^{-1}\int_{\frac{a_n}{2}}^\infty |p_j W|(t)dt \\ &\leq C_3 a_n^{-\frac{1}{2}}. \end{aligned}$$

Here we have used the bound (42), the bound for $\|p_n W\|_{L_1[\mathbb{R}]}$ in Lemma 2.2 (a), and also the form of ϕ (recall (83)), which guarantees that,

$$\int_0^\infty \frac{\phi(t)}{1+t}dt < \infty. \tag{16}$$

Next the bound (42) gives,

$$\begin{aligned} |I_2(x)| &\leq \int_0^{a_{2^{k-1}}} \frac{|p_j W\phi|(t)}{x-t}dt \\ &\leq C_4 a_n^{-\frac{1}{2}}\left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}} \int_0^{a_{2^{k-1}}} \frac{dt}{x-t} \\ &= C_4 a_n^{-\frac{1}{2}}\left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}} \log\left(1-\frac{a_{2^{k-1}}}{x}\right)^{-1} \end{aligned}$$

Now

$$1-\frac{a_{2^{k-1}}}{x} \geq 1-\frac{a_{2^{k-1}}}{a_{2^k}} \geq C_5\frac{1}{T\left(a_{2^k}\right)} \geq C_6\frac{1}{T(x)}$$

Thus,

$$|I_2(x)| \leq C_7 a_n^{-\frac{1}{2}}\left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}}\log\left(C_8 T(x)\right).$$

Step 3: Estimation of I_4 for $x \in \Im_k$

Now using our bound (42) again gives,

$$\begin{aligned}
|I_4(x)| &\leq \int_{a_{2^{k+2}}}^{\infty}\frac{|p_j W\phi|(t)}{t-x}dt \\
&\leq C_9[a_n^{-\frac{1}{2}}\int_{a_{2^{k+2}}}^{2a_{2^{k+2}}}\left|1-\frac{t}{a_n}\right|^{-\frac{1}{4}}\frac{dt}{t-x}+a_n^{-\frac{1}{2}}\int_{2a_{2^{k+2}}}^{\max\left\{2a_{2^{k+2}},\frac{a_n}{2}\right\}}\frac{\phi(t)}{t}dt \\
&\quad+\int_{\frac{a_n}{2}}^{\infty}\frac{|p_j W|(t)}{t}dt] \\
&\leq C_{10}a_n^{-\frac{1}{2}}[1+J]
\end{aligned}$$

where,

$$J := \int_{a_{2^{k+2}}}^{2a_{2^{k+2}}}\left|1-\frac{t}{a_n}\right|^{-\frac{1}{4}}\frac{dt}{t-x}$$

(We have used (98) and the bound on the L_1 norm of $p_n W$.) Here if $\left|1-\frac{t}{a_n}\right| \leq \frac{1}{2}\left(1-\frac{x}{a_n}\right)$, then,

$$|t-x| = a_n\left|\left(1-\frac{x}{a_n}\right)-\left(1-\frac{t}{a_n}\right)\right| \geq \frac{1}{2}a_n\left(1-\frac{x}{a_n}\right).$$

Thus,

$$\begin{aligned}
J &\leq C_{11}[\left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}}\int_{\substack{\left|1-\frac{t}{a_n}\right|\geq\frac{1}{2}\left(1-\frac{x}{a_n}\right)\\ t\in[a_{2^{k+2}},\ 2a_{2^{k+2}}]}}\frac{dt}{t-x} \\
&\quad+a_n^{-1}\left(1-\frac{x}{a_n}\right)^{-1}\int_{\substack{\left|1-\frac{t}{a_n}\right|\geq\frac{1}{2}\left(1-\frac{x}{a_n}\right)\\ t\in[a_{2^{k+2}},\ 2a_{2^{k+2}}]}}\left|1-\frac{t}{a_n}\right|^{-\frac{1}{4}}dt
\end{aligned}$$

$$
\begin{aligned}
&\leq C_{12}[\left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}}\log\left(1+\frac{a_{2^{k+2}}}{a_{2^{k+2}}-x}\right) \\
&\quad+\left(1-\frac{x}{a_n}\right)^{-1}\int_{|1-s|\leq\frac{1}{2}\left(1-\frac{x}{a_n}\right)}|1-s|^{-\frac{1}{4}}ds] \\
&\leq C_{13}\left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}}\log\left(C_{14}T(x)\right).
\end{aligned}
$$

