Mean Convergence of Extended Lagrange Interpolation for Exponential weights.

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Abstract. In this paper, we complete our investigations of mean convergence of Lagrange interpolation for fast decaying even and smooth exponential weights on the line. In doing so, we also present a summary of recent related work on the line and $[-1, 1]$ by the authors, Szabados, Vertesi, Lubinsky and Matjila. We also emphasize the important and fundamental ideas, applied in our proofs, that were developed by Erdős, Turan, Askey, Freud, Nevai, Szabados Vértesi and their students and collaborators. These methods include forward quadrature estimates, orthogonal expansions, Hilbert transforms, bounds on Lebesgue functions and the uniform boundedness principle.

Keywords: Lagrange interpolation, Mean convergence, Orthonormal polynomial, Weighted approximation.

1. Introduction and Statement of Results

The idea of this paper arose from recent work of the authors in [6], [7], [9],work of one of us and Lubinsky in [11] and [12], work of Lubinsky and Matjila in [22] and [23], work of Lubinsky in [15], [16], [17], work of Lubinsky and Mastroanni in [20] and [21] and Szabados [26]. The investigations involved studying weighted mean convergence of Lagrange interpolation for smooth, even and fast decaying exponential weights on the line and $[-1, 1]$ for two specific choices of interpolation nodes. Related results on uniform convergence, Hilbert transforms, converse quadrature, higher order interpolation and distribution of arbitrary interpolation arrays, can be found in [1], [2], [3], [5], [12], [7], [8], [10], [13], [15], [16], [17], [19], [21] [27], [29], [30] and the many references cited therein. All of our results rely heavily on bounds and estimates for the associated orthogonal polynomials and their zeroes, see [14] and we will refer to this latter excellent reference many times in what follows. We do not consider weighted mean convergence of Lagrange interpolation for non even exponential weights or exponential weights of less smoothness on the line, $[-1, 1]$ or arcs of $[-1, 1]$ but delay these for further investigations. See [4]. We also omit the topics of Lebesgue functions, Birkoff interpolation, and distribution of interpolation arrays for weights on the line, in the plane and $[-1,1]$. We refer the reader to the still largely cited books and survey [28], [24] and [25] for fundamental earlier work of Freud, Nevai, Bonan, Erdős, Turan, Muckenhoupt, Askey and Wagner, Szabados and Vértesi.

First, we require a general class of *strongly admissible weights* similar to those of [11]. The main feature of our weights is that they are of faster than smooth polynomial decay at infinity. Thus they differ from the well known Freud weight class which are of smooth polynomial decay at infinity. For a detailed perspective on this subject, see [18], [14] and the references cited therein.

DEFINITION 1.1. Let $w := e^{-Q}$ where $Q(x) : \mathbb{R} \to [0,\infty)$ is even, continuous, $Q''(x)$ exists in $(0,\infty)$, $Q^{(j)}(x) \geq 1$ in $(0,\infty)$, $j = 0,1,2$, and the function

$$T(x) := 1 + \frac{xQ''(x)}{Q'(x)}$$

is increasing in $(0,\infty)$ with

$$\lim_{x\to\infty} T(x) = \infty; \quad T(0^+) := \lim_{x\to 0^+} T(x) > 1.$$

Moreover, we assume that

$$T(x) \sim \frac{xQ'(x)}{Q(x)}$$

and for every $\varepsilon > 0$

$$T(x) \leq C\left(\log Q'(x)\right)^{1+\epsilon}. \tag{1.1}$$

Then w will be called *strongly admissible.*

Given w such as above, we let $p_n(x) := p_n(w^2, x)$ be the n-th orthonormal polynomial with a positive leading coefficient $\gamma_n > 0$ and let

$$U_n := \{-\infty < x_{n,n} < x_{n-1,n} < \cdots < x_{2,n} < x_{1,n} < \infty\}$$

be the set of zeros of $p_n(w^2, x)$. For each $n \geq 1$ and for the given weight w, we define an interpolatory matrix

$$V_{n+2} = U_n \cup \{y_0\} \cup \{-y_0\}$$

where y_0 maximizes $\|p_n w\|_{L_\infty(\mathbb{R})}$. The Lagrange interpolation polynomial of degree $n+1$ to a continuous $f:\mathbb{R}\to\mathbb{R}$ with respect to the array V_{n+2} is denoted by $L_{n+2}(f, V_{n+2})$. See for example [26].

To set the scene for our investigations, we begin with the main result of [7, Corollary 2.4].

THEOREM 1.2. *([7, Corollary 2.4]) Let w be strongly admissible and assume that* (1.1) *is replaced by the weaker condition:*
For every $\varepsilon > 0$

$$T(x) \leq C(Q(x))^{\epsilon}.$$

Let $0 < p < \infty$, $\Delta, \alpha \in \mathbb{R}$ and $\hat{\alpha} := \min\{\alpha, 1\}$. Then if $\alpha > 0$ and

$$\hat{\alpha} + \Delta > 1/p, \tag{1.2}$$

$$\lim_{n\to\infty} \left\| (L_{n+2}(f, V_{n+2}) - f)\, w(x)(1+|x|)^{-\Delta} \right\|_{L_p(\mathbb{R})} = 0 \tag{1.3}$$

for all continuous $f:\mathbb{R}\to\mathbb{R}$ with

$$\lim_{|x|\to\infty} fw(x)(1+|x|)^{\alpha} = 0. \tag{1.4}$$

Moreover, if (1.3) holds for every continuous function f satisfying (1.4) for $\alpha \in \mathbb{R}$ then necessarily (1.2) holds.

In particular, if we set $\Delta = 0$ in the above, we see that necessarily $\alpha > 1/p > 0$ which means that we cannot hope for Theorem 1.2 to hold for continuous functions f where fw is uniformly bounded. See for example [15] and [16] where this phenomenon occurs for exponential weights on $[-1, 1]$ and [17] and [20] where it fails for Freud type weights. We show that for Erdős weights, we can relax the polynomial decay condition on fw in (1.4) to allow for logarithmic decay. The price we pay is that in general, we obtain a stronger weight in the convergence norm given by (1.3). On its own, this observation is somewhat expected. See Theorem 1.6 below. The important observation is that the weight we need for convergence is much weaker than the one which appears in the main result of [11] under a logarithmic decay condition on f. The reason for this is that we use an extended system of interpolatory nodes which gives far better results. The idea of using such extended systems was first applied on the real line to Freud weights by Szabados in [26] and to Erdős type weights on the line and $[-1, 1]$ by Damelin in [1] and [2]. See Theorem 1.6 and its remark below. Our results below, essentially complete our current investigations for mean convergence of

Lagrange interpolation for fast decaying even Erdős weights on the line for the interpolation points defined above.

Following are our new results:

THEOREM 1.3. *Let w be strongly admissible, $0 < p < \infty$, $\hat{k} > 0$ and $\beta \in \mathbb{R}$. Then for*

$$\lim_{n\to\infty} \left\| (f - L_{n+2}(f, V_{n+2}))w(x)(1+|x|)^{-\beta}\big(\log(2+Q(x))\big)^{-1} \right\|_{L_p(\mathbb{R})} = 0 \tag{1.5}$$

to hold for every continuous $f : \mathbb{R} \to \mathbb{R}$ with

$$\lim_{|x|\to\infty} |f(x)|w(x)\big(\log|x|\big)^{1+\hat{k}} = 0 \tag{1.6}$$

it is necessary that $\beta \geq 1/p$. Moreover, if $p > 1$, it is also sufficient that $\beta \geq 1/p$ and if $0 < p \leq 1$, then it is sufficient that $\beta > 1/p$.

