BOUNDEDNESS AND UNIFORM NUMERICAL APPROXIMATION OF THE WEIGHTED HILBERT TRANSFORM ON THE REAL LINE

S. B. Damelin

Department of Mathematics and Computer Science,
Georgia Southern University, P.O. Box 8093,
Statesboro, GA 30460-8093.
email: damelin@gsu.cs.gasou.edu

Kai Diethelm

Institut für Angewandte Mathematik, Technische Universität
Braunschweig, Pockelsstraße 14, D-38106 Braunschweig,
Germany. email: K.Diethelm@tu-bs.de

Abstract

We establish the uniform boundedness of the weighted Hilbert transform in function spaces associated with a class of even weights on the real line with varying rates of smooth decay near $\pm\infty$. We then consider the numerical approximation of the weighted Hilbert transform and to this end we establish convergence results and error estimates which we prove are sharp. Our formulae are based on polynomial interpolation at the zeros of orthogonal polynomials associated with the weight function under consideration, augmented by two carefully chosen extra points. Typical examples of weights that are studied are: (a) $w_\alpha(x) := \exp(-|x|^\alpha)$, $\alpha > 1$, $x \in \mathbb{R}$; (b) $w_{k,\beta}(x) := \exp\left(-\exp_k\left(|x|^\beta\right)\right)$, $\beta > 0$, $k \geq 1$, $x \in \mathbb{R}$.

Mathematics Subject Classification (2000 Revision): 41A55, 65D30, 65R10.

Keywords: Cauchy principal value integral, Hilbert transform, Interpolatory quadrature formula, Erdős weight function, Error estimate, Freud weight function, Function of the second kind.

1 Introduction and Statement of Main Results

Let I denote the open interval $(-\infty, \infty)$ and $w : I \to (0, \infty)$ a weight function with all power moments

$$\int_I x^n w^2(x)dx,\ n \geq 0$$

finite. We consider the Banach space $(L_{\infty,w,I}, \|\cdot\|) =: L_{\infty,w}$ of continuous functions $f : I \to \mathbb{R}$ satisfying

$$\lim_{|x|\to\infty} f(x)w(x) = 0$$

with norm $\|f\| := \max\{|f(x)w(x)| : x \in I\}$ and define the weighted Hilbert transform

$$H[f; w^2](x) := \fint_I \frac{f(t)}{t-x} w^2(t)dt = \lim_{\varepsilon\to 0+} \int_{|t-x|\geq\varepsilon} \frac{f(t)}{t-x} w^2(t)dt, \quad x \in I, \tag{1}$$

i.e. the strongly singular integral in (1) is to be interpreted in the Cauchy principal value sense. It appears in numerous areas in analysis, in particular, weighted approximation, numerical quadrature, integrable systems, and orthogonal polynomials. We refer the reader to [3, 7, 8, 9, 10, 11, 13, 14, 16, 17, 20, 21, 24, 25, 28, 29, 30, 31, 32, 33, 37, 38] and the many references cited therein for a detailed account of this vast topic. Our interest in the operator given by (1) is to firstly establish its boundedness in suitable weighted subspaces of $L_{\infty,w}$ and secondly, to numerically approximate it by a quadrature procedure, which is based on polynomial interpolation at the zeros of orthogonal polynomials associated with the weight w under consideration augmented by two carefully chosen points. Interest in the numerical evaluation of the weighted Hilbert transform is primarily due to the fact that integral equations with Cauchy principal value integrals have shown to be an adequate tool for the modelling of many physical situations. However, only a small number of publications, see [7, 13], deal with this problem for the large classes of functions and weights presented here. Typically, our classes of functions are allowed to increase exponentially without bound near $\pm\infty$ and so our weights are chosen to counteract this growth.

1.1 Boundedness of the Weighted Hilbert Transform

It is a classical result, see [28, Chapter II, §2], that for finite intervals $[a, b]$, the unweighted Hilbert transform $H[\cdot; 1]$ is a bounded operator from $L_p[a, b]$ to $L_p[a, b]$ for $1 < p < \infty$ and is unbounded if $p = \infty$. In the latter case, results of [2] imply that $H[\cdot; 1]$ is bounded in suitable subspaces of $L_\infty[a, b]$. In a weighted setting, there are numerous old and quite recent results which deal with the boundedness of the weighted Hilbert transform in various weighted spaces from L_p to L_p and from L_p to L_∞. We refer the reader to [8, 10, 11, 20, 24, 25, 28, 30, 31, 32] and the references cited therein for an account of some of the results in this direction. The boundedness of $H[\cdot; w^2]$ in subspaces of $L_{\infty,w}$, however, is far less known. The only result in this direction, to our knowledge, is [13, Thm. 3.1] which deals with the Hermite weight $w(t) = \exp(-t^2/2)$, $t \in I$. This weight is a special case of the class of weights presented here.

In our first result, Theorem 1.2, we establish for large classes of exponential weights w on I, the boundedness of $H[\cdot; w^2]$ in suitable weighted Dini-Lipschitz type subpaces of $L_{\infty,w}$. Our weights are of the form

$$w(t) = \exp(-Q(t)),$$

where the function $Q : I \to I$ is smooth and of sufficiently fast growth near $\pm\infty$. Our class of weights include the canonical examples:

(a) $w_\alpha(t) := \exp\left(-|t|^\alpha\right)$, $\alpha > 1$, $t \in I$.

(b) $w_{k,\beta}(t) := \exp\left(-\exp_k\left(|t|^\beta\right)\right)$, $\beta > 0$, $k \geq 1$, $t \in I$, where $\exp_k$ denotes the kth iterated exponential.

For those who are not familiar, the weights listed above are respectively examples of Freud and Erdős weights. Freud weights are characterized by their smooth polynomial decay at infinity, and Erdős weights by their faster than smooth polynomial decay at infinity. For a detailed exposition of our classes of weights, we refer the reader to [4, 12, 19, 22] and the references cited therein.

In order to formulate our results, we need to define our class of admissible weights w and a suitable Dini-Lipschitz subspace of $L_{\infty,w}$. Let us agree from this point that I_+ will denote the interval $(0,\infty)$. For any two sequences (b_n) and (c_n) of non-zero real numbers we shall write $b_n \lesssim c_n$ ($b_n \gtrsim c_n$), if there exists a constant $C_1 > 0$ independent of n such that

$$b_n \leq C_1 c_n \ (b_n \geq C_1 c_n), \ n \to \infty$$

$b_n \sim c_n$ if

$$b_n \lesssim c_n \text{ and } c_n \lesssim b_n$$

and $b_n = o(c_n)$ if

$$\lim_{n\to\infty} \frac{b_n}{c_n} = 0.$$

Analoguous notation will be used for functions and sequences of functions. Throughout, C will denote a positive constant which may take on different values at different times, and Π_n will denote the class of polynomials of degree at most $n \geq 0$.

We need to define a class of admissible weights in the following sense.

Definition 1.1 *Let $w = \exp(-Q)$ where $Q : I \to I$ is even and continuous. The function w shall be called an admissible weight and we shall write $w \in \mathcal{A}$ if it has the following defining properties.*

(a) *Q''' and Q'' exist and are nonnegative in I^+. Moreover Q' exists and is strictly positive in I^+.*

(b) *The function*

$$T(x) := 1 + \frac{xQ''(x)}{Q'(x)}, \ x \in I_+$$

satisfies uniformly for $x \in I_+$

$$T(x) \sim \frac{Q'(x)}{Q(x)}.$$

Moreover T satisfies either:

(b1) *There exists* $A>1$ *and* $B>1$ *such that*

$$A \leq T(x) \leq B,\, x \in I_+,$$

or

(b2) T *is increasing in* I_+ *with*

$$\lim_{|x|\to\infty} T(x)=\infty, \qquad \lim_{x\to 0+} T(x)>0,$$

$$\forall \varepsilon>0,\; T(x) \lesssim (Q(x))^{\varepsilon},\, x \in I^+$$

and for $|x| \geq C$,

$$|T'(x)| \lesssim \frac{T^2(x)}{x}.$$

(c) *We have uniformly for large enough* $x \in I_+$,

$$\log Q(x+1) \lesssim \exp(Q(x-1)).$$

The essence of Definition 1.1 is that it covers as special cases the archetypal examples given above. In particular, (b1) forces the function Q (that is usually called the external field of the weight function w) to grow as a polynomial near infinity, so we have a Freud weight, while (b2) forces the external field to grow faster than a polynomial at infinity, which yields an Erdős weight. Thus we will treat simultaneously, even weights with different rates of decay near $\pm\infty$. The assumptions in (a) and (b) are weak smoothness and regularity assumptions which are at present sufficient for our methods of proof. Finally we note that we will often need to compare the behaviour of $Q(x+1)$ with $Q(x-1)$ uniformly for all large enough x. To this end we require the weak assumption (c) which is certainly satisfied by all our prime examples.

For what follows, we shall need to construct a suitable subspace of $L_{\infty,w}$. To this end, we shall define some important quantities:

Let a_u, $u>0$ be the positive root of the equation

$$u=\frac{2}{\pi}\int_0^1 \frac{a_u t Q'(a_u t)}{\sqrt{1-t^2}}dt$$

which exists under our hypotheses on w. For those who are are unfamiliar, one of its uses is the Mhaskar-Saff identity (e.g., [33])

$$\forall n \geq 1 \, \forall P \in \Pi_n : \|Pw\|_{L_\infty(I)} = \|Pw\|_{L_\infty[-a_n, a_n]},$$

which establishes where the sup norm of a weighted polynomial of degree at most n is supported. In particular, it is well known, see [22, 23], that for admissible Freud and Erdős weights,

$$\frac{a_u \log u}{u} \to 0, u \to \infty.$$

Now let

$$\sigma(u) := \inf \left\{ a_n : \frac{a_n}{n} \leq u \right\}, \, u > 0$$

and for $h > 0$, define

$$\Delta_h^1(f, x, I) := f(x + h/2) - f(x - h/2), \, x \pm h/2 \in I.$$

$\Delta_h^1(f, \cdot, I)$ is just the first symmetric difference operator, while σ, see [19, 12], typically satisfies the relation

$$\sigma\left(\frac{a_u}{u}\right) \sim a_u, \, u > 0. \tag{2}$$

Thus the function σ can be thought of as the inverse of the map

$$u \mapsto \frac{a_u}{u}.$$

Moreover, we shall be using a suitable modulus of continuity that will be defined as follows.

Let w first be a Freud weight. Then, see [19], we define:

$$\begin{aligned} \omega_{1,\infty}(f, w, u) &:= \sup_{0 < h \leq u} \|w \Delta_h^1(f, \cdot, I)\|_{L_\infty[-\sigma(h), \sigma(h)]} \\ &\quad + \inf_{P \in \Pi_0} \|(f - P)w\|_{L_\infty(I \setminus [-\sigma(u), \sigma(u)])}. \end{aligned} \tag{3}$$

Moreover, if w is an Erdős weight, see [12], we set for some fixed and large enough $L \geq L_0$

$$\begin{aligned} \omega_{1,\infty}(f, w, u) &:= \sup_{0 < h \leq u} \|w \Delta_{Lh\Phi_h(\cdot)}^1(f, \cdot, I)\|_{L_\infty[-\sigma(2h), \sigma(2h)]} \\ &\quad + \inf_{P \in \Pi_0} \|(f - P)w\|_{L_\infty(I \setminus [-\sigma(4u), \sigma(4u)])} \end{aligned} \tag{4}$$

where

$$\Phi_u(x) := \left|1 - \frac{|x|}{\sigma(u)}\right|^{1/2} + (T(\sigma(u)))^{-1/2} \quad \text{for } u > 0 \text{ and } x \in I.$$

We briefly recall, see [12, 4], that the function Φ_u illustrates endpoint improvements in the degree of weighted polynomial approximation for Erdős weights in the Mhaskar-Rakhmanov-Saff interval while no such effect is present for Freud weights.

We now set:

$$L_{\infty,w,\omega} := \left\{ f \in L_{\infty,w} : \int_0^1 \frac{\omega_{1,\infty}(f,w,u)}{u} du < \infty \right\} \tag{5}$$

and state our first result:

Theorem 1.2 *Let $w \in \mathcal{A}$. Then uniformly for $f \in L_{\infty,w,\omega}$*

$$\|H[f;w^2]\|_{L_\infty(I)} \lesssim \left[\|fw\|_{L_\infty(I)} + \int_0^1 \frac{\omega_{1,\infty}(f,w,u)}{u} du \right]. \tag{6}$$

We remark that, in particular, (6) shows that $H[\cdot;w^2]$ is a bounded map from $L_{\infty,w,\omega}$ to $L_{\infty,w}$.