Step 4: Estimation of $\|S_n[.]\|_{L_P[\Im_k]}$

Combining our estimates for I_j $j=1,2,4$ gives,

$$
|I_1+I_2+I_4|(x)\leq C_{14}a_n^{-\frac{1}{2}}\left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}}\log\left(C_{15}T(x)\right).
$$

Together with (92, 96 and 97), this gives,

$$
\begin{aligned}
&\left\|S_n\left[\sigma\phi W^{-1}\right]W(1+Q)^{-\Delta}\right\|_{L_P[\Im_k]} \\
\leq\ & Q\left(a_{2^k}\right)^{-\Delta}\left(1-\frac{a_{2^{k+1}}}{a_n}\right)^{-\frac{1}{4}} \\
&\times\left\{\begin{array}{c}\left(1-\frac{a_{2^{k+1}}}{a_n}\right)^{-\frac{1}{4}}\log\left(C_{15}T\left(a_{2^{k+1}}\right)\right)\left(a_{2^{k+1}}-a_{2^k}\right)^{\frac{1}{p}} \\ +a_n^{\frac{1}{2}}\sum_{j=n-1}^{n}\left\|P.V.\int_{a_{2^{k-1}}}^{a_{2^{k+2}}}\frac{\sigma\phi p_jW(t)}{x-t}dt\right\|_{L_P[\Im_k]}\end{array}\right\}
\end{aligned}
$$

We use M. Riez' theorem on the boundedness of the Hilbert transform from $L_p(\mathbb{R})$, to $L_p(\mathbb{R})$ [20] to deduce that,

$$
\begin{aligned}
&\left\|P.V.\int_{a_{2^{k-1}}}^{a_{2^{k+2}}}\frac{\sigma\phi p_jW(t)}{x-t}dt\right\|_{L_P[\Im_k]} \\
\leq\ & C_{17}\int_{a_{2^{k-1}}}^{a_{2^{k+2}}}|\sigma\phi p_jW|^p(t)dt \\
\leq\ & C_{17}a_n^{-\frac{p}{2}}\left(1-\frac{a_{2^{k+2}}}{a_n}\right)^{-\frac{p}{4}}\left(a_{2^{k+2}}-a_{2^{k-1}}\right).
\end{aligned}
$$

Next, note that, in view of (94), $n\geq 2^{k+3}$ for $k\leq l$, so,

$$
\left(1-\frac{a_{2^{k+1}}}{a_n}\right)\geq\left(1-\frac{a_{2^{k+2}}}{a_n}\right)\geq\left(1-\frac{a_{2^{k+2}}}{a_{2^{k+3}}}\right)\geq C_{18}\frac{1}{T\left(a_{2^k}\right)}
$$

Similarly by using the triangle inequality (if necessary) and (59) we can bound $(a_{2^{k+2}} - a_{2^{k-1}})$ and $(a_{2^{k+1}} - a_{2^k})^{\frac{1}{p}}$ to yield,

$$\begin{aligned} & \left\| S_n \left[\sigma\phi W^{-1} \right] W(1+Q)^{-\Delta} \right\|_{L_P[\Im_k]} \\ \leq \ & C_{20} Q\left(a_{2^k}\right)^{-\Delta} T\left(a_{2^k}\right)^{\frac{1}{2}} \log\left(C_{15} T\left(a_{2^{k+1}}\right)\right) \left(\frac{a_{2^k}}{T\left(a_{2^k}\right)} \right)^{\frac{1}{p}} \end{aligned} \tag{17}$$