THEOREM 1.4. *Let w be strongly admissible, $0 < p < \infty$, $\hat{k} > 0$ and $\beta \in \mathbb{R}$. Then for*

$$\lim_{n\to\infty} \left\| (f - L_{n+2}(f, V_{n+2}))w(x)(1+|x|)^{-1/p}\big(\log(2+Q(x))\big)^{-\beta} \right\|_{L_p(\mathbb{R})} = 0 \tag{1.7}$$

to hold for every continuous $f : \mathbb{R} \to \mathbb{R}$ with

$$\lim_{|x|\to\infty} |f(x)|w(x)\big(\log|x|\big)^{1+\hat{k}} = 0 \tag{1.8}$$

it is necessary that $\beta \geq 1$. Moreover, if $p > 1$, it also sufficient that $\beta \geq 1$, while if $0 < p \leq 1$, it is sufficient that $\beta > 1/p$.

REMARK 1.5. Theorems 1.3 and 1.4 are not artificial. Indeed, they constitute substantial improvements on earlier work for strongly admissible weights. To appreciate this, we find it appropriate to state the following Theorem which follows from the main result of [9] and [11] for strongly admissible weights and which is sharp for the different interpolation array, U_n.

THEOREM 1.6. *([9, 11]) Let w be strongly admissible, $0 < p < \infty$, $\Delta \in \mathbb{R}$ and $\kappa > 0$. Then for*

$$\lim_{n\to\infty} \left\| (f - L_n(f, U_n)))\, w\, (1+Q)^{-\Delta} \right\|_{L_p(\mathbb{R})} = 0$$

to hold for every continuous function $f : \mathbb{R} \to \mathbb{R}$ *satisfying,*

$$\lim_{|x|\to\infty} |fw|(x) \left(\log|x|\right)^{1+\kappa} = 0$$

it is necessary and sufficient that,

$$\Delta > \max\left\{0, \frac{2}{3}\left(\frac{1}{4} - \frac{1}{p}\right)\right\}.$$

We deduce that under a logarithmic decay condition on fw, a polynomial decay term together with a $\log(2+Q)$ decay term are necessary and sufficient for weighted convergence in $L_p(0 < p < \infty)$ with respect to the interpolatory matrix V_{n+2}. A comparison of Theorems 1.2-1.4 with Theorem 1.6 then show that in the sense of mean convergence, interpolation with the nodes V_{n+2} is more optimal than interpolation with the nodes U_n.

We close with a brief explanation of the structure of this paper. In Sections 2 and 3 we prove our sufficiency. To do this, we rely on two important old ideas. The first, see Section 2, is a bound for a Lebesgue function. The second, see Section 3, is splitting up our function into smaller pieces which vanish on carefully chosen intervals. We will also rely on forward quadrature estimates, Hilbert transforms and orthogonal expansions. In Section 4, we establish our necessity. Here we illustrate the idea of using the uniform boundedness principle. Our proofs use the methods above as they are applied in [11], [1], [7] and [26]. Many of the ideas originate earlier, see [24], [25] and [28].

2. The idea of the Lebesgue Function

In this section, we establish the sufficiency of our results. Throughout, for any two sequences (b_n) and (c_n) of nonzero real numbers, we shall write

$$b_n \lesssim c_n,$$

if there exists a constant $C > 0$, independent of n such that

$$b_n \le C c_n \quad \text{for } n \text{ large enough}$$

and we shall write $b_n \sim c_n$ if $b_n \lesssim c_n$ and $c_n \lesssim b_n$. Similar notation will be used for functions and sequences of functions. We will also often need technical estimates on $p_n(w^2)$, $n \ge 1$ and their zeroes. For these we refer the reader to [14].

We begin with the following auxiliary lemma which is a Lebesgue type estimate adapted from [11]. Here, if w satisfies the conditions of Theorem 1.2, we denote $w \in \mathcal{E}_1$.

LEMMA 2.1. *Let $w \in \mathcal{E}_1$, $\beta \in (0, 1/4)$ and let*

$$\Sigma_n(x) := \sum_{|x_{j,n}| \geq a_{\beta n}} |\ell_{j,n+2}(V_{n+2})| w^{-1}(x_{j,n}) w(x), \tag{2.1}$$

where $\{x_{n+1,n}, x_{n+2,n}\} := \{\pm y_0\}$. Then uniformly for $n \geq C$ and $x \in \mathbb{R}$

$$\Sigma_n(x) \lesssim \begin{cases} 1 & \text{if } |x| \leq a_{\frac{\beta n}{2}} \text{ or } |x| \geq a_{2n} \\ \log n & \text{if } a_{\frac{\beta n}{2}} \leq |x| \leq a_{2n}. \end{cases} \tag{2.2}$$

We remind the reader that in [2, Theorem 1.4], it was shown that if we sum over all the zeros in (2.1), we obtain a uniform order of $\log n$ in (2.2) and this order is sharp.

Proof of Lemma 2.1 By [2, Theorem 1.4], it is clear that (2.2) holds for $a_{\frac{\beta n}{2}} \leq |x| \leq a_{2n}$ and so it suffices to prove Lemma 2.1 for the range $|x| \leq a_{\frac{\beta n}{2}}$ or $|x| \geq a_{2n}$. To this end, we fix $x \in \mathbb{R}$ and let $k(x)$ denote the closest zero to x. Write for $n \geq 1$

$$\begin{aligned} \Sigma_n(x) &= \sum_{\substack{j:\, |x_{j,n}| \geq a_{\beta n} \\ j \in [k(x)+2, k(x)-2]}} |\ell_{j,n+2}(V_{n+2})| w^{-1}(x_{j,n}) w(x) & (2.3) \\ &+ \sum_{\substack{j:\, |x_{j,n}| \geq a_{\beta n} \\ j \notin [k(x)+2, k(x)-2]}} |\ell_{j,n+2}(V_{n+2})| w^{-1}(x_{j,n}) w(x) & (2.4) \\ &= \Sigma_{n,1}(x) + \Sigma_{n,2}(x). \end{aligned}$$

We first estimate $\Sigma_{n,1}(x)$. We have

$$\Sigma_{n,1}(x) \leq \sum_{\substack{|x_{j,n}| \geq a_{\beta n}, x_{jn} \in U_n \\ j \in [k(x)+2, k(x)-2]}} |\ell_{j,n+2}(V_{n+2})| w^{-1}(x_{j,n}) w(x)$$

$$+ |\ell_{n+1,n+2}(V_{n+2})| w^{-1}(y_0) w(x) + |\ell_{n+2,n+2}(V_{n+2})| w^{-1}(-y_0) w(x).$$

Then using the estimate

$$|\ell_{n+j,n+2}(V_{n+2})| w^{-1}(\pm y_0) w(x) \lesssim 1, \ j = 1, 2$$

which is easily established for the class $\mathcal{E}_1$, see [2, Lemma 2.5], we see that uniformly for $1 \leq j \leq n$

$$\Sigma_{n,1}(x) \lesssim 1 + \sum_{\substack{|x_{j,n}| \geq a_{\beta n}, x_{jn} \in U_n \\ j \in [k(x)+2, k(x)-2]}} |\ell_{j,n+2}(V_{n+2})| w^{-1}(x_{j,n}) w(x). \tag{2.5}$$

Next we need the following identities below which hold uniformly for x, $n \geq C$ and $j \in [k(x) + 2, k(x) - 2]$. They may be found using the results of [14] and ([2], (2.33) and (3.19)).