We now specialize Theorem 1.2 to Freud weights and discuss some simple but particularly important subsets X of $L_{\infty,w,\omega}$ with the property that the operator $H[\cdot;w^2]$ is a bounded map from X to $L_{\infty,w}$. This is contained in:

Corollary 1.3 *Let w be an admissible Freud weight. For $f \in L_{\infty,w,\omega}$ and $n \geq 1$, we set*

$$\begin{aligned} E_{n,\infty}[f,w] &:= \inf_{P \in \Pi_n} \|(f-P)w\|_{L_\infty(I)} \\ &= \|(f-P_n^*)w\|_{L_\infty(I)}. \end{aligned}$$

Here P_n^* denotes the best approximating polynomial where the infimum above is attained. Let X be the set of functions $f \in L_{\infty,w,\omega}$ satisfying

$$E_{n,\infty}[f,w] = O\left(\frac{a_n}{n}\right)^\alpha \tag{7}$$

for some $0 < \alpha < 1$, or

$$\|P_n^* w\|_{L_\infty(I)} = O(1). \tag{8}$$

Then $H[\cdot; w^2]$ is a bounded map from X to $L_{\infty,w}$. In particular, defining the Sobolev function space

$$L'_{\infty,w,\omega} := \left\{ f \in L_{\infty,w} : \|fw\|_{L_\infty(I)} + \|f'w\|_{L_\infty(I)} < \infty \right\},$$

we have that $H[\cdot; w^2]$ is a bounded map from $L'_{\infty,w,\omega}$ to $L_{\infty,w}$.

1.2 Numerical Integration of the Weighted Hilbert Transform on I

We consider the problem of the numerical approximation of the weighted Hilbert transform given by (1). The quadrature formulae we investigate are so-called *interpolatory product methods*, given by the following approach: For a given admissible weight w, we denote by

$$p_n(w^2, x) = \gamma_n(w^2)x^n + \dots, \qquad \gamma_n(w^2) > 0,\ n \geq 0$$

the unique orthonormal polynomials satisfying

$$\int_I p_n(w^2, x) p_m(w^2, x) w^2(x) dx = \delta_{mn}, \quad m, n = 0, 1, 2, \dots \tag{9}$$

and denote by

$$x_{n,n}(w^2) < x_{n-1,n}(w^2) < \dots < x_{2,n}(w^2) < x_{1,n}(w^2)$$

their n real simple zeros. Moreover, following [36, 5], we define $x_{0,n}$ to be a point with the property that

$$|p_n(x_{0,n})w(x_{0,n})| = \|p_n w\|_{L_\infty(I)}$$

and set $x_{n+1,n} = -x_{0,n}$. Here and in the following, we shall often use the abbreviated notation p_n for $p_n(w^2, \cdot)$. It is instructive to note that in [5, 36], it was shown that $\pm x_{0,n}$ are close to the largest and smallest zeros of $p_n(w^2, \cdot)$. The points $x_{j,n}$, $j = 0, 1, \dots n, n+1$ will serve as the nodes of our quadrature formula H_n which is therefore defined by

$$H_n[f; w^2](x) = \sum_{j=0}^{n+1} w_{j,n}(x) f(x_{j,n}). \tag{10}$$

The weights $w_{j,n}$ are chosen such that the quadrature error R_n satisfies

$$R_n[f; w^2](x) := H[f; w^2](x) - H_n[f; w^2](x) = 0$$

for every x and every $f \in \Pi_{n+1}$. It is clear then that H_n must be defined according to

$$H_n[f; w^2](x) := H[L_{n+2}[f]; w^2](x)$$

where $L_{n+2}[f]$ is the (Lagrange) interpolating polynomial for the function f with nodes $x_{j,n}$, $j = 0, 1, \ldots, n+1$. For simplicity, we will henceforth suppress the dependence of H_n on w. In other words, we shall write

$$H_n[f](x) := H_n[f; w^2](x).$$

We briefly mention that the quadrature formulae obtained according to this product integration approach, but using the zeros of the orthogonal polynomials only and not the extra points $x_{0,n}$ and $x_{n+1,n}$, have been investigated in [7] for a class of admissible Freud weights using estimates for functions of the second kind. Comparing our new results with those of that paper, we see that the introduction of the extra points leads to improved error estimates. See also Remark 4.1 below. The observation that the addition of the extra points gives rise to improved behavior in weighted polynomial approximation has also been made in connection with weighted mean and uniform convergence of various Lagrange interpolation processes whereas for weighted Hermite-Fejér interpolation this phenomenon is not present. A completely different approach for the numerical approximation of Hilbert transforms has been proposed in [18], albeit without a very detailed error analysis.

Our main results in this section are as follows:

Theorem 1.4 *Let $w \in \mathcal{A}$ and $f \in L_{\infty,w}$. Then uniformly for f and large enough n,*

$$\|L_{n+2}[f]w\|_{L_\infty(I)} \lesssim \|fw\|_{L_\infty(I)} \log n. \tag{11}$$

The above result indicates that $(L_{n+2})_{n=1}^{\infty}$ is an unbounded sequence of operators from $L_{\infty,w}$ to $L_{\infty,w}$ whose growth, however, is only very moderate; indeed, from classical results for corresponding problems on bounded intervals we may assume that a sequence of interpolation operators with slower growing norms does not exist. Weaker versions of this result have been proved in [5, 36] and L_p analogues can be found for example in [8].

Theorem 1.5 (a) *Let $w \in \mathcal{A}$ and $f \in L_{\infty,w}$. Then uniformly for f and large enough n*

$$\|H_n[f]\|_{L_\infty(I)} \lesssim \|fw\|_{L_\infty(I)} \log n \tag{12}$$

and

$$\|R_n[f]\|_{L_\infty(I)} \lesssim \int_0^1 \frac{\omega_{1,\infty}(f,w,u)}{u} du + \|fw\|_{L_\infty(I)} \log n. \tag{13}$$

(b) *Moreover, if w is an admissible Freud weight and $f^{(j)} \in L_{\infty,w,\omega}$ for $j = 0, 1$, then uniformly for f and large enough n*

$$\|R_n[f]\|_{L_\infty(I)} \lesssim \int_0^{a_n/n} \frac{\omega_{1,\infty}(f,w,u)}{u} du + E_{n+1,\infty}[f,w] \log n. \tag{14}$$

In particular, uniformly for f and large enough n

$$\|R_n[f]\|_{L_\infty(I)} \lesssim \frac{a_n \log n}{n}. \tag{15}$$

In order to assess the quality of the estimate (15), we now state the following theorem which shows, in particular, that the bound in (15) is sharp and cannot be improved in general. Moreover, it follows that no other quadrature rule can exist that allows a better estimate.

Theorem 1.6 *Let $s \in \mathbb{N}_0$, $n \in \mathbb{N}$, and $w \in \mathcal{A}$. If the quadrature rule Q_n is given by*

$$Q_n[f](x) = \sum_{j=1}^{n} a_{j,n}(x) f(x_{j,n})$$

with arbitrary functions $a_{j,n}$ and arbitrary real nodes $x_{j,n}$, then there exist two positive constants C_1, C_2, both of which are independent of n, and a function $f^* \in L_{\infty,w}$ (that may depend on n) with $\|w^2 f^{*(s)}\|_{L_\infty(I)} \le C_1$ such that

$$\sup_{x\in I} |H[f^*; w^2](x) - Q_n[f^*](x)| \ge C_2 \left(\frac{a_n}{n}\right)^s \log n. \tag{16}$$

We have thus shown the nonexistence of a quadrature rule of the form stated in the theorem that allows an error estimate of the form

$$\sup_{x\in I}|H[f;w^2](x) - Q_n[f](x)| = o\left(\left(\frac{a_n}{n}\right)^s \log n\right)$$

for all f with a bounded sth derivative. Actually, a careful inspection of the proof (that we shall give in Section 5), reveals that the statement holds for a much larger class of quadrature formulae, namely for formulae with multiple nodes of arbitrary order r, i.e. for methods of the form

$$\sum_{k=0}^{r-1}\sum_{j=1}^{n} a_{j,k,n}(x) f^{(k)}(x_{j,n}).$$

Using Theorem 1.6, we conjecture that (15) should hold for a suitable subspace of $L_{\infty,w}$ for admissible Erdős weights as well. We delay this investigation, however, for a future paper.

The remainder of this paper is organized as follows. In Section 2, we present the proofs of Theorem 1.2 and Corollary 1.3, and in Section 3 we present some auxiliary results which are of independent interest and the Proof of Theorem 1.4. In Section 4, we present the proof of Theorem 1.5 and in Section 5, we present the proof of Theorem 1.6. Section 6 contains the proof of a technical lemma, Lemma 5.1, which we use to prove Theorem 1.6.

2 The Proofs of Theorem 1.2 and Corollary 1.3

In this section, we present the proofs of Theorem 1.2 and Corollary 1.3. We begin with the former.

2.1 The Proof of Theorem 1.2

We first fix a positive constant C. For later use, we mention that it will be important to choose C large enough to ensure that

$$\forall x \geq C: \quad \frac{1}{Q'(x+1)} < 1.$$

Then we choose $x \in I$ and consider first the case that $|x| \geq C$. Without loss of generality, we may suppose that x is positive for the symmetric case $x \leq -C$ is similar. Let us write

$$\begin{aligned} H[f; w^2](x) &= \left(\int_{|t|>2x} + \int_{-2x}^{0} + \int_{0}^{2x} \right) \frac{f(t)w^2(t)}{t-x} dt \qquad (17) \\ &= I_1(x) + I_2(x) + I_3(x). \end{aligned}$$

We first estimate $I_1(x)$. In the region of integration for this integral, we note

$$|t-x| \geq |t| - |x| \geq |t| - |t|/2 = |t|/2.$$

Thus,

$$\begin{aligned} |I_1(x)| &\leq 2 \int_{|t|>2x} \frac{|f(t)|w^2(t)}{|t|} dt \\ &\leq 2 \int_{|t|>2} \frac{|f(t)|w^2(t)}{|t|} dt \\ &\leq 4\|fw\|_{L_\infty(I)} \int_2^\infty \frac{w(t)}{t} dt \\ &\lesssim \|fw\|_{L_\infty(I)}. \end{aligned}$$

Thus

$$|I_1(x)| \lesssim \|fw\|_{L_\infty(I)}. \qquad (18)$$

Next, we observe that

$$\begin{aligned} |I_2(x)| &= \left| \int_{-2x}^{0} \frac{f(t)w^2(t)}{t-x} dt \right| \\ &\leq \|fw\|_{L_\infty(I)} \int_{-2x}^{0} \frac{w(t)}{x-t} dt \\ &\leq \|fw\|_{L_\infty(I)} w(0) \int_x^{3x} \frac{du}{u} \\ &\lesssim \|fw\|_{L_\infty(I)}. \end{aligned}$$

Thus similarly we see that

$$|I_2(x)| \lesssim \|fw\|_{L_\infty(I)}. \qquad (19)$$

We now proceed to estimate $I_3(x)$. To this end, we find it convenient to set for the given x

$$A(x) = \frac{1}{Q'(x+1)}.$$

Observe that by choice of $A(x)$

$$w(y) \sim w(x) \tag{20}$$

uniformly for every $y \in I$ with $|x-y| \leq A(x)$. To see this, we recall that C was chosen large enough from the start to ensure that

$$\frac{1}{Q'(x+1)} < 1.$$

Then

$$\begin{aligned}
\frac{w(x)}{w(y)} &\leq \frac{w(x)}{w\left(x + \frac{1}{Q'(x+1)}\right)} \\
&= \exp\left(Q\left(x + \frac{1}{Q'(x+1)}\right) - Q(x)\right) \\
&\leq \exp\left(Q'\left(x + \frac{1}{Q'(x+1)}\right)\frac{1}{Q'(x+1)}\right) \\
&\leq \exp\left(Q'(x+1)\frac{1}{Q'(x+1)}\right) = \exp(1).
\end{aligned}$$

We then split $I_3(x)$ as follows: Write

$$\begin{aligned}
I_3(x) &= \int_0^{2x} \frac{f(t)w^2(t)}{t-x}dt \\
&= \left(\int_0^{x-1} + \int_{x-1}^{x-A(x)} + \fint_{x-A(x)}^{x+A(x)} + \int_{x+A(x)}^{2x}\right) \frac{f(t)w^2(t)}{t-x}dt \\
&= I_{31}(x) + I_{32}(x) + I_{33}(x) + I_{34}(x).
\end{aligned}$$