Step 5: Completion of the proof

The estimation of $S_n[.](x)$ for $x \in -\Im_k = [-a_{2^{k+1}} \ - a_{2^k}]$ is exactly the same as for $x \in \Im_k$. Since we have (96), and since $a_{2^k}, T\left(a_{2^k}\right)$ grow much slower than $Q\left(a_{2^k}\right)$ (Lemma 2.4 (a)), we obtain,

$$\begin{aligned} & \left\| S_n \left[\sigma\phi W^{-1} \right] W(1+Q)^{-\Delta} \right\|_{L_P\left[a_2 \leq |x| \leq a_{\frac{n}{8}}\right]} \\ \leq \ & \sum_{k=1}^{l} \left\| S_n \left[\sigma\phi W^{-1} \right] W(1+Q)^{-\Delta} \right\|_{L_P[\Im_k]} \\ \leq \ & C_{21} \sum_{k=1}^{l} 2^{\frac{-kp\Delta}{2}} \leq C_{22}. \end{aligned}$$

The estimation of $\left\| S_n \left[\sigma\phi W^{-1} \right] W(1+Q)^{-\Delta} \right\|_{L_P[|x| \leq a_2]}$ is similar but easier. We split,

$$H\left[\sigma\phi p_j W\right](x) = \left[\int_{-\infty}^{-2a_2} + P.V. \int_{-2a_2}^{2a_2} + \int_{2a_2}^{\infty}\right] \frac{\sigma\phi p_j W(t)}{x-t} dt.$$

The first and third integrals may be estimated as we did before, and the second is estimated as we did I_3. □

Armed with this lemma, we can complete the estimation of $L_n[g_n]$ over $[-a_{\beta n}, a_{\beta n}]$:

Lemma 4.4.

Let $\varepsilon \in (0,1)$. Let $\{g_n\}$ be as in Lemma 4.2, except that rather than (89), we assume that,

$$|g_n W|(x) \leq \varepsilon\phi(x), \ x \in \mathbb{R} \ n \geq 1. \tag{18}$$

Then,

$$\limsup_{n\to\infty} \left\| L_n[g_n] W(1+Q)^{-\Delta} \right\|_{L_P\left[|x|\leq a_{\frac{n}{8}}\right]} \leq C\varepsilon \tag{19}$$

where $C$is independent of $n, \{g_n\}$ and ε.

Proof.

Let,

$$\chi_n := \chi\left[-a_{\frac{n}{8}},\ a_{\frac{n}{8}}\right]; \sigma_n := sign\ S_n[h_n].$$

We shall show that,

$$\left\| L_n[g_n] W(1+Q)^{-\Delta} \right\|_{L_P\left[|x|\leq a_{\frac{n}{8}}\right]} \leq C\varepsilon \left\| S_n\left[\sigma\phi W^{-1}\right] W(1+Q)^{-\Delta} \right\|_{L_P\left[|x|\leq a_{\frac{n}{8}}\right]}$$

Then Lemma 4.3 gives the result. Let,

$$h_n := sign\left(L_n[g_n]\right) |L_n[g_n]|^{p-1} \chi_n W^{p-2}(1+Q)^{-\Delta p}.$$

Then using the orthogonality of $f - S_n[f]$ to $\mathcal{P}_{n-1}$, and the Guass quadrature formula, we see that,

$$\begin{aligned}
\left\| L_n[g_n] W(1+Q)^{-\Delta} \right\|_{L_P\left[|x|\leq a_{\frac{n}{8}}\right]} &= \int_{\mathbb{R}} L_n[g_n] h_n W^2 \\
&= \int_{\mathbb{R}} L_n[g_n] S_n[h_n] W^2 = \sum_{j=1}^{n} \lambda_{j,n} g_n(x_{j,n}) S_n[h_n](x_{j,n}) \\
&= \sum_{|x_{k,n}|<a_{\frac{n}{8}}} \lambda_{j,n} g_n(x_{j,n}) S_n[h_n](x_{j,n}) \\
&\leq \varepsilon \sum_{|x_{k,n}|<a_{\frac{n}{8}}} \lambda_{j,n} \phi(x_{j,n}) W^{-1}(x_{j,n}) |S_n[h_n](x_{j,n})| \\
&\leq C\varepsilon \int_{\mathbb{R}} \phi W |S_n[h_n]|
\end{aligned}$$