(a)

$$\ell_{j,n+2}(V_{n+2})(x) = \left(\frac{y_0^2 - x^2}{y_0^2 - x_{j,n}^2}\right) \ell_{j,n}(U_n)(x)$$
$$\lesssim \left(\frac{\left|1 - \frac{|x|}{a_n}\right| + L(nT(a_n))^{-2/3}}{\left|1 - \frac{|x_{jn}|}{a_n}\right| + L(nT(a_n))^{-2/3}}\right) \ell_{j,n}(U_n)(x). \tag{2.6}$$

(b)

$$\left(\frac{\left|1 - \frac{|x|}{a_n}\right| + L(nT(a_n))^{-2/3}}{\left|1 - \frac{|x_{j,n}|}{a_n}\right| + L(nT(a_n))^{-2/3}}\right) \sim 1. \tag{2.7}$$

(c)

$$|\ell_{j,n}(U_n)w(x)|w^{-1}(x_{j,n}) \lesssim 1. \tag{2.8}$$

Using (2.6)-(2.8), we see that (2.5) becomes,

$$\Sigma_{n,1}(x) \lesssim 1 \tag{2.9}$$

uniformly for $n \geq C$ and the given x.

Next, we estimate $\Sigma_{n,2}(x)$ so we assume henceforth that $j \notin [k(x)+2, k(x)-2]$. We will need the inequality, see [2, (2.31)],

$$\sum_{|x_{j,n}| \geq a_{\beta n}} |\ell_{j,n+2}(V_{n+2})|w^{-1}(x_{j,n})w(x)$$
$$\lesssim 1 + \sum_{\substack{|x_{j,n}| \geq a_{\beta n} \\ x_{jn} \in U_n}} \left(\frac{\left|1 - \frac{|x_{j,n}|}{a_n}\right| + L(nT(a_n))^{-2/3}}{\left|1 - \frac{|x|}{a_n}\right| + L(nT(a_n))^{-2/3}}\right)^{-3/4} \frac{\Delta x_{j,n}}{|x - x_{j,n}|} \tag{2.10}$$

together with two observations which follow using the methods of [11] and which hold uniformly for $n \geq C$, x and $1 \leq j \leq n$. Firstly

$$1 - \frac{|t|}{a_n} + L(nT(a_n))^{-2/3} \sim 1 - \frac{|x_{j,n}|}{a_n} + L(nT(a_n))^{-2/3}, \ t \in [x_{j+1,n}, x_{j,n}]$$

and secondly

$$|x - t| \sim |x - x_{j,n}|, \ t \in [x_{j+1,n}, x_{j,n}], \ x \notin [x_{j+2,n}, x_{j-2,n}].$$

In applying these identities we see with the help of [14] that

$$\Sigma_{n,2}(x) \lesssim \int\limits_{\substack{a_{\beta n}\leq |t|\leq a_n \\ |t-x|\geq C\frac{a_n\psi_n(x)}{n}}} \left(\frac{\left|1-\frac{|x|}{a_n}\right| + L(nT(a_n))^{-2/3}}{\left|1-\frac{|t|}{a_n}\right| + L(nT(a_n))^{-2/3}}\right)^{3/4} \frac{1}{|x-t|}dt \tag{2.11}$$

uniformly for the given x and $n \geq C$. Here for $|x| \leq a_n$, $n \geq 1$

$$\psi_n(x) :== \frac{\left|1-\frac{|x|}{a_n}\right| + (nT(a_n))^{-2/3} + T(a_n)^{-1}}{\sqrt{\left|1-\frac{|x|}{a_n}\right| + (nT(a_n))^{-2/3}}}$$

and $\psi_n(x) = \psi_n(a_n)$ for $|x| \geq a_n$.

Armed with the estimate (2.11), we realize that to establish Lemma 2.1, it suffices to estimate (2.11). We suppose without loss of generality that $0 \leq x \leq a_{\frac{\beta n}{2}}$ for the other case is similar. For notational simplicity, we set $S := \{j : j \notin [k(x)+2, k(x)-2]\}$. Then following [2, (3.20) - (3.23)] and a similar argument to the case $a_{\frac{\beta n}{2}} \leq |x| \leq a_{2n}$, we obtain uniformly for the given x and $n \geq C$,

$$\begin{aligned}
\Sigma_{n,2}(x) &\lesssim \sum_{j\in S} \frac{\Delta x_{j,n}}{|x-x_{j,n}|} + \qquad (2.12)\\
&+ \sum_{j\in S} \frac{\Delta x_{j,n}}{a_n^{3/4}|x-x_{j,n}|^{1/4}\left(\left|1-\frac{|x_{j,n}|}{a_n}\right|^{3/4} + L(nT(a_n))^{-2/3}\right)}\\
&\lesssim \int\limits_{\substack{a_{\beta n}\leq |t|\leq a_n \\ |t-x|\geq c\frac{a_n\psi_n(x)}{n}}} \frac{1}{|t-x|}dt\\
&\quad + \frac{1}{a_n^{3/4}} \int\limits_{\substack{a_{\beta n}\leq |t|\leq a_n \\ |t-x|\geq C\frac{a_n\psi_n(x)}{n}}} |t-x|^{-1/4}\left(1-\frac{|t|}{a_n} + L(nT(a_n))^{-2/3}\right)^{-3/4} dt\\
&\lesssim 1 + a_n^{-3/4} \int_{\frac{a_{\beta n}}{a_n}}^{1} \left|\frac{a_{\beta n}}{a_n} - \frac{a_{\frac{\beta n}{2}}}{a_n}\right|^{-1/4} \frac{1}{(1-s)^{3/4}}ds\\
&\lesssim 1 + a_n^{-3/4}T(a_n)^{-1/4}T(a_n)^{1/4} \lesssim 1. \qquad (2.13)
\end{aligned}$$

This last estimate proves Lemma 2.1. □

3. The idea of splitting

In this section, we establish the sufficiency of our theorems. The essential idea which goes back to [24], is to write the function as a sum of functions which vanish on carefully chosen intervals and centre our analysis on each of these subintervals. We now present the details of this analysis. We find it convenient to set for some fixed $\hat{k} > 0$,

$$\phi(x) := \left(\log(2 + x^2)\right)^{-1-\hat{k}}, \ x \in \mathbb{R}. \tag{3.1}$$

We begin with:

LEMMA 3.1. *Let $w \in \mathcal{E}_1$, $p > 1$ and let $\{f_n\} : \mathbb{R} \to \mathbb{R}$ be a sequence of measurable functions satisfying for $n \geq 1$,*

$$f_n(x) = 0, \ |x| < a_{\frac{n}{9}} \tag{3.2}$$

and

$$|f_n w|(x) \lesssim \phi(x), \ x \in \mathbb{R}. \tag{3.3}$$

Then,

$$\lim_{n\to\infty} \left\| L_{n+2}(f_n, V_{n+2})(x) w(x)(1 + |x|)^{-1/p} \left(\log(2 + Q(x))\right)^{-1} \right\|_{L_p(\mathbb{R})} = 0. \tag{3.4}$$

Proof. We distinguish two cases:

Suppose first that $|x| \leq a_{\frac{n}{18}}$ or $|x| \geq a_{2n}$. Then (2.1), (3.2) and (3.3) give

$$\begin{aligned} \left|L_{n+2}(f_n, V_{n+2})w\right|(x) &= \left| \sum_{|x_{j,n}| \geq a_{\frac{n}{9}}} \ell_{j,n}(V_{n+2}) w(x) f_n(x_{j,n}) \right| \\ &\lesssim \phi(a_n). \end{aligned} \tag{3.5}$$

(3.5) and the fact that Q grows faster than a polynomial then imply that

$$\begin{aligned} &\left\| L_{n+2}(f_n, V_{n+2})(x) w(x)(1 + |x|)^{-1/p} \left(\log(2 + Q(x))\right)^{-1} \right\|_{L_p\left(|x| \leq \frac{a_n}{18}\right)} \\ \lesssim\ & \phi(a_n) = o(1), \ n \to \infty. \end{aligned} \tag{3.6}$$

Next suppose that $a_{\frac{n}{18}} \le |x| \le a_{2n}$. Then (3.5) and [14] give,

$$\left\| L_{n+2}(f_n, V_{n+2})(x)w(x)(1+|x|)^{-1/p}\big(\log(2+Q(x))\big)^{-1}\right\|_{L_p\left(a_{\frac{n}{18}}\le|x|\le a_{2n}\right)}$$

$$\lesssim \frac{\log n\phi(a_n)}{\log nT(a_n)^{1/p}} = o(1),\ n\to\infty. \tag{3.7}$$

Combining our estimates (3.6) – (3.7) gives the lemma. □

Next, we treat functions that vanish for $|x| \ge a_{\frac{n}{9}}$.