Firstly,

$$|I_{31}(x)| \lesssim \|fw\|_{L_\infty(I)} \int_0^{x-1} \frac{w(t)}{x-t}dt \lesssim \|fw\|_{L_\infty(I)}\|w\|_{L_1(I)} \lesssim \|fw\|_{L_\infty(I)}.$$

Thus

$$|I_{31}(x)| \lesssim \|fw\|_{L_\infty(I)}. \tag{21}$$

Next,

$$\begin{aligned}|I_{32}(x)| &\lesssim \|fw\|_{L_\infty(I)} \int_{x-1}^{x-A(x)} \frac{w(t)}{x-t}dt \\ &\lesssim \|fw\|_{L_\infty(I)} w(x-1) \log Q'(x+1) \\ &\lesssim w(x-1)\log Q(x+1)\|fw\|_{L_\infty(I)} \lesssim \|fw\|_{L_\infty(I)}.\end{aligned}$$

In the last line, we used assumption (c) in Definition 1.1 and the fact that

$$Q'(y) \lesssim (Q(y))^C$$

for large enough y. This inequality follows immediately from assumption (b1) in Definition 1.1 in the Freud case. In the Erdős case we use the following argument. Since

$$\frac{d}{dx}\log Q'(x) = \frac{Q''(x)}{Q'(x)} \sim \frac{Q'(x)}{Q(x)} = \frac{d}{dx}\log Q(x)$$

and $Q(x) \to \infty$ as $|x| \to \infty$, it follows that

$$C_1 \log Q'(x) \leq \log Q(x) \leq C_2 \log Q'(x).$$

Thus

$$|I_{32}(x)| \lesssim \|fw\|_{L_\infty(I)}. \tag{22}$$

Similarly, we find that

$$\begin{aligned}|I_{34}(x)| &= \left|\int_{x+A(x)}^{2x} \frac{f(t)w^2(t)}{t-x}dt\right| \\ &\lesssim \|fw\|_{L_\infty(I)} w(x+A(x)) \int_{A(x)}^{x} \frac{1}{u}du \\ &\lesssim \|fw\|_{L_\infty(I)} w(x)(\log Q'(x+1) + \log x) \\ &\lesssim \|fw\|_{L_\infty(I)}.\end{aligned}$$

Thus the above estimate together with (21) and (22) yields for the given x

$$|I_{31}(x) + I_{32}(x) + I_{34}(x)| \lesssim \|fw\|_{L_\infty(I)}. \tag{23}$$

It remains to bound $|I_{33}(x)|$. This is the most difficult estimate and to this end, we will need the definition of our moduli of smoothness in (3) and (4).

We write

$$
\begin{aligned}
|I_{33}(x)| &= \left|\int_{x-A(x)}^{x+A(x)} \frac{f(t)w^2(t)}{t-x}dt\right| \\
&\leq \left|\int_{x-A(x)}^{x+A(x)} f(t)\frac{w^2(t)-w^2(x)}{t-x}dt\right| \\
&\quad + w^2(x)\left|\int_{x-A(x)}^{x+A(x)} \frac{f(t)-f(x)}{t-x}dt\right| \\
&= |I_{331}(x)| + |I_{332}(x)|. \qquad (24)
\end{aligned}
$$

Here we used the triangle inequality and the identity

$$
\int_{x-A(x)}^{x+A(x)} \frac{1}{t-x}dt = 0.
$$

Suppose first that w is a Freud weight; the problem for Erdős weights will be discussed below. We begin by making the substitution $t = u/2 + x$ into $I_{332}(x)$ in (24). Recall that x is fixed large enough and so both u and t vary. Thus

$$
\begin{aligned}
|I_{332}(x)| &\lesssim w^2(x)\int_0^{2A(x)} \left|\frac{f(x+u/2)-f(x-u/2)}{u}\right| du \\
&\lesssim w^2(x)\left[\int_{\substack{0<u\leq 2A(x)\\ \sigma(u)\geq x}} + \int_{\substack{0<u\leq 2A(x)\\ \sigma(u)<x}}\right]\left|\frac{f(x+u/2)-f(x-u/2)}{u}\right| du \\
&= I_{3321}(x) + I_{3322}(x). \qquad (25)
\end{aligned}
$$

First observe that

$$
\begin{aligned}
I_{3321}(x) &\lesssim \int_0^{2A(x)} \max_{|y|\leq\sigma(u)} \left|\Delta_u^1(f,y,I)w^2(y)\right| \frac{1}{u}du \\
&\leq \int_0^{2A(x)} \sup_{0<h\leq u} \left\|\Delta_h^1(f,y,I)w(y)\right\|_{L_\infty(|y|\leq\sigma(h))} \frac{1}{u}du \\
&\leq \int_0^1 \frac{\omega_{1,\infty}(f,w,u)}{u}du. \qquad (26)
\end{aligned}
$$

Next we estimate $I_{3322}(x)$. Observe that as $x > \sigma(u)$ and $0 < u/2 \leq A(x)$, we may apply (20) to deduce that

$$
\begin{aligned}
\left|\Delta_u^1(f,x,I)w^2(x)\right| &\leq \left|f(x+u/2)w^2(x)\right| + \left|(f(x-u/2))w^2(x)\right| \\
&\leq \|fw\|_{L_\infty(I)} \max\left\{w^2(x)w^{-1}(x+u/2), w(x)\right\} \\
&\lesssim \|fw\|_{L_\infty(I)}w(x) \lesssim \|fw\|_{L_\infty(I)}w(\sigma(u)). \qquad (27)
\end{aligned}
$$

Combining (26) and (27), we see that

$$
\begin{aligned}
I_{332}(x) &\lesssim \int_0^1 \frac{\omega_{1,\infty}(f,w,u)}{u}du + \|fw\|_{L_\infty(I)} \int_0^{A(x)} \frac{w(\sigma(u))}{u}du \\
&\lesssim \int_0^1 \frac{\omega_{1,\infty}(f,w,u)}{u}du + \|fw\|_{L_\infty(I)} \int_0^1 \frac{1}{u^{1/2}}du \\
&\lesssim \int_0^1 \frac{\omega_{1,\infty}(f,w,u)}{u}du + \|fw\|_{L_\infty(I)}. \qquad (28)
\end{aligned}
$$

In the last line we used the fact that for u sufficiently small, and thus for $\sigma(u)$ sufficiently large, $w(\sigma(u)) \le u^{1/2}$. We may ensure this by simply choosing C large enough at the start.

We now proceed with the estimate of $|I_{331}(x)|$. To this end, we write

$$
\begin{aligned}
|I_{331}(x)| &= \left| \int_{x-A(x)}^{x+A(x)} f(t) \frac{w^2(t) - w^2(x)}{t-x} dt \right| \\
&\lesssim \|fw\|_{L_\infty(I)} \int_{x-A(x)}^{x+A(x)} w^{-1}(t) |(w^2)'(\eta)| dt
\end{aligned}
$$

for some $\eta \in (x,t)$. Using (20), we may continue this as:

$$
\begin{aligned}
|I_{331}(x)| &\lesssim A(x)\|wQ'\|_{L_\infty(I)} w(x-A(x)) w^{-1}(x+A(x)) \|fw\|_{L_\infty(I)} \\
&\lesssim \|fw\|_{L_\infty(I)}. \qquad (29)
\end{aligned}
$$

Finally combining (29) with (28) and (23) shows that, for the given $x \ge C$, we have the estimate

$$
\left|H[f;w^2](x)\right| \lesssim \|fw\|_{L_\infty(I)} + \int_0^1 \frac{\omega_{1,\infty}(f,w,u)}{u}du. \qquad (30)
$$

Suppose now that $1 \le x < C$. Then (18) and (19) follow in exactly the same way. For I_3, we write

$$
\begin{aligned}
I_3(x) &= \int_0^{2x} \frac{w^2(t)f(t)}{t-x}dt \\
&= \left(\int_0^{x-1/x} + \fint_{x-1/x}^{x+1/x} + \int_{x+1/x}^{2x} \right) \frac{w^2(t)f(t)}{t-x}dt \\
&= I_{31}(x) + I_{32}(x) + I_{33}(x).
\end{aligned}
$$

Observe that we do not need (20) as x is bounded. Proceeding as before, we obtain

$$\begin{aligned}\left|H[f;w^2](x)\right| &\lesssim \left(x\|w\|_{L_\infty(I)} + w(1/x)\log x + 1\right)\|fw\|_{L_\infty(I)} \\ &\quad + \int_0^1 \frac{\omega_{1,\infty}(f,w,u)}{u}du \\ &\lesssim \|fw\|_{L_\infty(I)} + \int_0^1 \frac{\omega_{1,\infty}(f,w,u)}{u}du.\end{aligned}$$

If $0 \le x < 1$, we proceed much as above except we split

$$H[f;w^2](x) = \int_{-\infty}^{\infty} \frac{f(t)w^2(t)}{t-x}dt = \left(\int_{-\infty}^{x-1} + \int_{x-1}^{x+1} + \int_{x+1}^{\infty}\right)\frac{w^2(t)f(t)}{t-x}dt.$$

Thus we have shown that if w is an admissible Freud weight, we have uniformly for every $x \in I$,

$$\left|H[f;w^2](x)\right| \lesssim \|fw\|_{L_\infty(I)} + \int_0^1 \frac{\omega_{1,\infty}(f,w,u)}{u}du.$$

As the right hand side of the above equation is independent of x, we may take the maximum over all x and deduce (6).

Suppose now that w is an admissible Erdős weight. We only dicuss explicitly the case that x is sufficiently large for the other cases are similar. Inspecting the proof above, we find that the only difference between the two types of weight functions arises in the bounds for $I_{33}(x)$. Firstly, a cursory look at the proof of the estimate for $I_{331}(x)$ shows that by its construction, it works simultaneously for both admissible Freud and Erdős weights. Thus we consider (25) for large enough x and make some suitable modifications. Let us write

$$\begin{aligned}|I_{332}(x)| &\lesssim w^2(x)\int_0^{2A(x)}\left|\frac{f(x+u/2)-f(x-u/2)}{u}\right|du \\ &\lesssim w^2(x)\left(\int_{\substack{0<u\le 2A(x)\\ \sigma(2u)\ge x}} + \int_{\substack{0<u\le 2A(x)\\ \sigma(2u)<x}}\left|\frac{f(x+u/2)-f(x-u/2)}{u}du\right|\right) \\ &= I_{3321}(x) + I_{3322}(x).\end{aligned}$$

The estimate for $I_{3322}(x)$ is similar to the Freud case so it amounts to estimating $I_{3321}(x)$. We may write $x = a_s$ for some large enough $s > 0$ and so we learn, see [10, (2.16) and (2.21)], that

$$\frac{1}{Q'(x+1)} \leq \frac{1}{Q'(x)} \lesssim \frac{a_s}{s}\Phi_{2s}(x).$$

We may now make the substitution $u = Lv\Phi_v(x)$ into $I_{3321}(x)$ for some large enough but fixed $L \geq L_0$ and apply the method of [12, Theorem 1.3] to deduce that

$$I_{3321}(x) \lesssim \int_0^{\frac{a_{3t}}{3t}} \left|\Delta^1_{Lv\Phi_v(x)}(f,x,I)w^2(x)\right| \frac{1}{v}dv.$$

We remark that the choice of L ensures that upper bound on the integral is $\frac{a_{3t}}{3t}$. As t is sufficiently large, this latter fact and [10, (2.21)] allows us to argue that

$$2v \leq 2\frac{a_{3t}}{3t} \leq \frac{5}{6}\left(\frac{a_{3t}}{5t/4}\right) < (5t/4)a_{5t/4}.$$

Thus using [4, (2.13)], we deduce that

$$\sigma(2v) \geq x.$$

It follows then that we have

$$\begin{aligned} I_{3321}(x) &\leq \int_0^{\frac{a_{3t}}{3t}} \sup_{0<h\leq v} \left\|\Delta^1_{Lh\Phi_h(x)}(f,y,I)w(y)\right\|_{L_\infty(|y|\leq\sigma(2h))} \frac{1}{v}dv \\ &\lesssim \int_0^1 \frac{\omega_{1,\infty}(f,w,u)}{u}du. \end{aligned}$$

□

We observe that we actually proved more than (6) and to this end we record:

Theorem 2.1 *Assume the hypotheses of Theorem* 1.2. *Then given any* $\varepsilon > 0$ *sufficiently small,*

$$\|H[f;w^2]\|_{L_\infty(I)} \lesssim \left[\|fw\|_{L_\infty(I)} + \int_0^\varepsilon \frac{\omega_{1,\infty}(f,w,u)}{u}du\right]\log\frac{1}{\varepsilon}. \tag{31}$$

2.2 The Proof of Corollary 1.3

Assume first that $f \in L'_{\infty,w,\omega}$. We recall the inequality, see [19, Corollary 1.8],

$$\omega_{1,\infty}(f,w,u) \lesssim u\|f'w\|_{L_\infty(I)} \tag{32}$$

which holds for sufficiently small u. Now choose fixed $\varepsilon > 0$ sufficiently small so that (32) holds for all $u < \varepsilon$. Applying (31) with this ε gives the result in this case.