by Lemma 3.2. Note that it is easy to verify the approximation property in Lemma 3.2 for ϕ (in fact Jacksons Theorem gives polynomials of degree $o(n)$ satisfying (76)). We can continue this as,

$$\begin{aligned}
&= C\varepsilon \int_{\mathbb{R}} \phi\sigma_n W^{-1} W^2 |S_n[h_n]| \\
&= C\varepsilon \int_{\mathbb{R}} h_n S_n\left[\phi\sigma_n W^{-1}\right] W^2 \\
&= C\varepsilon \int_{-a_{\frac{n}{8}}}^{a_{\frac{n}{8}}} h_n S_n\left[\phi\sigma_n W^{-1}\right] W^2
\end{aligned}$$

for $h_n$has its support inside $\left[-a_{\frac{n}{8}}, a_{\frac{n}{8}}\right]$. Using Holders Inequality with $q = \frac{p}{p-1}$, we continue this as,

$$\begin{aligned}
&\leq \quad C\varepsilon \left(\left(\int_{-a_{\frac{n}{8}}}^{a_{\frac{n}{8}}} \left| h_n W(1+Q)^{\Delta} \right|^q \right)^{\frac{1}{q}} \int_{-a_{\frac{n}{8}}}^{a_{\frac{n}{8}}} \left| S_n \left[\phi \sigma_n W^{-1} \right] W(1+Q)^{-\Delta} \right|^p \right)^{\frac{1}{p}} \\
&= \quad C\varepsilon \left\| L_n \left[g_n \right] W(1+Q)^{-\Delta} \right\|_{L_P\left[|x| \leq a_{\frac{n}{8}}\right]}^{p-1} \left\| S_n \left[\phi \sigma_n W^{-1} \right] W(1+Q)^{-\Delta} \right\|_{L_P\left[|x| \leq a_{\frac{n}{8}}\right]}
\end{aligned}$$

Cancelling the $p-1$th power of $\|L_n ...\|$ gives (102). $\square$

We can now turn to the,

Proof of the Sufficiency Part of Theorem 1.3.

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and satisfy (16). Let $\varepsilon > 0$. We can choose a polynomial P such that,

$$\left\| (f - P) W \phi^{-1} \right\|_{L_\infty[\mathbb{R}]} \leq \varepsilon.$$

(Compare (5)). Then for n large enough,

$$\begin{aligned}
&\left\| f - L_n[f] W(1+Q)^{-\Delta} \right\|_{L_P[\mathbb{R}]} \qquad (20) \\
\leq \quad &\left\| (f - P) W(1+Q)^{-\Delta} \right\|_{L_P[\mathbb{R}]} + \left\| (P - L_n[f]) W(1+Q)^{-\Delta} \right\|_{L_P[\mathbb{R}]} \\
\leq \quad &\varepsilon \left\| \phi(1+Q)^{-\Delta} \right\|_{L_P[\mathrm{R}} + \left\| (L_n[P - f]) W(1+Q)^{-\Delta} \right\|_{L_P[\mathbb{R}]}
\end{aligned}$$

The first norm in (103) is finite as $\Delta > 0$, and as Qfaster than any power of x.