LEMMA 3.2. *Let $w \in \mathcal{E}_1$, $p > 1$ and $\varepsilon \in (0,1)$. Let $\{g_n\} : \mathbb{R} \to \mathbb{R}$ be a sequence of measurable functions such that for $n \ge 1$,*

$$g_n(x) = 0,\ |x| \ge a_{\frac{n}{9}} \tag{3.8}$$

and

$$|g_n w|(x) \lesssim \varepsilon,\ x \in \mathbb{R}. \tag{3.9}$$

Then,

$$\lim_{n\to\infty}\left\| L_{n+2}(g_n, V_{n+2})(x)w(x)(1+|x|)^{-1/p}\big(\log(2+Q(x))\big)^{-1}\right\|_{L_p\left(|x|\ge a_{\frac{n}{8}}\right)} = 0. \tag{3.10}$$

Proof. Let $n \ge C$ and fix $|x| \ge a_{n/8}$. We may further assume that $|x| \le a_{2n}$. Then for such x we may apply the estimate (2.10) together with the identities (3.8) and (3.9) to obtain

$$\begin{aligned}
|L_{n+2}(g_n, V_{n+2})w|\,(x) &\lesssim \varepsilon \sum_{|x_{j,n}|\le a_{\frac{n}{9}}} |\ell_{j,n}(V_{n+2})(x)|\, w^{-1}(x_{j,n}) \\
&\lesssim \varepsilon \sum_{|x_{j,n}|\le a_{\frac{n}{9}}} \frac{\Delta x_{j,n}}{|x-x_{j,n}|}\left(\frac{\left|1-\frac{|x|}{a_n}\right| + L(nT(a_n))^{-2/3}}{\left|1-\frac{|x_{j,n}|}{a_n}\right| + L(nT(a_n))^{-2/3}}\right)^{3/4} \\
&\lesssim \varepsilon T(a_n)^{-3/4} a_n^{3/4}\int_0^{a_{\frac{n}{9}}}(a_n-t)^{-7/4}dt \\
&\lesssim \varepsilon.
\end{aligned} \tag{3.11}$$

Thus using (3.11) and the fact that Q grows faster than a polynomial gives,

$$\begin{aligned}
&\left\| L_{n+2}(g_n, V_{n+2})(x)w(x)(1+|x|)^{-1/p}\big(\log(2+Q(x))\big)^{-1}\right\|_{L_p\left(|x|\ge a_{\frac{n}{8}}\right)} \\
&\lesssim \varepsilon \left\|(1+|x|)^{-1/p}\big(\log(2+Q(x))\big)^{-1}\right\|_{L_p\left(|x|\ge a_{\frac{n}{8}}\right)} \\
&\lesssim \varepsilon.
\end{aligned} \tag{3.12}$$

Letting $\varepsilon \to 0^+$ in (3.12) gives the lemma. □

Now we estimate the L_p norm in (3.10) for the range $|x| \le a_{\frac{n}{8}}$. Here we follow ideas and methods from ([11], Lemmas 4.3 and 4.4). More precisely we shall prove the following:

LEMMA 3.3. *Let w be strong admissible, $p > 1$, $\varepsilon \in (0,1)$ and let $\{g_n\} : \mathbb{R} \to \mathbb{R}$ be a sequence of measurable functions satisfying (3.8) and*

$$|g_n w|(x) \lesssim \varepsilon\phi(x),\ x \in \mathbb{R}. \tag{3.13}$$

Then

$$\lim_{n\to\infty} \left\| L_{n+2}(g_n, V_{n+2})(x)w(x)(1+|x|)^{-1/p}\big(\log(2+Q(x))\big)^{-1} \right\|_{L_p\left(|x|\le a_{\frac{n}{18}}\right)} = 0. \tag{3.14}$$

Proof. We first establish the following inequality which is of independent interest.

Orthogonal expansion and Hilbert transform lemma :

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. Then uniformly for $u \ge C$ and σ

$$\left\| S_n[\sigma\phi w^{-1}]w(1+|x|)^{-1/p}\big(\log(2+Q(x))\big)^{-1)} \right\|_{L_p\left(|x|\le a_{\frac{n}{8}}\right)} \tag{3.15}$$

$$\lesssim \|\sigma\|_{L_\infty(\mathbb{R})}.$$

Here, $S_n[;]$ denotes the nth partial orthonormal expansion of ;.

To see this, we assume without loss of generality, that $\|\sigma\|_{L_\infty(\mathbb{R})} = 1$ and write as in ([11], (4.10)) for $x \in \mathbb{R}$ and $n \ge 1$

$$\left| S_n[\sigma\phi w^{-1}]w \right|(x) \lesssim a_n^{1/2}\left(1 - \frac{|x|}{a_n}\right)^{-1/4} \sum_{j=n-1}^{n} |H[\sigma\phi p_j w]|(x).$$

Here for suitable $f : \mathbb{R} \to \mathbb{R}$

$$H[f](x) := \lim_{\varepsilon\to 0^+} \int_{|x-t|\ge\varepsilon} \frac{f(t)}{x-t}\,dt$$

denotes the Hilbert transform of f.

We choose $\ell = \ell(n)$ with

$$2^{2^\ell} \le \frac{n}{8} \le 2^{2^\ell+1}$$

so that uniformly for $n \geq C$,

$$\ell \sim \log\log n$$

and

$$a_n \geq a_{2^{2^k+3}},\ 1 \leq k \leq \ell.$$

Moreover we define

$$\tau_k := \left[a_{2^{2^k}}, a_{2^{2^k+1}}\right],\ k \geq 1.$$

Our choice of ℓ is motivated in part by the following identity which follows using [14] below. Uniformly for $k \geq 1$,

$$\left(\log\left(2 + Q(a_{2^{2^k}})\right)\right) \sim 2^k. \tag{3.16}$$

It now follows exactly as in ([11], Lemma 4.3), that we obtain uniformly for $k \geq 1$ and $n \geq C$,

$$\begin{aligned} &\left\| S_n[\sigma\phi w^{-1}]w(1+|x|)^{-1/p}\big(\log(2+Q(x))\big)^{-1}\right\|_{L_p(\tau_k)} \\ &\quad\lesssim \left[\big[\log(2+Q(a_{2^{2^k}}))\big]^{-1}\log\left(T(a_{2^{2^k+1}})\right)T(a_{2^{2^k}})^{1/2-1/p}\right]. \end{aligned} \tag{3.17}$$

Applying (1.1) for suitably small $\varepsilon > 0$ which we may, yields a $\delta > 0$ so that for the given $n \geq C$ and uniformly for $k \geq 1$,

$$\left\| S_n[\sigma\phi w^{-1}]w(1+|x|)^{-1/p}\big(\log(2+Q(x))\big)^{-1}\right\|_{L_p(\tau_k)} \lesssim 2^{-k\delta}. \tag{3.18}$$

Summing (3.18) then gives for the given n

$$\begin{aligned} &\left\| S_n[\sigma\phi w^{-1}]w(1+|x|)^{-1/p}\big(\log(2+Q(x))\big)^{-1}\right\|_{L_p\left(a_4 \leq |x| \leq a_{\frac{n}{8}}\right)} \\ &\lesssim \sum_{k=1}^{\ell} \left\| S_n[\sigma\phi w^{-1}]w(1+|x|)^{-1/p}\big(\log(2+Q(x))\big)^{-1/2-\delta}\right\|_{L_p(\tau_k)} \\ &\lesssim \sum_{k=1}^{\ell} 2^{-k\delta} \lesssim 1. \end{aligned} \tag{3.19}$$

Similarly and more easily it also follows that uniformly for $n \geq C$,

$$\left\| S_n[\sigma\phi w^{-1}]w(1+|x|)^{-1/p}\big(\log(2+Q(x))\big)^{-1}\right\|_{L_p(0 \leq |x| \leq a_4)} \lesssim 1. \tag{3.20}$$

Thus (3.15) is established.