Suppose next that (7) or (8) hold. Then using [19, Corollary 1.6] or [6, Theorem 2.3], we deduce that for sufficiently small u and for any $0 < \alpha \leq 1$

$$\omega_{1,\infty}(f, w, u) \lesssim u^{\alpha}. \tag{33}$$

The result then follows again. $\square$

3 The Proof of Theorem 1.4

In this section, we present the proof of Theorem 1.4. We begin by fixing some notation and stating some auxiliary results that will be helpful in what follows and of independent interest.

We begin by recalling the Christoffel function λ_n, $n \geq 1$, for w^2 defined by

$$\lambda_n(w^2)(x) := \inf_{P \in \Pi_{n-1} \setminus \{0\}} \int_{-\infty}^{\infty} P^2(t) w^2(t) dt / P^2(x),\ x \in I$$

and we let

$$\lambda_{kn}(w^2) := \lambda_n(w^2)(x_{kn}),\ 1 \leq k \leq n$$

be the corresponding Cotes numbers. In describing the spacing between successive zeros of $p_n(w^2)$, we will also use the following sequences of functions:

Define for every $n \geq 1$ the sequence

$$\delta_n := (nT(a_n))^{-2/3}$$

and for $|x| \leq a_n$ let

$$\psi_n(x) := \left|1 - \frac{|x|}{a_n}\right| + \delta_n$$

and

$$\phi_n(x) := \frac{\psi_n(x) + T(a_n)^{-1}}{\sqrt{\psi_n(x)}} = \frac{\left|1 - \frac{|x|}{a_n}\right| + \delta_n + T(a_n)^{-1}}{\sqrt{\left|1 - \frac{|x|}{a_n}\right| + \delta_n}}.$$

We set $\phi_n(x) = \phi_n(a_n)$ for $|x| \geq a_n$. We recall that $T(x) > 1$ for all x by definition. Therefore, $\delta_n = o(1)$. Hence it is an immediate consequence of these definitions that, for all x,

$$\delta_n \leq \psi_n(x) \leq 1 + \delta_n \lesssim 1 \tag{34}$$

and thus

$$\begin{aligned} |\phi_n(x)| &= \sqrt{\psi_n(x)} + \frac{1}{T(a_n)\sqrt{\psi_n(x)}} \lesssim 1 + \sqrt{\delta_n^{-1}} \\ &\lesssim (nT(a_n))^{-1/3} \lesssim n^{1/3}T(a_n)^{-2/3} \end{aligned} \tag{35}$$

in view of the fact that $T(a_n) \lesssim n^2$, which was shown in [12, Lemma 2.2(e)] for Erdős weights and is fulfilled trivially because T is bounded by definition for Freud weights.

We will establish the following fundamental lemma.

Lemma 3.1 *For a given $x \in I$, let x_{dn}, $1 \leq d \leq n$ denote the node of the quadrature formula that is closest to x, and let*

$$D_n(x) := \begin{cases} \lambda_{dn}(w^2)\dfrac{L_{n+2}[f](x_{dn}) - L_{n+2}[f](x)}{x_{dn} - x} & \textit{if } x_{dn} \neq x, \\ \lambda_{dn}(w^2)L'_{n+2}[f](x_{dn}) & \textit{if } x_{dn} = x. \end{cases}$$

Then, uniformly for $|x| < a_n(1+\delta_n)$,

$$|D_n(x)| \lesssim \|wL_{n+2}[f]\|_{L_\infty(I)}. \tag{36}$$

For the proof of Lemma 3.1, we require the following rather technical result.

Lemma 3.2 (a) *The Christoffel numbers $\lambda_{dn}(w^2)$ satisfy*

$$\lambda_{dn}(w^2) \sim \frac{a_n}{n} w^2(x_{dn})\phi_n(x_{dn}) \tag{37}$$

uniformly for $n \geq 1$ and $1 \leq d \leq n$.

(b) *Uniformly for $n \geq C$ and $P \in \Pi_n$ the Markov-Bernstein inequality*

$$\|P'\Psi_n w\|_{L_\infty(I)} \lesssim \frac{n}{a_n}\|Pw\|_{L_\infty(I)} \tag{38}$$

holds, where

$$\Psi_n(x) = \begin{cases} \left(1 - \frac{|x|}{a_n}\right)^{1/2} + T(a_n)^{-1/2} & \textit{for } |x| \leq a_n, \\ T(a_n)^{-1/2} & \textit{for } |x| > a_n. \end{cases}$$

(c) *Uniformly for $n \geq 1$ and $1 \leq j \leq n-1$,*

$$x_{jn} - x_{j+1,n} \sim \frac{a_n}{n}\phi_n(x_{jn}) \tag{39}$$

and

$$\left|1 - \frac{x_{1,n}}{a_n}\right| \lesssim \delta_n.$$

(d) *Uniformly for $u > 0$ and $j = 0, 1$,*

$$Q^{(j)}(a_u) \sim \frac{uT(a_u)^{j-1/2}}{a_u^j}. \tag{40}$$

The Freud case of part (a) of this lemma follows from [22, Thm. 1.1]; the Erdős case is discussed in [23, Thm. 1.2] (see also [10, Lemma 2.1(a)]). As far as part (b) is concerned, the Erdős case is discussed in [4, Thm. 3.1]. For the Freud case, from [1, Lemma 4.3] we have

$$\|P'w\|_{L_\infty(I)} \lesssim \frac{n}{a_n}\|Pw\|_{L_\infty(I)}$$

which implies (38) in view of the fact that, for Freud weights, $T(x) \sim 1$ uniformly for all x, and hence also $\Psi_n(x) \sim 1$ uniformly with respect to x and n. The proof of part (c) for Freud weights is contained in [1, eq. (4.17)]. For Erdős weights, we take this result from [23, Corollary 1.3(b) and the remark following it]. Finally, part (d) for Freud weights follows from [1, eq. (4.3)], and for Erdős weights we take it from [12, Lemma 2.2(b)].

We may now present:

The Proof of Lemma 3.1. Owing to symmetry, we may restrict our attention to the case $x \geq 0$.

By the Mean Value Theorem, we see that

$$D_n(x) = \lambda_{dn}(w^2)L'_{n+2}[f](\xi)$$

for some ξ located between x and x_{dn}. Thus, using (37) and the Markov-Bernstein inequality (38), we deduce that

$$\begin{aligned} |D_n(x)| &= \lambda_{dn}(w^2)|L'_{n+2}[f](\xi)| \\ &\sim \frac{a_n}{n}w^2(x_{dn})\phi_n(x_{dn})|L'_{n+2}[f](\xi)| \end{aligned} \tag{41}$$

$$
\begin{aligned}
&\leq \frac{a_n}{n} w^2(x_{dn}) \frac{1}{w(\xi)} \phi_n(x_{dn}) \Psi_n(x_{dn})^{-1} \|wL'_{n+2}[f]\|_{L_\infty(I)} \\
&\lesssim w^{3/2}(x_{dn}) w^{-1}(\xi) \phi_n(x_{dn}) \Psi_n(x_{dn})^{-1} \\
&\quad \times w^{1/2}(x_{dn}) \|wL_{n+2}[f]\|_{L_\infty(I)}. \qquad (42)
\end{aligned}
$$

We now prove that

$$
w^{3/2}(x_{dn}) w^{-1}(\xi) \Psi_n(x_{dn})^{-1} \lesssim 1 \qquad (43)
$$

and

$$
w^{1/2}(x) \phi_n(x) \lesssim 1. \qquad (44)
$$

Combining (43) and (44) with (42) then gives the lemma. Thus we proceed to prove (43) and (44). We begin with (43). Here we take into consideration that the distance between x_{dn} and ξ is bounded from above by the distance between x_{dn} and its neighbour node $x_{d-1,n}$. This latter distance is by (39) bounded by $Ca_n n^{-1} \phi_n(x_{dn})$. Thus, again using the Mean Value Theorem,

$$
\begin{aligned}
w^{3/2}(x_{dn}) w^{-1}(\xi) &= w^{1/2}(x_{dn}) \exp(Q(\xi) - Q(x_{dn})) \\
&= w^{1/2}(x_{dn}) \exp((\xi - x_{dn}) Q'(\eta)) \\
&\lesssim w^{1/2}(x_{dn}) \exp(C a_n n^{-1} \phi_n(x_{dn}) Q'(x_{dn})).
\end{aligned}
$$

Using (39) and (40), the continuity of the map

$$
u \mapsto a_u, \; u > 0
$$

and the fact (see Lemma 3.2(c)) that the nodes $x_{0,n}$ and $x_{n+1,n}$ are in the interval $[-a_{2n}, a_{2n}]$ for large enough n, allows us to write $x_{dn} = a_s$ for some $s > 0$. (We may assume that x is sufficiently large for this purpose, otherwise (43) and (44) are trivial.) Hence we deduce

$$
w^{3/2}(x_{dn}) w^{-1}(\xi) \Psi_n(x_{dn})^{-1} \lesssim w^{1/2}(a_s) \exp(C n^{-1} \phi_n(a_s) Q(a_s)) \Psi_n(a_s)^{-1}.
$$

Recalling that $\phi_n(x) \leq n^{1/3} T(a_n)^{-2/3}$ for all x, see (35), we continue the estimation above by

$$
\begin{aligned}
w^{3/2}(x_{dn}) w^{-1}(\xi) \Psi_n(x_{dn})^{-1} &\lesssim w^{1/2}(a_s) \exp(C \delta_n Q(a_s)) \Psi_n(a_s)^{-1} \\
&\lesssim \exp\left(-\frac{s}{4T(a_s)^{1/2}}\right) T(a_s)^{1/2}.
\end{aligned}
$$

For Freud weights, T is uniformly bounded, and therefore we see that the last expression is bounded by $\exp(-Cs) \lesssim 1$ uniformly in s. For Erdős weights we recall [12, Lemma 2.2(b)] that states that we have some $\epsilon > 0$ such that $T(a_s) \lesssim s^{2-\epsilon}$. This implies

$$\exp\left(-\frac{s}{4T(a_s)^{1/2}}\right) T(a_s)^{1/2} \lesssim \exp\left(-\frac{s^{\epsilon/2}}{4}\right) s^{1-\epsilon/2}$$

which is bounded uniformly in s proving (43).

Now we prove (44). First observe that

$$\phi_n(x) \sim \begin{cases} \sqrt{1-|x|/a_n+\delta_n}, & \text{if } |x| \le a_n(1-1/T(a_n)+\delta_n), \\ \dfrac{1}{T(a_n)\sqrt{1-\frac{|x|}{a_n}+\delta_n}}, & \text{if } a_n(1-\frac{C}{T(a_n)}+\delta_n) \le |x| \le a_n(1-\delta_n), \\ n^{1/3}/T^{2/3}(a_n), & \text{if } |x| \ge a_n(1-\delta_n). \end{cases}$$

Now we consider several cases. Suppose first that $|x| \le a_n/2$. Then it is easily seen that

$$w^{1/2}(x)\phi_n(x) \le w^{1/2}(0)\sqrt{1+\delta_n} \lesssim 1.$$

Suppose next that $|x| \ge a_{n/2}$. Then using [10, (2.21)], we recall that $|x| \ge a_n(1-C/T(a_n))$ and thus using (35)

$$\begin{aligned} w^{1/2}(x)\phi_n(x) &\le w^{1/2}(a_{n/2})\frac{n^{1/3}}{T^{2/3}(a_n)} \\ &\lesssim \exp\left(-\frac{n}{2T^{1/2}(a_n)}\right)\frac{n^{1/3}}{T^{2/3}(a_n)} \\ &\lesssim 1. \end{aligned}$$

Finally we consider the case $a_n/2 \le x \le a_{n/2}$. This is the most difficult part as the sequence ϕ_n changes its behaviour in this range and because for Erdős weights, $w(a_n/2)$ is in general much smaller than $w(a_{n/2})$. We proceed as follows. Observe that (44) holds at the points $a_n/2$ and $a_{n/2}$. Thus it suffices to show that the sequence $w^{1/2}(x)\phi_n(x)$ remains bounded pointwise in x as x varies from $a_n/2$ to $a_{n/2}$. Let us put $x = a_s$ for some $s > 0$. We may assume without loss of generality that $s \le 2n$ where n is large enough. Moreover it is clear that we may assume that s is sufficiently large. Then using (40), we see after some calculations that

$$\begin{aligned} f(s) &:= w^{1/2}(a_s)\phi_n(a_s) \\ &\lesssim (1-s/n)^{1/2}\exp(-s^{1/2})T(a_s)^{1/2} \\ &=: g(s). \end{aligned}$$

As s is large enough, we find that $g'(s) < 0$ and so $f(s)$ remains bounded. Thus (44) is established and we have proved the lemma. □

Next we prove:

Lemma 3.3 *For a given $x \in I$, let x_{dn}, $1 \leq d \leq n$ denote the node of the quadrature formula that is closest to x, and let*

$$S_n(x) := \sum_{j=1, j\neq d}^{n} \frac{\lambda_{jn}(w^2)}{w(x_{jn})|x_{jn}-x|}.$$

Then, for $|x| < a_n(1+\delta_n)$,

$$0 < S_n(x) \leq C \log n$$

where C is independent of x, n, and d.