Next, let

$$\chi_n := \chi \left[-a_{\frac{n}{9}}, a_{\frac{n}{9}} \right]$$

and write,

$$P - f = (P - f)\chi_n + (p - f)(1 - \chi_n) =: g_n + f_n$$

By Lemma 4.1,

$$\lim_{n \to \infty} \left\| L_n \left[f_n \right] W(1+Q)^{-\Delta} \right\|_{L_P[\mathbb{R}]} = 0$$

Also Lemmas 4.2 and 4.4, together give,

$$\limsup_{n \to \infty} \left\| L_n \left[g_n \right] W(1+Q)^{-\Delta} \right\|_{L_P[\mathbb{R}]} \leq C\varepsilon,$$

with C independent of ε. Substituting the estimates for $L_n\left[f_n\right]$, and $L_n\left[g_n\right]$ into (103) and then letting $\varepsilon \to 0$, gives (15). $\square$

5 Proof of the Neccessary Conditions

We begin with a lemma,

Lemma 5.1.
Let $0 < p < \infty, \beta \in (0,1]$. Let $0 < A < B < \infty$, and $\phi : \mathbb{R} \to (0,\infty)$ be a continous function such that for $1 \leq s,\, t \leq \infty$ with $\frac{1}{2} \leq \frac{s}{t} \leq 2$,

$$A \leq \frac{\phi\left(a_s\right)}{\phi\left(a_t\right)} \leq B \tag{21}$$

Then for $n \geq 1$,

$$\left\|p_n W \phi\right\|_{L_P\left[-a_{\beta n}\ ,a_{\beta n}\right]} \geq C a_n^{-\frac{1}{2}} \left\| \phi(t) \left(1 - \frac{|t|}{a_n} + L\delta_n\right)^{-\frac{1}{4}} \right\|_{L_P\left[-a_{\beta n}\ ,a_{\beta n}\right]} . \tag{22}$$

Here C depends on A, B, but is independent of n, and ϕ.

Proof.
By Lemma 2.2 (d), for $x \in [x_{j+1,n}\ x_{j,n}]$,

$$\max\left\{ l_{j,n}(x) W^{-1}\left(x_{j,n}\right) W(x),\ l_{j+1,n}(x) W^{-1}\left(x_{j+1,n}\right) W(x) \right\} \geq \frac{1}{2}$$

and hence,

$$\begin{aligned} \left|p_n W\right|(x) &\geq \frac{1}{2} \min\left\{\left|x - x_{j,n}\right| p_n' W\left(x_{j,n}\right), \left|x - x_{j+1,n}\right| p_n' W\left(x_{j+1,n}\right)\right\} \\ &\geq C_1 \frac{n}{a_n^{\frac{3}{2}}} \Psi_n^{-1}\left(x_{j,n}\right) \left(1 - \frac{\left|x_{j,n}\right|}{a_n} + L\delta_n\right)^{-\frac{1}{4}} \min\left\{\left|x - x_{j,n}\right|, \left|x - x_{j+1,n}\right|\right\}, \end{aligned}$$

by Lemma 2.1 (f), (g). Let,

$$\Im_{j,n} := \left[x_{j+1,n} + \frac{1}{4}\left(x_{j,n} - x_{j+1,n}\right), x_{j,n} + \frac{1}{4}\left(x_{j,n} - x_{j+1,n}\right)\right].$$

We see from the spacing of the zeros in Lemma 2.1 (b), that for $x \in \Im_{j,n}$,

$$\left|p_n W\right|(x) \geq C_2 a_n^{-\frac{1}{2}} \left(1 - \frac{\left|x_{j,n}\right|}{a_n} + L\delta_n\right)^{-\frac{p}{4}} \int_{\Im_{j,n}} \phi^p.$$

The result follows if we can show that,

$$\int_{\Im_{j,n}} \phi^p \geq C_4 \int_{x_{j+1,n}}^{x_{j,n}} \phi^p$$

To do this, it suffices to show that,

$$\phi(t) \sim \phi\left(x_{j,n}\right),\ t \in \left[x_{j+1,n}\ x_{j,n}\right].$$