Next we shall show that given $\epsilon \in (0,1)$ and measurable $g : \mathbb{R} \to [0,\infty)$, we have uniformly for $n \geq C$

$$\|L_{n+2}(g_n, V_{n+2})wg\|_{L_p\left(|x|\leq a_{\frac{n}{8}}\right)} \lesssim \varepsilon \left\|S_n[\sigma_n \phi w^{-1}]wg\right\|_{L_p\left(|x|\leq a_{\frac{n}{8}}\right)}. \tag{3.21}$$

Once this is established, (3.15) and the above prove the lemma. In order to proceed to prove (3.21), we find it convenient to introduce some notation.

Letting sgn and χ denote the usual signum and indicator functions respectively, we define for $n \geq 1$ the following quantities:

$$H_n[g_n](x) := \sum_{j=1}^{n} (y_0^2 - x_{j,n}^2)^{-1} \ell_{j,n}(U_n)(x) g_n(x_{j,n}),\ x \in \mathbb{R}. \tag{3.22}$$

$$\chi_n := \chi_{\left[-a_{\frac{n}{8}}, a_{\frac{n}{8}}\right]}. \tag{3.23}$$

$$h_n(x) := \tag{3.24}$$
$$\operatorname{sgn}(H_n[g_n]) \left|H_n[g_n]\right|^{p-1}(x)\chi_n(x) w^{p-2}(x) g^p(x) |y_0^2 - x^2|^{p-1},\ x \in \mathbb{R}.$$

$$\sigma_n := \operatorname{sgn}(S_n[h_n]). \tag{3.25}$$

First note that using (3.8), we may conclude that $g_n(\pm y_0) = 0$. Thus

$$\begin{aligned}
&\|L_{n+2}(g_n, V_{n+2})(x)w(x)g(x)\|^p_{L_p\left(|x|\leq a_{\frac{n}{8}}\right)} \\
&= \int_{|x|\leq a_{\frac{n}{8}}} \left|L_{n+2}(g_n, V_{n+2})\right|^p(x) w^p(x) g^p(x)\,dx \\
&= \int_{|x|\leq a_{\frac{n}{8}}} \left|\sum_{j=1}^{n} \left(\frac{y_0^2 - x^2}{y_0^2 - x_{j,n}^2}\right) \ell_{j,n}(U_n)(x) g_n(x_{j,n})\right|^p w^p(x) g^p(x)\,dx \\
&= \int_{|x|\leq a_{\frac{n}{8}}} |y_0^2 - x^2|\, H_n[g_n](x_{j,n}) h_n(x) w^2(x)\,dx \\
&\lesssim \frac{a_n^2}{T(a_n)^2} \int_{\mathbb{R}} H_n[g_n](x) h_n(x) w^2(x)\,dx.
\end{aligned} \tag{3.26}$$

Now recalling that for $n \geq 1$, $h_n - S_n[h_n] \perp \Pi_{n-1}$ and $H_n[g_n]S_n[h_n] \in \Pi_{2n-1}$, we may apply Gauss quadrature and continue (3.26) as

$$\begin{aligned}
&\|L_{n+2}(g_n, V_{n+2})w(x)g(x)\|^p_{L_p\left(|x| \leq a_{\frac{n}{8}}\right)} \\
&\lesssim \frac{a_n^2}{T(a_n)^2} \int_{\mathbb{R}} H_n[g_n](x) S_n[h_n](x) w^2(x) dx \\
&\lesssim \frac{a_n^2}{T(a_n)^2} \sum_{j=1}^{n} \lambda_{j,n} H_n[g_n](x_{j,n}) S_n[h_n](x_{j,n}) \\
&\lesssim \frac{a_n^2}{T(a_n)^2} \sum_{|x_{j,n}| \leq a_{\frac{n}{9}}} \frac{\lambda_{j,n} g_n(x_{j,n})}{y_0^2 - x_{j,n}^2} S_n[h_n](x_{j,n}) \\
&\lesssim \varepsilon \frac{a_n^2}{T(a_n)^2} \sum_{|x_{j,n}| \leq a_{\frac{n}{9}}} \lambda_{j,n} |S_n(x_{j,n})| w^{-1}(x_{j,n}) \frac{\phi(x_{j,n})}{y_0^2 - x_{j,n}^2} \\
&\lesssim \varepsilon \sum_{|x_{j,n}| \leq a_{\frac{n}{9}}} \lambda_{j,n} |S_n[h_n](x_{j,n})| w^{-1}(x_{j,n}) \phi(x_{j,n}). \qquad (3.27)
\end{aligned}$$

Now by ([11], Lemma 3.2), we may continue (3.27) as

$$\begin{aligned}
&\|L_{n+2}(g_n, V_{n+2})(x)w(x)g(x)\|^p_{L_p\left(|x| \leq a_{\frac{n}{8}}\right)} \\
&\lesssim \varepsilon \int_{\mathbb{R}} |S_n[h_n]|(x) \phi w^{-1} w^2(x) dx \\
&\lesssim \varepsilon \int_{\mathbb{R}} \sigma_n(x) S_n[h_n](x) \phi(x) w^{-1}(x) w^2(x) dx \\
&\lesssim \varepsilon \int_{\mathbb{R}} h_n(x) S_n[\sigma_n \phi w^{-1}](x) w^2(x) dx \\
&\lesssim \varepsilon \int_{|x| \leq a_{\frac{n}{8}}} h_n(x) S_n[\sigma_n \phi w^{-1}](x) w^2(x) dx \\
&\lesssim \varepsilon \left(\int_{|x| \leq a_{\frac{n}{8}}} |h_n w g^{-1}|^q(x) dx \right)^{1/q} \left(\int_{|x| \leq a_{\frac{n}{8}}} \left| S_n[\sigma_n \phi w^{-1}] w(x) g(x) \right|^p dx \right)^{1/p} \\
&\lesssim \varepsilon \, \|L_{n+2}(g_n, V_{n+2})(x)w(x)g(x)\|^{p-1}_{L_p\left(|x| \leq a_{\frac{n}{8}}\right)} \times \\
&\times \left\| S_n[\sigma_n \phi w^{-1}](x) w(x) g(x) \right\|_{L_p\left(|x| \leq a_{\frac{n}{8}}\right)}. \qquad (3.28)
\end{aligned}$$

(3.28) then establishes (3.21) and hence the lemma. □

Armed with Lemmas (3.1) – (3.3) we are ready for the

The Sufficiency Proof of Theorems 1.3 and 1.4 for $p > 1$.