Proof. The positivity of $S_n(x)$ is obvious. By (37), we find that

$$S_n(x) \lesssim \frac{a_n}{n}\sum_{j=1}^{d-1}\frac{w(x_{jn})\phi_n(x_{jn})}{|x_{jn}-x|} + \frac{a_n}{n}\sum_{j=d+1}^{n}\frac{w(x_{jn})\phi_n(x_{jn})}{|x_{jn}-x|}. \tag{45}$$

Looking at the first sum in (45), we see that using (39) we have

$$\begin{aligned} \frac{a_n}{n}\sum_{j=1}^{d-1}\frac{w(x_{jn})\phi_n(x_{jn})}{|x_{jn}-x|} &\lesssim \sum_{j=1}^{d-1}\frac{w(x_{jn})(x_{j-1,n}-x_{jn})}{|x_{jn}-x|} \\ &\lesssim \sum_{j=1}^{d-1}\int_{x_{jn}}^{x_{j-1,n}}\frac{w(u)}{|u-x|}du \lesssim \int_{x_{d-1,n}}^{\infty}\frac{w(u)}{u-x}du. \end{aligned}$$

To bound this integral, we note that

$$\begin{aligned} \int_{x_{d-1,n}}^{\infty}\frac{w(u)}{u-x}du &= \int_{x+1}^{\infty}\frac{w(u)}{u-x}du + \int_{x_{d-1,n}}^{x+1}\frac{w(u)}{u-x}du \\ &\leq \int_{x+1}^{\infty}w(u)du + \|w\|_{L_\infty(I)}\int_{x_{d-1,n}}^{x+1}\frac{1}{u-x}du \\ &\lesssim 1-\log(x_{d-1,n}-x). \end{aligned}$$

We point out here that the last expression cannot be negative for sufficiently large n in view of the definition of d and the well known spacing properties of the x_{jn}.

In view of the definition of x_{dn}, we find (using (39)) that

$$x_{d-1,n} - x > \frac{1}{2}(x_{d-1,n} - x_{dn}) \sim \frac{a_n}{n}\phi_n(x_{d-1,n}).$$

A combination of the last two bounds yields

$$\int_{x_{d-1,n}}^{\infty} \frac{w(u)}{u-x} du \lesssim \log \frac{n}{a_n \phi_n(x_{d-1,n})} \leq C \log n.$$

This yields

$$\frac{a_n}{n} \sum_{j=1}^{d-1} \frac{w(x_{jn})\phi_n(x_{jn})}{|x_{jn} - x|} \lesssim \log n,$$

uniformly for the given x under consideration. The second sum in inequality (45) can be bounded in an analogous manner, completing the proof. □

Next, we recall some Markov-Stieltjes inequalities for our class $\mathcal{A}$, see [26, §3].

Lemma 3.4 *Let $d \in \{1, 2, \ldots, n\}$.*

(a) *If the function g is such that $g^{(j)}(x) \geq 0$ for $j = 0, 1, \ldots, 2n-1$ and $x \leq x_{dn}$, then*

$$\sum_{j=d+1}^{n} \lambda_{jn}(w^2) g(x_{jn}) \leq \int_{-\infty}^{x_{dn}} g(x) w^2(x) dx \leq \sum_{j=d}^{n} \lambda_{jn}(w^2) g(x_{jn}).$$

(b) *If the function g is such that $(-1)^j g^{(j)}(x) \geq 0$ for $j = 0, 1, \ldots, 2n-1$ and $x \geq x_{dn}$, then*

$$\sum_{j=1}^{d-1} \lambda_{jn}(w^2) g(x_{jn}) \leq \int_{x_{dn}}^{\infty} g(x) w^2(x) dx \leq \sum_{j=1}^{d} \lambda_{jn}(w^2) g(x_{jn}).$$

This enables us to prove:

Lemma 3.5 *For a given $x \in I$, let x_{dn}, $1 \leq d \leq n$ denote the node of the quadrature formula that is closest to x. Then, for $|x| < a_n(1+\delta_n)$,*

$$\left| \int_{-\infty}^{\infty} \frac{w^2(t)}{t-x} dt - \sum_{j=1, j \neq d}^{n} \frac{\lambda_{jn}(w^2)}{x_{jn} - x} \right| \leq C w(x)$$

where C is independent of x, n, and d.

Proof. Assume that $x_{d+1,n} < x_{dn} \le x < x_{d-1,n}$. The case that x is located on the other side of x_{dn} can be treated in an analoguous manner. We use the Markov-Stieltjes inequalities from Lemma 3.4 and derive that

$$\left| \fint_{-\infty}^{\infty} \frac{w^2(t)}{t-x}dt - \sum_{j=1,j\neq d}^{n} \frac{\lambda_{jn}(w^2)}{x_{jn}-x} \right| \le \frac{\lambda_{d+1,n}(w^2)}{x-x_{d+1,n}} + \left| \fint_{x_{d+1,n}}^{x_{d-1,n}} \frac{w^2(t)}{t-x}dt \right|.$$

We bound each of the two summands separately. For the first one we conclude, in view of $x - x_{d+1,n} > x_{dn} - x_{d+1,n}$ and the relations (37) and (39) that

$$\begin{aligned} 0 < \frac{\lambda_{d+1,n}(w^2)}{x-x_{d+1,n}} &\lesssim \frac{a_n}{n} w^2(x_{d+1,n}) \frac{\phi_n(x_{d+1,n})}{x_{dn}-x_{d+1,n}} \\ &\lesssim w^2(x_{d+1,n}) \lesssim w(x). \end{aligned}$$

For the second summand, we split up the interval of integration according to

$$\begin{aligned} \left| \fint_{x_{d+1,n}}^{x_{d-1,n}} \frac{w^2(t)}{t-x}dt \right| &\le \left| \fint_{x_{d+1,n}}^{2x-x_{d+1,n}} \frac{w^2(t)}{t-x}dt \right| + \left| \int_{2x-x_{d+1,n}}^{x_{d-1,n}} \frac{w^2(t)}{t-x}dt \right| \\ &=: J + J^*. \end{aligned}$$

Note that the singular point is now the midpoint of the first interval of integration, and the second integral is regular. For these two integrals we estimate

$$\begin{aligned} J &= \left| \int_{x_{d+1,n}}^{2x-x_{d+1,n}} \frac{w^2(t)-w^2(x)}{t-x}dt \right| \le \int_{x_{d+1,n}}^{2x-x_{d+1,n}} \left| \frac{w^2(t)-w^2(x)}{t-x} \right| dt \\ &\le \int_{x_{d+1,n}}^{2x-x_{d+1,n}} \left|(w^2)'(\xi(t))\right| dt = 2\int_{x_{d+1,n}}^{2x-x_{d+1,n}} w(\xi(t))|w'(\xi(t))|dt \\ &= 2\int_{x_{d+1,n}}^{2x-x_{d+1,n}} Q'(\xi(t))w^2(\xi(t))dt. \end{aligned}$$

Now since wQ' is bounded, this implies that

$$J \lesssim \int_{x_{d+1,n}}^{2x-x_{d+1,n}} w(\xi(t))dt \lesssim (x-x_{d+1,n})w(x_{d+1,n}) \lesssim w(x).$$

To complete the proof of Lemma 3.5, we need to find a corresponding bound for J^*. This is done in the following way. In view of $x_{d-1,n} - x \le x_{d-1,n} - x_{d+1,n}$ and $x - x_{d+1,n} \ge (x_{dn} - x_{d+1,n})/2$ (which follows from the choice of d), we have

$$\begin{aligned} J^* &\le w^2(2x-x_{d+1,n}) \int_{2x-x_{d+1,n}}^{x_{d-1,n}} \frac{dt}{t-x} = w^2(x) \log \frac{x_{d-1,n}-x}{x-x_{d+1,n}} \\ &\le w^2(x) \log 2 \left(\frac{x_{d-1,n}-x_{d+1,n}}{x_{dn}-x_{d+1,n}} \right). \end{aligned}$$

Here we see that, since $x_{d-1,n} - x_{d+1,n} \sim x_{dn} - x_{d+1,n}$, the argument of the logarithm is uniformly bounded, and therefore

$$J^* \lesssim w^2(x) \lesssim w(x)$$

as required. $\square$

Finally we end this section with the proof of Theorem 1.4 whose statement we will need in Section 4 below.

The Proof of Theorem 1.4. Throughout, let $l_{j,n+2}(V_{n+2})$, $j = 0, \ldots, n+1$, denote the fundamental Lagrange polynomials of degree $n+1$ at the points x_{jn}, $0 \le j \le n+1$, and denote by $l_{j,n}(U_n)$, $j = 1, \ldots, n$, the corresponding fundamental polynomials of degree $n-1$ at the points x_{jn}, $1 \le j \le n$. It is well known, see for example [5], that

$$l_{j,n+2}(V_{n+2})(x) = \left(\frac{x_{0,n}^2 - x^2}{x_{0,n}^2 - x_{j,n}^2}\right) l_{j,n}(U_n)(x), \quad 1 \le j \le n,$$

$$l_{0,n+2}(V_{n+2})(x) = \frac{p_n(x)(x_{0,n} + x)}{2x_{0,n}p_n(x_{0,n})}$$

and

$$l_{n+1,n+2}(V_{n+2})(x) = \frac{p_n(x)(x_{n+1,n} - x)}{2x_{n+1,n}p_n(x_{n+1,n})}.$$

Thus,

$$\begin{aligned} L_{n+2}[f](x)w(x) &= \frac{(x_{0,n} - x)w(x)p_n(x)f(-x_{0,n})}{2x_{0,n}p_n(-x_{0,n})} \\ &\quad + \frac{(x_{0,n} + x)w(x)p_n(x)f(x_{0,n})}{2x_{0,n}p_n(x_{0,n})} \\ &\quad + w(x)\sum_{j=1}^{n} \frac{(x_{0,n}^2 - x^2)l_{j,n+2}(x)f(x_{j,n})}{x_{0,n}^2 - x_{j,n}^2} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

First observe that using [5, (3.9), (3.16) and (3.20)], we may write

$$\begin{aligned} |I_3| &\lesssim \|fw\|_{L_\infty(I)} \sum_{j=1}^{n} \frac{\left(\left|1 - \frac{|x|}{a_n}\right| + \delta_n\right)^{3/4}}{\left(\left|1 - \frac{|x_{j,n}|}{a_n}\right| + \delta_n\right)^{3/4}} \frac{\Delta x_{j,n}}{|x - x_{j,n}|} \\ &\lesssim \|fw\|_{L_\infty(I)} \log n. \end{aligned}$$

Next we estimate $|I_1 + I_2|$. Firstly, it is well known that

$$|p_n w(x)| \lesssim n^{1/6} T(a_n)^{1/6} a_n^{-1/2}, \quad x \in I.$$

Secondly we also know that there exists η such that uniformly for $1 \le j \le n$ and $|x - x_{j,n}| \le \eta \frac{a_n}{n} \phi_n(x)^{1/2}$,

$$\left| \frac{p_n(x) w(x)}{x - x_{j,n}} \right| \sim a_n^{-3/2} \phi_n(x)^{1/4}.$$

Applying this with $x = \pm x_{0,n}$ and $j = 1$ shows that

$$|p_n w|(\pm x_{0,n}) \gtrsim n^{1/6} T(a_n)^{1/6} a_n^{-1/2}$$

for large enough n. We deduce that for $|x| \le a_{2n}$,

$$|I_1 + I_2| \lesssim \|fw\|_{L_\infty(I)}.$$

Thus

$$\|L_{n+2}[f]w\|_{L_\infty(I)} \lesssim \|L_{n+2}[f]w\|_{L_\infty(|x| \le a_{2n})} \lesssim \|fw\|_{L_\infty(I)} \log n. \qquad \square$$

4 Proof of Theorem 1.5

In this section we present the proof of Theorem 1.5.