In view of (104), it suffices to show that if $x_{j+1,n} = a_s$ and $x_{j,n} = a_t$, with s, $t \geq s_0$ (some fixed large s_0), then,

$$1 \leq \frac{s}{t} \leq 2 \tag{23}$$

Now in view of the spacing Lemma 2.1 (b),

$$\begin{aligned}
0 &\leq 1 - \frac{a_s}{a_t} = \frac{\left(x_{j,n} - x_{j+1,n}\right)}{x_{j+1,n}} \\
&\leq C_5 \frac{a_n}{n} \frac{\Psi_n\left(x_{j,n}\right)}{a_1} \leq C_6 \frac{a_n}{n} \frac{\Psi\left(a_n\right)}{a_1} \\
&= C_6 \frac{a_n}{n} \max\left(\left(L\delta_n\right)^{\frac{1}{2}} T\left(a_n\right)^{-1}, \left(L\delta_n\right)^{\frac{1}{2}}\right) = C_6 \frac{a_n}{a_1 n} n^{\frac{1}{3}} T\left(a_n\right)^{-\frac{2}{3}} \\
&= o\left(\frac{1}{T\left(a_n\right)}\right)
\end{aligned}$$

using the definition (35) and the fact that $T\left(a_n\right)$ grows slower than any power of n. Thus it follows that for $n \geq n_0$ and s, $t \geq s_0$, Lemma 2.3 (d) it guarantees (106). $\square$

Proof of the necessity part of Theorem 1.3.

Fix $\Delta \in \mathbb{R}$, $\kappa > 0$, $\delta > 1 + \kappa$ and assume the conclusion of Theorem 1.3 is true. Let X be the space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with,

$$\|f\|_X := \sup_{x \in \mathbb{R}} |fW|(x) \left(\log\left(2 + |x|\right)\right)^{\delta} < \infty.$$

Moreover, let Y be the space of all measurable functions $f : \mathbb{R} \to \mathbb{R}$ with,

$$\|f\|_Y := \left\| fW(1+Q)^{-\Delta} \right\|_{L_p(\mathbb{R})} < \infty.$$

Each $f \in X$ satisfies (15), so the conclusion of Thm 1.3 ensures that,

$$\lim_{n\to\infty} \|f - L_n[f]\|_Y = 0$$

Since X is a Banach space, the uniform boundedness principle gives,

$$\|f - L_n[f]\|_Y \leq C\|f\|_X \tag{24}$$

with C independent of n and f. In particular as $L_1[f] = f(0)$ (recall that $p_1(x) = \gamma_1(x)$), we deduce that for $f \in X$ with $f(0) = 0$,

$$\|f\|_Y \leq C\|f\|_X.$$

So for such f,

$$\|L_n[f]\|_Y \leq 2C\|f\|_X \tag{25}$$

Choose g_n continuous in $\mathbb{R}$, with $g_n = 0$ in $[0,\infty) \cup (-\infty, \frac{-1}{2}a_n]$, with,

$$\|g_n\|_X = \sup_{x\in\mathbb{R}} |g_n W|(x) \left(\log(2+|x|)^\delta\right) = 1,$$

and for $x_{j,n} \in \left(-\frac{1}{2}a_n, 0\right)$,

$$g_n W(x_{j,n}) \left(\log(2+|x_{j,n}|)^\delta \, sign\left(p_n'(x_{j,n})\right)\right) = 1.$$

For example, $\left(g_n W(x)\left(\log(2+|x|)\right)^\delta\right)$ can be chosen to be piecewise linear. Then for $x \in [1.a_n]$,

$$\begin{aligned}
|L_n[g_n](x)| &= \left| \sum_{x_{j,n}\in[-\frac{1}{2}a_n,0)} g_n(x_{j,n}) \frac{p_n(x)}{p_n'(x_{j,n})(x - x_{j,n})} \right| \\
&= |p_n(x)| \sum_{x_{j,n}\in[-\frac{1}{2}a_n,0)} \frac{\left(\log(2+|x_{j,n}|)\right)^{-\delta}}{|p_n' W|(x_{j,n})(x + |x_{j,n}|)} \\
&\geq C_1 a_n^{\frac{1}{2}} |p_n(x)| (\log a_n)^{-\delta} x^{-1} \sum_{x_{j,n}\in[-\frac{1}{2}a_n,0)} (x_{j,n} - x_{j+1,n}) \\
&\quad \text{(by lemma 2.1 (g) and (b))} \\
&\geq C_2 a_n^{\frac{1}{2}} |p_n(x)| (\log a_n)^{-\delta}
\end{aligned}$$