Let $\varepsilon \in (0, 1)$. We may choose a polynomial P such that,

$$\left\|(f-P)w\phi^{-1}\right\|_{L_\infty(\mathbb{R})} < \epsilon. \tag{3.29}$$

See for example [12, pg 752]. Let $n \geq C$ and for $x \in \mathbb{R}$, let

$$g_n(x) := (P-f)(x)\chi_{\left[-a_{\frac{n}{9}},a_{\frac{n}{9}}\right]}(x) \tag{3.30}$$

and

$$f_n(x) := (P-f)(x)\left(1-\chi_{\left[-a_{\frac{n}{9}},a_{\frac{n}{9}}\right]}\right)(x). \tag{3.31}$$

Note that $g_n(x) = 0$ for $|x| \geq a_{\frac{n}{9}}$, $f_n(x) = 0$ for $|x| \leq a_{\frac{n}{9}}$ and

$$f - P = f_n + g_n.$$

Thus Lemmas 3.1-3.3 show that

$$\lim_{n\to\infty}\left\|L_n(P-f, V_{n+2})(x)w(x)(1+|x|)^{-1/p}\big(\log(2+Q(x))^{-1}\big)\right\|_{L_p(\mathbb{R})} = 0. \tag{3.32}$$

Moreover by choice of P we have

$$\begin{aligned} &\left\|(f-P)(x)w(x)(1+|x|)^{-1/p}\big(\log(1+Q(x))\big)^{-1}\right\|_{L_p(\mathbb{R})} \\ &\quad\lesssim \varepsilon\left\|(1+|x|)\phi^p(x)\right\|^{1/p}_{L_1(\mathbb{R})} \lesssim \varepsilon. \end{aligned} \tag{3.33}$$

Now write,

$$\begin{aligned} &\left\|(f-L_{n+2}(f,V_{n+2}))(x)w(x)(1+|x|)^{-1/p}\big(\log(2+Q(x))\big)^{-1}\right\|_{L_p(\mathbb{R})} \\ \leq\ &\left\|(f-P)(x)w(x)(1+|x|)^{-1/p}\big(\log(2+Q(x))\big)^{-1}\right\|_{L_p(\mathbb{R})} \\ +\ &\left\|L_n(P-f,V_{n+2})(x)w(x)(1+|x|)^{-1/p}\big(\log(2+Q(x))\big)^{-1}\right\|_{L_p(\mathbb{R})}. \end{aligned}$$

Then (3.32) and (3.33) give,

$$\limsup_{n\to\infty}\left\|(f-L_{n+2}(f,V_{n+2}))(x)w(x)(1+|x|)^{-1/p}\big(\log(2+Q(x))\big)^{-1}\right\|_{L_p(\mathbb{R})} \lesssim \varepsilon.$$

Letting $\varepsilon \to 0^+$ gives the result. □

We now present

The Sufficiency Proof of Theorems 1.3 and 1.4 for $0 < p \le 1$

We begin with Theorem 1.3. Let us choose some constant $q > 1$ satisfying $pq > 1$. Since $\beta > 1/p$, we may choose some β_1 satisfying

$$\beta - 1/p + 1/pq > \beta_1 \ge 1/pq.$$

Now by Hölder's inequality, with $1/q + 1/q' = 1$, we have

$$\begin{aligned}
&\left\| (f - L_{n+2}(f, V_{n+2}))\, w(x)(1+|x|)^{-\beta} \left(\log(2+Q(x))\right)^{-1} \right\|^p_{L_p(\mathbb{R})} \\
= &\int \left| (f - L_{n+2}(f, V_{n+2}))\, w(x)(1+|x|)^{-\beta} \left(\log(2+Q(x))\right)^{-1} \right|^p dx \\
\le &\left(\int \left| (f - L_{n+2}(f, V_{n+2}))\, w(x)(1+|x|)^{-\beta_1} \left(\log(2+Q(x))\right)^{-1} \right|^{pq} dx \right)^{1/q} \\
&\qquad \times \left(\int 1/(1+|x|)^{(\beta-\beta_1)pq'} dx \right)^{1/q'} .
\end{aligned}$$

Since $\beta_1 \ge 1/pq$ and $(\beta - \beta_1)pq' > 1$,

$$\int \frac{1}{(1+|x|)^{-(\beta-\beta_1)pq'}} dx < \infty$$

and by the result of Theorem 1.3 for $p > 1$, we have

$$\lim_{n\to\infty} \left(\int \left| (f - L_{n+2}(f, V_{n+2}))\, w(x)(1+|x|)^{-\beta_1} \left(\log(2+Q(x))\right)^{-1} \right|^{pq} dx \right)^{1/q} = 0.$$

Therefore, we have the result for this case. For Theorem 1.4, let β satisfy $1/\beta < p \le 1$. Then $1/p < \frac{1}{1-\beta p+p}$ and so we may choose q with $1/p < q < \frac{1}{1-\beta p+p}$. This implies that $\beta - 1/p + 1/pq > 1$ and so we may choose β_1 with $1 \le \beta_1 < \beta - 1/p + 1/pq$. Summarizing the above imply that $pq > 1$ and $(\beta - \beta_1)pq' > 1$. Thus we may proceed as in Theorem 1.3 and deduce that

$$\begin{aligned}
&\left\| (f - L_{n+2}(f, V_{n+2}))\, w(x)(1+|x|)^{-1/p} \left(\log(2+Q(x))\right)^{-\beta} \right\|^p_{L_p(\mathbb{R})} \\
\le &\left\| (f - L_{n+2}(f, V_{n+2}))\, w(x)(1+|x|)^{-1/pq} \left(\log(2+Q(x))\right)^{-\beta_1} \right\|^p_{L_{pq}(\mathbb{R})} \\
\times &\left(\int \frac{1}{(1+|x|)(\log(2+Q(x)))^{(\beta-\beta_1)pq'}} dx \right)
\end{aligned}$$

and so Theorem 1.4 holds for this case as well. □.

4. Necessity

In this section, we establish our necessary conditions. The argument we shall use, relies on an application of the generalized uniform boundedness principle. The original idea should be credited to Paul Nevai in [24] and later modifications have appeared in [20], [23], [11], [12], [7] and [9]. For our purposes, we require additional new ideas and the analysis is presented below. We shall deal with Theorem 1.3, Theorem 1.4 is similar.

We begin with a lemma which we will need in the sequel.

LEMMA 4.1. *Let $w \in \mathcal{E}_1$ and $0 < p < \infty$. Then uniformly for $n \geq 1$,*

$$a_n^{1/p-1/2} \lesssim \|p_n w\|_{L_p[\frac{a_n}{4}, \frac{a_n}{2}]}. \tag{4.1}$$

Proof. Using ([12], Lemma 5.1), for every interval $\tau_n \subset [-a_n, a_n]$ containing at least two zeros of $p_n(w^2; \ )$ we have

$$\|p_n w\|_{L_p(\tau_n)} \gtrsim a_n^{-1/2} \left\| \left(\left| 1 - \frac{|t|}{a_n} \right| + (nT(a_n))^{-2/3} \right)^{-1/4} \right\|_{L_p(\tau_n)}$$

uniformly for $n \geq 1$. Applying this with $\tau_n := [\frac{a_n}{4}, \frac{a_n}{2}]$ and using [14] below gives the lemma. □

We are ready for the

Proof of Theorem 1.3.

Fix $w \in \mathcal{E}_1, \ 0 < p < \infty, \ \hat{k} > 0$ and $\beta \in \mathbb{R}$. Assume moreover that we have convergence in (1.5) for every continuous f satisfying (1.6). Let $\eta : \mathbb{R} \to (0, \infty)$ be a positive even continuous function decreasing in $(0, \infty)$ with limit 0 at ∞. For clarity, we choose its rate of decay later.