4.1 The Proof of Theorem 1.5(a)

We suppose without loss of generality that $x \ge 0$ and consider two cases, namely $x > a_n(1 + \delta_n)$ and $x \le a_n(1 + \delta_n)$, where we recall the sequence $\delta_n = (nT(a_n))^{-2/3}$ from Section 3.

In the first case, $x > a_n(1 + \delta_n)$, we proceed as follows. For the given x, define $A(x) = 1/Q'(x + 1)$ as in the proof of Theorem 1.2. We then split the integral $H_n[f](x) = I_1 + I_2 + I_3 + I_4 + I_5$ according to

$$\begin{aligned} I_1 &:= \int_{|t| > 2x} \frac{L_{n+2}[f](t) w^2(t)}{t - x} dt, \\ I_2 &:= \int_{-2x}^{0} \frac{L_{n+2}[f](t) w^2(t)}{t - x} dt, \end{aligned}$$

$$
\begin{aligned}
I_3 &:= \int_0^{x-A(x)} \frac{L_{n+2}[f](t)w^2(t)}{t-x}dt,\\
I_4 &:= \fint_{x-A(x)}^{x+A(x)} \frac{L_{n+2}[f](t)w^2(t)}{t-x}dt,\\
I_5 &:= \int_{x+A(x)}^{2x} \frac{L_{n+2}[f](t)w^2(t)}{t-x}dt.
\end{aligned}
$$

Looking at I_1, we see that, since $|t| > 2x$, we have

$$
|t-x| \geq |t| - |x| > |t| - \frac{|t|}{2} = \frac{|t|}{2}.
$$

Thus,

$$
\begin{aligned}
|I_1| &\leq \|wL_{n+2}[f]\|_{L_\infty(I)} \int_{|t|>2x} \frac{w(t)}{|t-x|}dt\\
&\leq 2\|wL_{n+2}[f]\|_{L_\infty(I)} \int_{|t|>2x} \frac{w(t)}{|t|}dt\\
&= 4\,\|wL_{n+2}[f]\|_{L_\infty(I)} \int_{2x}^{\infty} \frac{w(t)}{t}dt\\
&\leq 4\,\|wL_{n+2}[f]\|_{L_\infty(I)} \int_{2a_n}^{\infty} \frac{w(t)}{t}dt.
\end{aligned}
$$

Since $a_n \to \infty$ as $n \to \infty$, the integral is bounded in n. Thus we may apply Theorem 1.4 and deduce that

$$
|I_1| \lesssim \|wf\|_{L_\infty(I)} \log n.
$$

Turning our attention to I_2, we find that

$$
\begin{aligned}
|I_2| &\leq \|wL_{n+2}[f]\|_{L_\infty(I)} \int_{-2x}^{0} \frac{w(t)}{x-t}dt\\
&\lesssim \|wf\|_{L_\infty(I)} \log n \int_{-2x}^{0} \frac{dt}{x-t}\\
&= C\,\|wf\|_{L_\infty(I)} \log n \int_x^{3x} \frac{du}{u} = (\log 3)C\,\|wf\|_{L_\infty(I)} \log n.
\end{aligned}
$$

Next, we estimate I_3, where we argue using Definition 1.1(c) that

$$
|I_3| \lesssim \|wf\|_{L_\infty(I)} \log n \left(\int_0^{x-1} \frac{w(t)}{x-t}dt + \int_{x-1}^{x-A(x)} \frac{w(t)}{x-t}dt \right)
$$

$$
\begin{aligned}
&\lesssim \|wf\|_{L_\infty(I)} \log n \left(\|w\|_{L_1(I)} + w(x-1) \int_{1/x}^{1} \frac{du}{u} \right) \\
&\lesssim \|wf\|_{L_\infty(I)} \log n \left(\|w\|_{L_1(I)} + w(x-1) \log Q'(x+1) \right) \\
&\lesssim \|wf\|_{L_\infty(I)} \log n.
\end{aligned}
$$

Thus

$$
|I_3| \lesssim \|wf\|_{L_\infty(I)} \log n.
$$

For I_5 we derive similarly that

$$
\begin{aligned}
|I_5| &\lesssim \|wf\|_{L_\infty(I)} \int_{x+A(x)}^{2x} \frac{w(t)}{x-t} dt \log n \\
&\lesssim \|wf\|_{L_\infty(I)} w\left(x + A(x)\right) \int_{A(x)}^{x} \frac{du}{u} \log n \\
&\lesssim \|wf\|_{L_\infty(I)} w(x) \log Q'(x+1) \log n \\
&\lesssim \|wf\|_{L_\infty(I)} \log n.
\end{aligned}
$$

Finally we come to I_4 for which we find that $I_4 = I_6 + I_7$ with

$$
I_6 = \int_{x-A(x)}^{x+A(x)} \frac{L_{n+2}[f](t)w^2(t) - L_{n+2}[f](t)w^2(x)}{t-x} dt
$$

and

$$
I_7 = \int_{x-A(x)}^{x+A(x)} \frac{L_{n+2}[f](t)w^2(x) - L_{n+2}[f](x)w^2(x)}{t-x} dt.
$$

Here,

$$
\begin{aligned}
|I_7| &\leq w^2(x) \int_{x-A(x)}^{x+A(x)} \left| \frac{L_{n+2}[f](t) - L_{n+2}[f](x)}{t-x} \right| dt \\
&= w^2(x) \int_{x-A(x)}^{x+A(x)} \left| L'_{n+2}[f](\xi_t) \right| dt.
\end{aligned}
$$

Taking into consideration the Markov-Bernstein inequality (38), we may continue the bound on $|I_7|$ by

$$
\begin{aligned}
|I_7| &\lesssim w^2(x) \int_{x-A(x)}^{x+A(x)} \left| w(\xi_t) L'_{n+2}[f](\xi_t) \right| w^{-1}(\xi_t) dt \\
&\leq w^2(x) \left\| w L'_{n+2}[f] \right\|_{L_\infty(I)} \int_{x-A(x)}^{x+A(x)} w^{-1}(\xi_t) dt
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim w^2(x)\frac{nT(a_n)^{1/2}}{a_n}\|wL_{n+2}[f]\|_{L_\infty(I)}\,2A(x)w^{-1}(x+A(x))dt\\
&\lesssim w^2(x)\frac{nT(a_n)^{1/2}}{a_n}\log n\,\|wf\|_{L_\infty(I)}\,2A(x)w^{-1}(x+A(x))\\
&\lesssim w(a_n)\frac{nT(a_n)^{1/2}}{a_n}\frac{1}{Q'(a_n)}\log n\,\|wf\|_{L_\infty(I)}\\
&\lesssim \|wf\|_{L_\infty(I)}.
\end{aligned}
$$

Thus

$$|I_7| \lesssim \|wf\|_{L_\infty(I)}\log n.$$

The remaining part is I_6 for which we derive that

$$
\begin{aligned}
|I_6| &\leq \|wL_{n+2}[f]\|_{L_\infty(I)}\left|\int_{x-A(x)}^{x+A(x)} w^{-1}(t)\frac{w^2(t)-w^2(x)w^{-1}(t)}{t-x}dt\right|\\
&\lesssim \|wf\|_{L_\infty(I)}\log n\int_{x-A(x)}^{x+A(x)} w^{-1}(t)\left|(w^2)'(\xi_t)\right|dt\\
&\lesssim \|wf\|_{L_\infty(I)}\log n\int_{x-A(x)}^{x+A(x)} w^{-1}(t)w(\xi_t)\left|w'(\xi_t)\right|dt\\
&\lesssim \|wf\|_{L_\infty(I)}\log n\int_{x-A(x)}^{x+A(x)} w^{-1}(t)w^2(\xi_t)Q'(\xi_t)dt.
\end{aligned}
$$

As above we see that $w^{-1}(t)w^2(\xi_t)\sim w(t)$ which implies that

$$|I_6| \lesssim \|wf\|_{L_\infty(I)}\log n.$$

Combining all of the above estimates, we find that

$$|H_n[f](x)| \lesssim \|wf\|_{L_\infty(I)}\log n$$

uniformly for $|x|>a_n(1+\delta_n)$.

Looking at the remaining range of x, we proceed as follows. Let us write

$$
\begin{aligned}
H_n[f](x) &= \fint_{-\infty}^{\infty}\frac{w^2(t)L_{n+2}[f](t)}{t-x}dt\\
&= L_{n+2}[f](x)\fint_{-\infty}^{\infty}\frac{w^2(t)}{t-x}dt\\
&\quad+\int_{-\infty}^{\infty}w^2(t)\frac{L_{n+2}[f](t)-L_{n+2}[f](x)}{t-x}dt.
\end{aligned}
$$

Here we note that the quotient in the last integral is a polynomial of degree n. The fundamental result on Gaussian quadrature tells us that

$$\sum_{j=1}^{n} \lambda_{jn}(w^2)\pi(x_{jn}) = \int_{-\infty}^{\infty} w^2(t)\pi(t)dt$$

for every polynomial $\pi \in \Pi_{2n-1}$, where $\lambda_{jn}(w^2)$ are the Christoffel numbers for the weight w^2. Thus,

$$H_n[f](x) = L_{n+2}[f](x) \fint_{-\infty}^{\infty} \frac{w^2(t)}{t-x}dt + \sum_{j=1}^{n} \lambda_{jn}(w^2) \frac{L_{n+2}[f](x_{jn}) - L_{n+2}[f](x)}{x_{jn} - x}$$

where we tacitly assume that the proper limits are taken in the case $x = x_{jn}$ for some j. Now for the given x, we denote by x_{dn}, $1 \le d \le n$ the node of the quadrature formula that lies closest to x. Then, observing that $L_{n+2}[f](x_{jn}) = f(x_{jn})$, we see that

$$H_n[f](x) = J_1(x) + J_2(x) + D_n(x)$$

with

$$\begin{aligned}
J_1(x) &= L_{n+2}[f](x) \left(\fint_{-\infty}^{\infty} \frac{w^2(t)}{t-x}dt - \sum_{j=1, j\neq d}^{n} \frac{\lambda_{jn}(w^2)}{x_{jn} - x} \right), \\
J_2(x) &= \sum_{j=1, j\neq d}^{n} \lambda_{jn}(w^2) \frac{f(x_{jn})}{x_{jn} - x}, \\
D_n(x) &= \lambda_{dn}(w^2) \frac{L_{n+2}[f](x_{dn}) - L_{n+2}[f](x)}{x_{dn} - x}.
\end{aligned}$$

In Lemma 3.1, we have shown that uniformly for the given x and n

$$|D_n(x)| \lesssim \|wf\|_{L_\infty(I)} \log n.$$

Moreover, we find that

$$\begin{aligned}
|J_2(x)| &= \left| \sum_{j=1, j\neq d}^{n} \frac{\lambda_{jn}(w^2)}{w(x_{jn})} \frac{w(x_{jn})f(x_{jn})}{x_{jn} - x} \right| \\
&\lesssim \|wf\|_{L_\infty(I)} \sum_{j=1, j\neq d}^{n} \frac{\lambda_{jn}(w^2)}{w(x_{jn})|x_{jn} - x|}.
\end{aligned}$$

Thus using Lemma 3.3, we find that uniformly for the given x and n

$$|J_2(x)| \lesssim \|wf\|_{L_\infty(I)} \log n.$$

Finally, Lemma 3.5 asserts that

$$\left| \int_{-\infty}^{\infty} \frac{w^2(t)}{t-x} dt - \sum_{j=1, j\neq d}^{n} \frac{\lambda_{jn}(w^2)}{x_{jn} - x} \right| \lesssim w(x),$$

which implies using Theorem 1.4 that

$$|J_1(x)| \lesssim \|wL_{n+2}[f]\|_{L_\infty(I)} \lesssim \|wf\|_{L_\infty(I)} \log n.$$

We may now combine the bounds for $J_1(x)$, $J_2(x)$, and $D_n(x)$ to see that

$$|H_n[f](x)| \lesssim \|wf\|_{L_\infty(I)} \log n$$

uniformly for this x and n. Taking the maximum over all such x establishes (12).