Then by (108),

$$\begin{aligned}
2C &= 2C\|g_n\|_X \geq \|L_n[g_n]\|_Y \\
&\geq C_3 a_n^{\frac{1}{2}} (\log a_n)^{-\delta} \left\| p_n W (1+Q)^{-\Delta} \right\|_{L_P[1,a_n]} \\
&\geq C_4 a_n^{\frac{1}{p}} (\log a_n)^{-\delta} Q(a_n)^{-\max\{\Delta,0\}} \begin{array}{ll} 1 & ,p<4 \\ (\log n)^{\frac{1}{4}} & ,p=4 \\ (nT(a_n))^{\frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right)} & ,p>4 \end{array}
\end{aligned}$$

Here we used the monotonicity of Q, Lemma 2.2 (a) and Lemma 2.1 (d). Note that $[-1,1]$ does not give a big contribution to the L_p norm of $p_n W$. Since $a_n^{\frac{1}{p}} (\log a_n)^{-\delta}$ grows to ∞, we see that $\Delta > 0$ is necessary for $p \leq 4$. Also, for $p > 4$, we obtain from Lemma 2.3 (b),

$$2C \geq C_5 a_n^{\frac{1}{p}} (\log a_n)^{-\delta} T(a_n)^{\frac{\Delta}{2}+\frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right)} n^{-\Delta+\frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right)}$$

Since the terms invoving a_n and $T(a_n)$ grow to ∞ with n, we see that necessarily,

$$\Delta > \frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right).\square$$

Proof of Theorem 1.4.

This is similar to the previous proof. We let X be the Banach space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ vanishing outside $[-2,2]$, with norm,

$$\|f\|_X := \|f\|_{[-2,2]}.$$

We let Y be the space of all measurable $f : \mathbb{R} \to \mathbb{R}$ with,

$$\|f\|_Y := \|fWU\|_{L_P[\mathbb{R}]} < \infty.$$

Assume that we cannot find f sayisfying (19). Then the uniform boundedness principle gives (107) for all $f \in X$. Again, when $f(0) = 0$, we obtain (108). We now choose $g_n \in X$, with

$$(g_n W)(x_{j,n})\, sign\,(p_n'(x_{j,n})) = 1$$

in $\left[-1,-\frac{1}{2}\right]$, $g_n = 0$in $(-\infty,-2]\cup[0,\infty)$ and

$$g_n W\left(x_{j,n}\right) sign\left(p_n'\left(x_{j,n}\right)\right) \geq 0$$

in $[-2,2t]$. Much as before, we deduce that for $x \geq 1$,

$$\left|L_n\left[g_n\right](x)\right| \geq C a_n^{\frac{1}{2}} \frac{|p_n(x)|}{x}$$

Also by hypothesis, there exists C_1 and C_2 such that,

$$U(x) \geq C_1 x^{\frac{3}{2}-\frac{1}{p}} Q(x)^{-\frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right)}, x \geq C_2.$$

Hence by (108),

$$\begin{aligned} 2C &= 2C\left\|g_n\right\|_X \geq \left\|L_n\left[g_n\right]\right\|_Y \\ &\geq C_1\left\|L_n\left[g_n\right](x)W(x)x^{\frac{3}{2}-\frac{1}{p}}Q(x)^{-\frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right)}\right\|_{L_P[C_2,a_n]} \\ &\geq C_2 a_n^{\frac{1}{2}-\frac{1}{p}} Q\left(a_n\right)^{-\frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right)}\left\|p_n W\right\|_{L_P\left[a_{\frac{n}{2}},\, a_n\right]} \\ &\geq C_3 T\left(a_n\right)^{\frac{1}{4}-\frac{1}{p}}, \end{aligned}$$

much as before, by Lemma 2.2 (a) and (42). Of course this is impossible for large n and we have a contridiction. □

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