Let X be the Banach space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with,

$$\|f\|_X := \sup_{x \in \mathbb{R}} |fw|(x) \big(\log(2 + |x|) \big)^{-1-\hat{k}} \eta(x)^{-1} < \infty.$$

Moreover, let Y be the space of measurable functions $f : \mathbb{R} \to \mathbb{R}$ for $1 \leq p < \infty$ with

$$\|f\|_Y := \left\| f(x)w(x)(1 + |x|)^{-\beta} \big(\log(2 + Q(x)) \big)^{-1} \right\|_{L_p(\mathbb{R})} < \infty.$$

We note that Y is a metric space for $1 < p < \infty$ with metric

$$\|f-g\| := \left\|(f-g)(x)w(x)(1+|x|)^{-\beta}\big(\log(2+Q(x))\big)^{-1}\right\|_{L_p(\mathbb{R})}.$$

For $0 < p < 1$, let Y be the space of measurable functions $f : \mathbb{R} \to \mathbb{R}$ with

$$\|f\|_Y := \left\|f(x)w(x)(1+|x|)^{-\beta}\big(\log(2+Q(x))\big)^{-1}\right\|_{L_p(\mathbb{R})}^p < \infty.$$

We then note that Y is a metric space with a metric

$$\|f-g\| := \left\|(f-g)(x)w(x)(1+|x|)^{-\beta}\big(\log(2+Q(x))\big)^{-1}\right\|_{L_p(\mathbb{R})}^p.$$

Since for $0 < p < 1$, the method is the same as for $1 \le p < \infty$, we only prove for $1 \le p < \infty$. Now each $f \in X$ satisfies (1.6) so we have

$$\lim_{n\to\infty} \left\|(f - L_{n+2}(f, V_{n+2}))\right\|_Y = 0.$$

That is, for each $f \in X$, there exists $\varepsilon > 0$ such that for $n \ge 1$

$$\left\|(f - L_n[f, V_{n+2}])\right\|_Y \le \varepsilon.$$

Then by the generalized uniform boundedness principle, the metric of the operator $I - L_n[;]$ is uniformly bounded in n. That is, for each $f \in X$ if we let $\tilde{f} := \frac{f}{\|f\|_X}$, there exists a constant m independent of f and n, so that for every $n \ge 1$

$$\left\|\tilde{f} - L_{n+2}(\tilde{f}, V_{n+2})\right\|_Y \le m.$$

Hence,

$$\|f - L_{n+2}(f, V_{n+2})\|_Y \le m\|f\|_X \tag{4.2}$$

for every $f \in X$ and for every $n \ge 1$. Now we recall that

$$\begin{aligned} L_{n+2}(f, V_{n+2})(x) &= \sum_{j=1}^{n} \ell_{j,n+2}(V_{n+2})(x) f(x_{j,n}) \\ &\quad + \ell_{n+1,n+2}(V_{n+2})(x) f(y_0) + \ell_{n+2,n+2}(V_{n+2})(x) f(-y_0) \end{aligned}$$

where

$$\ell_{n+1,n+2}(V_{n+2})(x) = \frac{p_n(x)(y_0+x)}{2y_0 p_n(y_0)}$$

and

$$\ell_{n+2,n+2}(V_{n+2})(x) = \frac{p_n(x)(y_0-x)}{2y_0 p_n(-y_0)}.$$

Then a straightforward calculation shows that for functions f satisfying $f(0) = 0$ and $f(\pm y_0) = 0$

$$L_3(f, U_3)(x) = 0 \text{ for every } x \in \mathbb{R}.$$

Thus applying (4.2) for such f with $n = 1$ gives,

$$\|f - L_3(f, U_3)\|_Y \le m\|f\|_X$$

which implies that $\|f\|_Y \le m\|f\|_X$.

It follows that we have,

$$\|L_{n+2}(f, V_{n+2})\|_Y \le 2m\|f\|_X \tag{4.3}$$

for every $f \in X$ with $f(0) = f(\pm y_0) = 0$. Here m does not depend on f so we fix it.

We will now show that (4.3) with a careful choice of f implies our necessary conditions. To this end, we proceed to define a special sequence of functions and establish some of its important properties. Choose $\{g_n\} : \mathbb{R} \to \mathbb{R}$ continuous with

(i)

$$g_n(x) = 0,\ x \in \left(-\infty, \frac{a_n}{2}\right] \cup \left[\frac{3a_n}{4}, \infty\right).$$

(ii) For $x_{j,n} \in \left(\frac{a_n}{2}, \frac{3a_n}{4}\right)$,

$$\eta(x_{j,n})^{-1}(g_n w)(x_{j,n})\mathrm{sgn}(p_n'(x_{j,n}))\phi(x_{j,n})^{-1} = 1$$

and $\|g_n\|_X = 1$.

In particular, it is easy to see that $g_n(0) = 0$ and $g_n(\pm y_0) = 0$.

Moreover, by definition of g_n, we have for $n \ge C$ and for every $x \in \left[\frac{a_n}{4}, \frac{a_n}{2}\right]$,

$$\begin{aligned}
&|L_{n+2}(g_n, V_{n+2})|(x) \\
&= \left| \sum_{x_{j,n} \in \left(\frac{a_n}{2}, \frac{3a_n}{4}\right)} \left(\frac{y_0^2 - x^2}{y_0^2 - x_{j,n}^2}\right) \left(\frac{p_n(x)}{p_n'(x_{j,n})(x - x_{j,n})}\right) g_n(x_{j,n}) \right| \\
&= |p_n(x)|(y_0^2 - x^2) \sum_{x_{j,n} \in \left(\frac{a_n}{2}, \frac{3a_n}{4}\right)} \frac{\phi(x_{j,n})\eta(x_{j,n})}{(x_{j,n} - x)|p_n'(x_{j,n})w(x_{j,n})|(y_0^2 - x_{j,n}^2)} \\
&\gtrsim |p_n(x)| \left(\frac{y_0^2 - x^2}{y_0^2}\right) \phi(a_n)\eta\left(\frac{3a_n}{4}\right) \times \\
&\quad \sum_{x_{j,n} \in \left(\frac{a_n}{2}, \frac{3a_n}{4}\right)} (x_{j,n} - x)^{-1} |p_n'(x_{j,n})w(x_{j,n})|^{-1}.
\end{aligned} \tag{4.4}$$

Indeed, using [14], we see that for $n \geq C$, and for every $x \in [\frac{a_n}{4}, \frac{a_n}{2}]$

$$\begin{aligned}
&\big|L_{n+2}(g_n, V_{n+2})\big|(x) \\
&\gtrsim \frac{a_n^{3/2}}{n} |p_n(x)| \phi(a_n) \eta\left(\frac{3a_n}{4}\right) \sum_{x_{j,n} \in \left(\frac{a_n}{2}, \frac{3a_n}{4}\right)} \frac{1}{x_{j,n} - x} \\
&\gtrsim a_n^{1/2} |p_n(x)| \phi(a_n) \eta\left(\frac{3a_n}{4}\right) \sum_{k=C_1}^{c_{2n}} \frac{1}{k}.
\end{aligned}$$

Thus we learn that for the special sequence $\{g_n\}$ we have:

(i) $g_n(0) = g_n(\pm y_0) = 0$.