The bound (13) then follows from (12) and (6). □

Remark 4.1 In view of the proof of (12), it is natural to ask where exactly in the proof we required the use of the modified interpolation array. Indeed, a careful reading of our proof reveals that we worked mostly with the zeroes of the orthogonal polynomials $p_n(w^2)$. The idea of using the sequence $\{L_{n+2}\}$ was merely to obtain the correct order $\log n$ which follows by Theorem 1.4. In [7], we used another idea, namely estimates of functions of the second kind although we did not obtain as a good an estimate as $\log n$. We believe that this latter idea will ultimately lead to the correct order $\log n$ as well.

4.2 The Proof of Theorem 1.5(b)

Firstly by the construction of the quadrature formula, $R_n[p; w^2] \equiv 0$ whenever p is a polynomial of degree at most $n+1$. Let us denote by π_{n+1} the polynomial of degree $n+1$ that gives the best uniform approximation to f with respect to the weight function w, i.e.

$$\|w(f - \pi_{n+1})\|_{L_\infty(I)} = E_{n+1}[f]_{w,\infty}.$$

Then for every $x \in I$

$$\begin{aligned} R_n[f;w^2](x) &= R_n[f-\pi_{n+1};w^2](x) + R_n[\pi_{n+1};w^2](x) \\ &= R_n[f-\pi_{n+1};w^2](x) \\ &= H[f-\pi_{n+1};w^2](x) - H_n[f-\pi_{n+1};w^2](x). \end{aligned}$$

Firstly by (12), we find that

$$\max_{x\in I} |H_n[f-\pi_{n+1};w^2](x)| \lesssim \|w(f-\pi_{n+1})\|_{L_\infty(I)} \log n = E_{n+1,\infty}[f,w] \log n.$$

Thus it remains to prove that

$$\max_{x\in I} \left|H[f-\pi_{n+1};w^2](x)\right| \lesssim \int_0^{a_n/n} \frac{\omega_{1,\infty}(f,w,u)}{u} du + E_{n+1,\infty}[f,w]\log n. \quad (46)$$

Firstly, by (31), we have that

$$\begin{aligned} \left\|H[f-\pi_{n+1};w^2]\right\|_{L_\infty(I)} \lesssim & \int_0^{a_n/n} \frac{\omega_{1,\infty}(f-\pi_{n+1},w,u)}{u} du \qquad (47) \\ & + E_{n+1,\infty}[f,w]\log n. \end{aligned}$$

Thus we must estimate the first term in (47) above. To this end, we apply the Marchaud inequality [19, Corollary 1.6] and write for $u > 0$

$$\begin{aligned} &\omega_{1,\infty}(f-\pi_{n+1},u) \\ &\lesssim u \sum_{j=1}^{[\log_2(n+1)]} \frac{2^j}{a_j} E_{2^j,\infty}[f-\pi_{n+1},w] + u \sum_{j=[\log_2(n+1)]+1}^{[-\log_2 u]} \frac{2^j}{a_j} E_{2^j,\infty}[f,w] \\ &= \Sigma_1 + \Sigma_2. \qquad (48) \end{aligned}$$

Here $[x]$ denotes the largest integer $\leq x$. We proceed to first estimate Σ_1. The crucial observation here is that for any fixed $1 \leq j \leq [\log_2(n+1)]$,

$$\begin{aligned} E_{2^j,\infty}[f-\pi_{n+1},w] &\leq E_{2,\infty}[f-\pi_{n+1},w] \\ &= \inf_{Q\in\Pi_2} \|(f-\pi_{n+1}-Q)w\|_{L_\infty(I)} \leq E_{n+1,\infty}[f,w]. \end{aligned}$$

Applying this identity gives that

$$\Sigma_1 = u \sum_{j=1}^{[\log_2(n+1)]} \frac{2^j}{a_j} E_{2^j,\infty}[f-\pi_{n+1},w]$$

$$\leq \quad u\left(\frac{n}{a_n}\right) E_{n+1,\infty}[f,w] \sum_{j=1}^{[\log_2(n+1)]} \left(\frac{a_n/n}{a_{2^j}/2^j}\right)$$
$$\lesssim \quad u\left(\frac{n}{a_n}\right) E_{n+1,\infty}[f,w] \log n.$$

Thus
$$\int_0^{a_n/n} \frac{\Sigma_1}{u} du \lesssim E_{n+1,\infty}[f,w] \log n. \tag{49}$$

Next let us set $v_j := \frac{2^j}{a_{2^j}}$, $j \geq 1$. Then we see that

$$\begin{aligned}
\int_0^{a_n/n} \frac{\Sigma_2}{u} du &\lesssim \int_0^{a_n/n} \sum_{j=[\log_2(n+1)]+1}^{[-\log u]} v_j \omega_{1,\infty}(f,w,v_j^{-1}) \\
&\lesssim \int_0^{a_n/n} \sum_{j=[\log_2(n+1)]+1}^{[-\log u]} \int_{v_{j-1}}^{v_j} \omega_{1,\infty}(f,w,v^{-1}) dv\, du \\
&\lesssim \int_0^{a_n/n} \int_{\frac{n+1}{a_{n+1}}}^{\beta(u)} \omega_{1,\infty}(f,w,v^{-1}) dv\, du
\end{aligned}$$

where $\beta(u) := \frac{1}{u a_{1/u}}$, $u > 0$. An application of Fubini's theorem then yields

$$\begin{aligned}
\int_0^{a_n/n} \frac{\Sigma_2}{u} du &\lesssim \int_{\frac{n+1}{a_{n+1}}}^{\infty} \int_0^{v^{-1}} \omega_{1,\infty}(f,w,v^{-1}) du\, dv \\
&\lesssim \int_0^{a_n/n} \frac{\omega_{1,\infty}(f,w,z)}{z} dz.
\end{aligned} \tag{50}$$

Inserting (49) and (50) into (47) then proves (14). (15) then follows from (14) and from [19, Corollary 1.8]. □

5 Proof of Theorem 1.6

In this section, we prove Theorem 1.6. For the proof, we use a function $B_s : I \to \mathbb{R}$ with the following properties.

- $B_s(x) = 0$ for $|x| \geq 1$.
- $B_s(x) > 0$ for $|x| < 1$.
- B_s is s-times continuously differentiable.

It is easily seen that a function with these properties exists; as an example we may choose a B-spline of order $s+1$ [34, §4.3]. Note that, as a consequence of these requirements, the function $B_s^{(s)}$ is bounded on I. Moreover, because B_s is a nonvanishing, continuous and nonnegative function with bounded support, the integral $\int_{-\infty}^{\infty} B_s(x)dx$ exists and is strictly positive.

Then, for $j = 0, 1, \ldots, 2n$, we define $t_j := -a_n + ja_n/n$. In this way, we partition the interval $(-a_n, a_n)$ into $2n$ subintervals of equal length. We now have to discuss how the interpolation nodes x_{jn} are distributed over these subintervals. The crucial result here is the following Lemma, obtained from [15, Lemma 3.6] by a transformation to the interval $[-a_n, a_n]$. It seems that a proof of the Lemma is not available in the English literature; therefore we provide it in Section 6.

For an arbitrary real number x, we shall use the notation $[x]$ to denote the largest integer not exceeding x.

Lemma 5.1 *Let x_{jn} and t_j be given as stated above. Then, there exists $y \in [-a_n, a_n]$ and there exist m pairwise different indices $k_1, k_2, \ldots, k_m \in \{1, 2, \ldots, 2n\}$ with*

(a) $m \geq n/6$,

(b) $x_{jn} \notin (t_{k_\mu - 1}, t_{k_\mu})$ *for all* $j = 1, 2, \ldots, n$ *and all* $\mu = 1, 2, \ldots, m$, *and*

(c) $0 \leq \xi - y < 3\mu a_n/n$ *for all* $\xi \in (t_{k_\mu - 1}, t_{k_\mu})$ *and all* $1 \leq \mu \leq n/6$.

Proof of Theorem 1.6. We make use of these intervals $(t_{k_\mu - 1}, t_{k_\mu})$ and define

$$f^*(t) := \begin{cases} \dfrac{B_s(4\frac{n}{a_n}(t - t_{k_\mu}) + 2)}{(n/a_n)^s w^2(t_{k_\mu})}, & t \in (t_{k_\mu - 1}, t_{k_\mu}) \text{ for some } \mu, \\ 0, & \text{else.} \end{cases}$$

Because of Lemma 5.1(b), this construction ensures that no x_{jn} is in any of the intervals $(t_{k_\mu - 1}, t_{k_\mu})$, and therefore $f^*(x_{jn}) = 0$ for all j. This implies that $Q_n[f^*; x] = 0$ for all x. Thus,

$$\sup_{x \in I} |H[f^*; x] - Q_n[f^*; x]| \geq |H[f^*; y] - Q_n[f^*; y]| = |H[f^*; y]|$$

$$
\begin{aligned}
&= \left|\int_{-\infty}^{\infty} \frac{f^*(t)w^2(t)}{y-t}dt\right| = \sum_{\mu=1}^{m}\int_{t_{k_{\mu}-1}}^{t_{k_\mu}} \frac{f^*(t)w^2(t)}{t-y}dt \\
&\gtrsim \frac{n}{a_n}\sum_{\mu=1}^{[n/6]}\int_{t_{k_{\mu}-1}}^{t_{k_\mu}} \frac{f^*(t)w^2(t)}{\mu}dt \\
&\gtrsim \frac{n}{a_n}\sum_{\mu=1}^{[n/6]}\frac{1}{\mu}\int_{t_{k_{\mu}-1}}^{t_{k_\mu}} \frac{B_s(4\frac{n}{a_n}(t-t_{k_\mu})+2)}{(n/a_n)^s}dt
\end{aligned}
$$

by definition of f^*, Lemma 5.1(c) and using the fact that for $t \in [t_{k_\mu-1}, t_{k_\mu}]$, we have $w(t) \sim w(t_{k_\mu})$. Thus, introducing the change of variables $u = 4\frac{n}{a_n}(t - t_{k_\mu}) + 2$, we find that

$$
\begin{aligned}
\sup_{x\in I}|H[f^*;x]-Q_n[f^*;x]| &\gtrsim \left(\frac{a_n}{n}\right)^{s-1}\sum_{\mu=1}^{[n/6]}\frac{1}{\mu}\int_{t_{k_{\mu}-1}}^{t_{k_\mu}} B_s\left(\frac{4n}{a_n}(t-t_{k_\mu})+2\right)dt \\
&= \frac{1}{4}\left(\frac{a_n}{n}\right)^{s}\sum_{\mu=1}^{[n/6]}\frac{1}{\mu}\int_{-2}^{2} B_s(u)du \\
&= \frac{1}{4}\left(\frac{a_n}{n}\right)^{s}\sum_{\mu=1}^{[n/6]}\frac{1}{\mu}\int_{-1}^{1} B_s(u)du
\end{aligned}
$$

where, in the last equality, we used the fact that $B_s(u) = 0$ for $u \notin [-1, 1]$. Now we see that the value of the integral depends only on the function B_s (and therefore on the given and fixed number s) but neither on the number n of nodes of the quadrature formula nor on the the summation index μ. It thus follows that

$$
\sup_{x\in I}|H[f^*;x]-Q_n[f^*;x]| \gtrsim \left(\frac{a_n}{n}\right)^{s}\sum_{\mu=1}^{[n/6]}\frac{1}{\mu} \gtrsim \left(\frac{a_n}{n}\right)^{s}\log n
$$

because of Lemma 5.1(a). We have thus proved Theorem 1.6 provided we can show that f^* has the other desired properties in the statement of the theorem. Obviously, f^* has finite support, and therefore $f^*(x)w(x) \to 0$ as $|x| \to \infty$. Moreover, on every interval $(t_{k_\mu-1}, t_{k_\mu})$, we have

$$
\left\|f^{*(s)}\right\|_{L_\infty(t_{k_\mu-1},t_{k_\mu})} = 4^s \left\|B_s^{(s)}\right\|_{L_\infty(I)} w^{-2}(t_{k_\mu}).
$$

Since, for t in such an interval, $w(t) \sim w(t_{k_\mu})$ uniformly, we derive that

$$
\left\|w^2 f^{*(s)}\right\|_{L_\infty(t_{k_\mu-1},t_{k_\mu})} \sim 4^s \left\|B_s^{(s)}\right\|_{L_\infty(I)}
$$

which implies (because $f^{*(s)} \equiv 0$ outside these intervals) that

$$\left\| w^2 f^{*(s)} \right\|_{L_\infty(I)} \sim 4^s \left\| B_s^{(s)} \right\|_{L_\infty(I)} \sim 1$$

independent of n. This completes the proof of Theorem 1.6. $\square$

6 Proof of Lemma 5.1

The proof that we shall give here is a slight modification of the proof given (in German) by Strauß [35, Lemma 2]. As mentioned there, some of the key ideas are already contained in the paper of Makovoz and Šeško [27]. Since an English translation does not seem to be readily available for either of these two papers, we provide the fundamental details here. Many parts of the arguments are combinatorial or geometric in nature. The reader is encouraged to draw a sketch of the situation in order to clarify the ideas.

For simplicity, we introduce the notation $\Delta_i := (t_{i-1}, t_i)$ for the subintervals under consideration. All these intervals Δ_i that do not contain any of the x_{jn} will be called "intervals of type A". Let p be the number of intervals of type A. Note that we have $2n$ intervals Δ_i altogether, and they are pairwise disjoint. Moreover we have n points x_{jn}; and thus at least n subintervals are of type A. In other words, $p \geq n$.

For $0 \leq k \leq 2n-1$ and $1 \leq l \leq 2n-k$, let $\beta(k,l)$ be the number of intervals of type A from $\{\Delta_{k+1}, \Delta_{k+2}, \ldots, \Delta_{k+l}\}$, and let

$$\Theta(k,l) := \frac{\beta(k,l)}{l}.$$

Then there exists some $\hat{k} < 3n/2$ such that

$$\min_{1 \leq l \leq 2n-k} \Theta(\hat{k}, l) \geq \frac{1}{3}. \tag{51}$$

We prove the inequality (51) in an indirect fashion, i.e. we assume that

$$\forall k < \frac{3}{2}n : \min_l \Theta(k,l) < \frac{1}{3}.$$

This implies, in particular, that $\min_l \Theta(0,l) < 1/3$, and thus there exists some $l_1 \geq 1$ such that $\Theta(0, l_1) < 1/3$. If $l_1 \geq 3n/2$, then we stop here; otherwise, by assumption, there exists some $l_2 \geq 1$ such that $\Theta(l_1, l_2) < 1/3$. If now $l_1 + l_2 \geq 3n/2$, then we stop again; otherwise we continue the iteration and find that there exists $l_3 \geq 1$ with $\Theta(l_1 + l_2, l_3) < 1/3$, and so on. Since all the $l_\nu \geq 1$, the sum $\sum_{\nu=1}^{\kappa} l_\nu$ is a strictly increasing function of κ, and thus we must, at some point, reach an l_ν with the following properties:

$$\Theta\left(\sum_{\mu=1}^{\kappa} l_\mu, l_{\kappa+1}\right) < \frac{1}{3} \qquad (\kappa = 0, 1, \ldots, \nu)$$

and

$$\sum_{\mu=1}^{\nu} l_\mu < \frac{3}{2}n \leq \sum_{\mu=1}^{\nu+1} l_\mu. \tag{52}$$

Let q now be the number of intervals of type A from $\{\Delta_1, \ldots, \Delta_{\sum_{\mu=1}^{\nu+1} l_\mu}\}$. By the definition of Θ and our assumption,

$$q = \sum_{\mu=1}^{\nu+1} l_\mu \Theta\left(\sum_{\kappa=1}^{\mu-1} l_\kappa, l_\mu\right) < \frac{1}{3}\sum_{\mu=1}^{\nu+1} l_\mu,$$

and thus

$$n \leq p \leq q + 2n - \sum_{\mu=1}^{\nu+1} l_\mu < 2n - \frac{2}{3}\sum_{\mu=1}^{\nu+1} l_\mu$$

which implies

$$\frac{2}{3}\sum_{\mu=1}^{\nu+1} l_\mu < n$$

and so contradicts (52). Thus we have shown (51).

Now we choose $y := -a_n + a_n\hat{k}/n$ with the $\hat{k}$ whose existence we have just deduced. Denote the intervals of type A from $\{\Delta_{\hat{k}+1}, \ldots, \Delta_{2n+2}\}$ (in their natural order) by $\Delta_{k_1}, \Delta_{k_2}, \ldots, \Delta_{k_m}$ (this implicitly defines the values $k_1, k_2, \ldots, k_m$). Then, (51) implies

$$m = (2n - \hat{k})\Theta(\hat{k}, 2n - \hat{k}) \geq \frac{1}{3}\left(2n - \frac{3}{2}n\right) = \frac{n}{6}.$$

This proves parts (a) and (b) of the Lemma.

For $\xi \in \Delta_{k_\mu} = (t_{k_\mu - 1}, t_{k_\mu})$ we then have, by construction, $0 \leq \xi - y$. It remains to prove $t_{k_\mu} - y < 3\mu a_n/n$. Again we do this indirectly: We assume that there exists some μ, $1 \leq \mu \leq n/6$, with the property that $t_{k_\mu} \geq y + 3\mu a_n/n$.

This implies that the set $\{\Delta_{\hat{k}+1}, \Delta_{\hat{k}+2}, \ldots, \Delta_{\hat{k}+3\mu}\}$ does not contain the interval Δ_{k_μ} (by construction of y), and thus the set contains less than μ intervals of type A. Hence $3\mu\Theta(\hat{k}, 3\mu) < \mu$, i.e. $\Theta(\hat{k}, 3\mu) < 1/3$ in contradiction to (51). This proves statement (c) of the Lemma. $\Box$

Acknowledgments

The research work described in this paper was performed while the second author visited the Center for Applicable Analysis and Number Theory at the University of the Witwatersrand whose support and hospitality are gratefully acknowledged. The second author also thanks the Deutsche Forschungsgemeinschaft for financial assistance. The research of the first author was supported in part by 1998 URC grant Rexaandmlnepen.

References

[1] G. Criscuolo, B. Della Vecchia, D. S. Lubinsky, and G. Mastroianni, *Functions of the second kind for Freud weights and series expansions of Hilbert transforms*, J. Math. Anal. Appl. **189** (1995), 256–296.

[2] G. Criscuolo and G. Mastroianni, *On the convergence of the Gauss quadrature rules for Cauchy principal value integrals*, Ricerche Mat. **XXXV** (1986), 45–60.

[3] G. Criscuolo and G. Mastroianni, *On the convergence of an interpolatory product rule for evaluating Cauchy principal value integrals*, Math. Comp. **48** (1987), 725–735.

[4] S. B. Damelin, *Converse and smoothness theorems for Erdős weights in L_p $(0 < p \leq \infty)$*, J. Approx. Theory **93** (1998), 349–398.

[5] S. B. Damelin, *The Lebesgue function and Lebesgue constant of Lagrange interpolation for Erdős weights.*, J. Approx. Theory **94** (1998), 235–262.

[6] S. B. Damelin, *A characterisation of smoothness for Freud weights*, J. Comput. Appl. Math. **99** (1998), 463–473.

[7] S. B. Damelin and K. Diethelm, *Interpolatory product quadratures for Cauchy principal value integrals with Freud weights*, Numer. Math. **83** (1999), 87–105.

[8] S. B. Damelin, H. S. Jung and K. H. Kwon, *Converse quadrature sum inequalities and mean convergence of Lagrange interpolation.* Preprint.

[9] S. B. Damelin and A. B. J. Kuijlaars, *The support of the equilibrium measure in the presence of a monomial external field on* $[-1,1]$, Trans. Amer. Math. Soc. **351** (1999), 4561–4584.

[10] S. B. Damelin and D. S. Lubinsky, *Necessary and sufficient conditions for mean convergence of Lagrange interpolation for Erdős weights*, Canad. J. Math. **48** (1996), 710–736.

[11] S. B. Damelin and D. S. Lubinsky, *Necessary and sufficient conditions for mean convergence of Lagrange interpolation for Erdős weights II*, Canad. J. Math. **48** (1996), 737–757.

[12] S. B. Damelin and D. S. Lubinsky, *Jackson theorems for Erdős weights in* L_p $(0 < p \leq \infty)$, J. Approx. Theory **94** (1998), 333–382.

[13] M. C. De Bonis, B. Della Vecchia, and G. Mastroianni, *Approximation of the weighted Hilbert transform on the real line*, submitted for publication (1996).

[14] P. Deift, *Orthogonal Polynomials and Random Matrices: a Riemann–Hilbert approach*, Courant Lecture Notes in Mathematics, Courant Institute, New York (1999).

[15] K. Diethelm, *Uniform convergence of optimal order quadrature rules for Cauchy principal value integrals*, J. Comput. Appl. Math. **56** (1994), 321–329.

[16] K. DIETHELM, *Gaussian quadrature formulae of the third kind for Cauchy principal value integrals: Basic properties and error estimates*, J. Comput. Appl. Math. **65** (1995), 97–114.

[17] K. DIETHELM, *The order of convergence of modified interpolatory quadratures for singular integrals of Cauchy type*, Z. Angew. Math. Mech. **75** (1995), S621–S622.

[18] K. DIETHELM, *A method for the practical evaluation of the Hilbert transform on the real line*, J. Comput. Appl. Math. **112** (1999), 45–53.

[19] Z. DITZIAN AND D. S. LUBINSKY, *Jackson and smoothness theorems for Freud weights in* $L_p(0 < p < \infty)$, Constr. Approx. **13** (1997), 99–152.

[20] H. KÖNIG, *Vector valued Lagrange interpolation and mean convergence of Hermite series.* In *Functional Analysis* (K. D. Bierstedt, A. Pietsch, W. M. Ruess, and D. Vogt, eds.), Marcel Dekker, New York (1994), 227–247.

[21] A. B. J. KUIJLAARS AND P. D. DRAGNEV, *Equilibrium problems associated with fast decreasing polynomials*, Proc. Amer. Math. Soc. **127** (1999), 1065–1074.

[22] A. L. LEVIN AND D. S. LUBINSKY, *Christoffel functions, orthogonal polynomials, and Nevai's conjecture for Freud weights*, Constr. Approx. **8** (1992), 463–535. Erratum: Constr. Approx. **11** (1995), 417–418.

[23] A. L. LEVIN, D. S. LUBINSKY, AND T. Z. MTHEMBU, *Christoffel functions and orthogonal polynomials for Erdős weights on* $(-\infty, \infty)$, Rend. Mat. Appl. (7) **14** (1994), 199–289.

[24] D. S. LUBINSKY, *Mean convergence of Lagrange interpolation for exponential weights on* $[-1, 1]$, Canad. J. Math. **50** (1998), 1273–1297.

[25] D. S. LUBINSKY AND D. M. MATJILA, *Necessary and sufficient conditions for mean convergence of Lagrange interpolation for Freud weights*, SIAM J. Math. Anal. **26** (1995), 238–262.

[26] D. S. Lubinsky and P. Rabinowitz, *Rates of convergence of Gaussian quadrature for singular integrands*, Math. Comp. **43** (1984), 219–242.

[27] Y. I. Makovoz and M. A. Šeško, *Estimation of the error of a quadrature formula for a singular integral*, Vescī Akad. Navuk BSSR Ser. Fīz.-Mat. Navuk (1977), 36–41 and 140 (in Russian).

[28] S. G. Mikhlin and S. Prössdorf, *Singular Integral Operators*, Springer, Berlin (1986).

[29] G. Monegato, *On the weights of certain quadratures for the numerical evaluation of Cauchy principal value integrals and their derivatives*, Numer. Math. **50** (1987), 273–281.

[30] B. Muckenhoupt, *Mean convergence of Jacobi series*, Proc. Amer. Math. Soc. **23** (1970), 306–310.

[31] P. Nevai, *Mean convergence of Lagrange interpolation III*, Trans. Amer. Math. Soc. **282** (1984), 669–698.

[32] P. Nevai, *Geza Freud, Orthogonal polynomials and Christoffel functions: A case study*, J. Approx. Theory **48** (1986), 3–167.

[33] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Springer, New York (1997).

[34] L. L. Schumaker, *Spline Functions: Basic Theory*, J. Wiley & Sons, New York (1981).

[35] R. Strauss, *Eine Interpolationsquadratur für Cauchy-Hauptwert-integrale*, Rostock. Math. Kolloq. **22** (1983), 57–66.

[36] J. Szabados, *Weighted Lagrange and Hermite-Fejér interpolation on the real line*, J. Inequal. Appl. **1** (1997), 99–123.

[37] V. Totik, *Weighted Approximation with Varying Weight*, Lect. Notes in Math. 1569, Springer, Berlin (1994).

[38] F. G. Tricomi, *Integral Equations*, Interscience Publishers, New York (1957).