(ii) For every $x \in [\frac{a_n}{2}, \frac{a_n}{2}]$ and $n \geq C$,

$$\big|L_{n+2}(g_n, V_{n+2})\big|(x) \gtrsim a_n^{1/2} |p_n(x)| \phi(a_n) (\log n) \eta\left(\frac{3a_n}{4}\right). \tag{4.5}$$

Now we apply (4.3) to g_n and use (4.5) and (4.1). This gives for $n \geq C$,

$$\begin{aligned}
2m &= 2m\|g_n\|_X \geq \big\|L_{n+2}(g_n, V_{n+2})\big\|_Y \\
&= \Big\|L_{n+2}(g_n, V_{n+2})(x) w(x) (1+|x|)^{-\beta} \big(\log(2 + Q(x))\big)^{-1}\Big\|_{L_p(\mathbb{R})} \\
&\gtrsim (\log n)^{-1} a_n^{-\max(\beta,0)} \big\|L_{n+2}(g_n, V_{n+2}) w\big\|_{L_p[1,a_n]} \\
&\gtrsim a_n^{-\max(\beta,0)} \eta\left(\frac{3a_n}{4}\right) (\log n)^{-1} \log n \phi(a_n) a_n^{1/2} \|p_n w\|_{L_p[\frac{a_n}{4}, \frac{a_n}{2}]} \\
&\gtrsim a_n^{-\max(\beta,0)} (\log a_n)^{-1-\hat{k}} a_n^{1/p} \eta\left(\frac{3a_n}{4}\right).
\end{aligned} \tag{4.6}$$

We now choose η at the start so that $\eta\left(\frac{3a_n}{4}\right) \geq \big(\log\log(a_n)\big)^{-1}$.
Then (4.6) implies that for $n \geq C$,

$$a_n^{1/p - \max(\beta,0)} (\log a_n)^{-1-\hat{k}} \big(\log\log(a_n)\big)^{-1} \lesssim 2m. \tag{4.7}$$

Suppose that $\beta \leq 0$. Then (4.7) implies that for $n \geq C$,

$$a_n^{1/p} \big(\log(a_n)\big)^{-1-\hat{k}} \big(\log\log(a_n)\big)^{-1} \lesssim 2m.$$

But this is impossible for $n \geq C$. So necessarily $\beta > 0$. Then (4.7) implies for $n \geq C$,

$$a_n^{1/p-\beta} \big(\log(a_n)\big)^{-1-\hat{k}} \big(\log\log(a_n)\big)^{-1} \lesssim 2m.$$

If $\beta < 1/p$ we again obtain a contradiction. So necessarily $\beta \geq 1/p$. $\square$

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References

1. S.B. Damelin, *The weighted Lebesgue constant of Lagrange interpolation for exponential weights on* $[-1, 1]$, Acta-Mathematica (Hungarica)., **81**(3), (1998), pp 211-228.
2. S. B. Damelin, *The Lebesgue function and Lebesgue constant of Lagrange interpolation for Erdős weights.*, J. Approx. Theory, **94**(2), (1998), pp 235-262.
3. S. B. Damelin, *On the distribution of general interpolation arrays for exponential weights*, Electronic Transactions of Numerical Analysis, **13**(2002), pp 12-21.
4. S. B. Damelin, *Marcinkiewicz-Zygmund inequalities and the Numerical approximation of singular integrals for non symmetric exponential weights*, submitted.
5. S. B. Damelin, *The Hilbert transform and orthonormal expansions for exponential weights, Approximation Theory X: Abstract and Classical Analysis, Chui, Schumaker and Stoekler (eds), Vanderbilt Univ. Press (2002), pp 117-135, Erratum in Necessary and Sufficient conditions for uniform convergence of orthonormal expansions on the line and uniform bounds on weighted Hilbert transforms, submitted.*
6. S. B. Damelin, H. S. Jung and K. H. Kwon, *Necessary conditions for mean convergence of Lagrange interpolation for exponential weights*, J. Comput. Appl. Math., **132**(2), (2001), pp 357-369.
7. S. B. Damelin, H. S. Jung and K. H. Kwon, *Converse Marcinkiewicz-Zygmund inequalities on the real line with applications to mean convergence of Lagrange interpolation.*, Analysis, **22**(1), (2002), pp 33–56.
8. S. B. Damelin, H. S. Jung and K. H. Kwon, *Convergence of Hermite and Hermite-Fejer interpolation of higher order for Freud weights*, J. Approx. Theory, **113**(1), (2001), pp 21–58.
9. S. B. Damelin, H. S. Jung and K. H. Kwon, *A note on mean convergence of Lagrange interpolation in* $L_p (0 < p \leq 1)$, J. Comput. Appl. Math., **133**(1-2), (2001), pp 277-282.
10. S. B. Damelin, H. S. Jung and K. H. Kwon, *On mean convergence of Hermite and Hermite Féjer for Erdős weights*, J. Comput. Appl. Math., **137**(1), (2001), pp 71–76.
11. S. B. Damelin and D. S. Lubinsky, *Necessary and sufficient conditions for mean convergence of Lagrange interpolation for Erdős weights.*, Can. J. Math., **48**(4), (1996), pp 710–736.

12. S. B. Damelin and D. S. Lubinsky, *Necessary and sufficient conditions for mean convergence of Lagrange interpolation for Erdős weights II.*, Can. J. Math., **48**(4), (1996), pp 737–757.
13. H. König, *Vector valued Lagrange interpolation and mean convergence of Hermite series.*, Proc. Essen Conference on Functional Analysis, North Holland.
14. E. Levin and D.S. Lubinsky, *Orthogonal polynomials for exponential weights.*, CMS Books in Mathematics/Ouvrages de Mathematiques de la SMC, 4. Springer-Verlag, New York, 2001.
15. D. S. Lubinsky, *Mean convergence of Lagrange interpolation for exponential weights on* $[-1, 1]$., Can. J. Math., **50**(6), (1998), pp 1273–1297.
16. D. S. Lubinsky, *On Converse Marcinkiewicz-Zygmund inequalities in* $L_p, p > 1$, Const. Approx., **15**(4), (1999), pp 577–610
17. D. S. Lubinsky, *Converse quadrature sum inequalities for polynomials with Freud weights.*, Acta Sci. Math. (Szeged), **60**, (1995), pp 525–557.
18. D. S. Lubinsky, *An Update on Orthogonal Polynomials and Weighted Approximation on the Real line*, Acta Applc. Math., **33**, (1993), 121–164.
19. D. S. Lubinsky, *On mean convergence of Lagrange interpolation for general arrays*, J. Approx. Theory, **104**, (2000), pp 220-225.
20. D. S. Lubinsky and G. Mastroianni, *Mean convergence of extended Lagrange interpolation with Freud weights.*, Acta Sci. Math. **84**(1-2), (1999), pp 47–63.
21. D. S. Lubinsky and G. Mastroinanni, *Converse quadrature sum inequalities for Freud weights II*, preprint.
22. D.S. Lubinsky and D.M. Matjila, *Necessary and Sufficient Conditions for Mean Convergence of Lagrange Interpolation for Freud Weights*, SIAM J. Math. Anal., **26**,(1995), pp 238-262
23. D.M. Matjila, *Convergence of Lagrange interpolation for Freud weights in weighted* $L_p(\mathbb{R})$, $0 < p \leq 1$., Nonlinear Numerical Methods and Rational Approximation II, A. Cuyt ed, Kluwer Academic Publishers, (1994), pp 25-35.
24. P. Nevai, *Orthogonal Polynomials, "Memoirs Amer.Math.Soc."*, **213**, Amer.Math.Soc, Providence, R.I, 1979.
25. P. Nevai, *Geza Freud, orthogonal polynomials and Christoffel functions: a case study*, J. Approx. Theory, **48**, (1986), 3–167.
26. J. Szabados, *Weighted Lagrange interpolation and Hermite–Fejer interpolation on the real line.*, J. of Ineq. and Appl., **1**, (1997), pp 99–123.
27. J. Szabados, *Where are the nodes of 'good' interpolation polynomials on the real line?*, J. Approx. Theory, **103**, (2000), pp 357-359.
28. J. Szabados and P. Vértesi, *Interpolation of Functions*, World Scientific, Singapore, 1991.
29. P. Vértesi, *On the Lebesgue function of weighted Lagrange interpolation, (Freud-Type weights)*, Const. Approx., **15**(3), (1999), pp 355-367.
30. P. Vértesi, *On the Lebesgue function of weighted Lagrange interpolation II*, J. Austr. Math. Soc. Ser. A, **65**, (1998), 2, pp 145-162.