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Local Paley–Wiener theorems for functions analytic on unit spheres

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Abstract

The purpose of this paper is to provide new and simplified statements of local Paley–Wiener theorems on the $(n-1)$-dimensional unit sphere realized as a subset of $n = 2, 3$ Euclidean space. More precisely, given a function $f : \mathbb{C}^n \to \mathbb{C}, n = 2, 3$, whose restriction to an $n-1$ sphere is analytic, we establish necessary and sufficient conditions determining whether f is the Fourier transform of a compactly supported, bounded function $F : \mathbb{R}^n \to \mathbb{C}$. The essence of this investigation is that, because of the local nature of the problem, the mapping $f \to F$ is not in general invertible and so the problem cannot be studied via a Fourier integral. Our proofs are new.

1. Introduction

A problem of interest in radiation and scattering problems is that of determining the support of a compactly supported, square integrable scattering potential, $F : \mathbb{R}^n \to \mathbb{C}, n = 2, 3$, from far field data given by a function $f : \mathbb{C}^n \to \mathbb{C}$.

Suppose, *a priori*, that we know that for every vector $\mathbf{z} \in S^{n-1}$, the $(n-1)$-dimensional unit sphere, realized as a proper subset of $\mathbb{R}^n$, the far field data f are given *locally* by an integral such as

$$f(\mathbf{z}) = \int_\tau F(\mathbf{x})\, \mathrm{e}^{-\mathrm{i}k(\mathbf{z}\cdot\mathbf{x})}\, \mathrm{d}^n(x) \tag{1.1}$$

for some bounded set $\tau \in \mathbb{R}^n$ and bounded, compactly supported $F : \mathbb{R}^n \to \mathbb{C}$ with support in τ. In practice, $\mathbf{z}$ is the unit vector in the direction where f is measured and k is an absolute real constant. The inverse support problem, as treated for example in [4, 5, 10, 21] and the references cited therein, studies the problem of determining bounds for the support of the set τ, assuming the model (1.1). For many applications, assumption (1.1) on the given f is strong and often not obvious from the given data.

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The purpose of this paper is to provide new and simplified statements of *local Paley–Wiener* theorems on S^{n-1}. More precisely, given a function $f : \mathbb{C}^n \to \mathbb{C}$, $n = 2, 3$, whose restriction to S^{n-1} is analytic, we establish necessary and sufficient conditions determining whether f is the Fourier transform of a compactly supported, bounded function $F : \mathbb{R}^n \to \mathbb{C}$. The essence of this investigation is that, because of the *local* nature of the problem, the mapping $f \to F$ is not in general invertible and so the problem cannot be studied via a Fourier integral. Our proofs are new and use a beautiful interplay between plane wave expansions and Euclidean geometrical arguments.

We will show that, provided the restriction of f to S^{n-1} is analytic and f satisfies a growth condition of exponential type at infinity, then (1.1) holds for some compact set $\tau \in \mathbb{R}^n$ and bounded $F : \mathbb{R}^n \to \mathbb{C}$ with compact support in τ. We also describe the smallest support sets for which our results are best possible. Results of this type are typically known in the literature as Paley–Wiener theorems [PW], see [8, 10, 17, 18, 21] and the references cited therein. In this paper, we seek generalizations of [PW] in that we do not assume that f is entire nor that its restriction to $\mathbb{R}^n$ is square integrable. The later assumptions are basic in [PW]. Indeed, we show that (1) a growth condition of f of exponential type at infinity and (2) an assumption that the restriction of f to S^{n-1} is analytic are enough to deduce (1.1). In [PW], the function F obtained is square integrable on $\mathbb{R}^n$, compactly supported but not necessarily bounded.

1.1. Notation and structure of the paper

In order to motivate what follows, we need to introduce some further background and machinery. Throughout, for any non-zero real sequences a_n and b_n, we shall write $a_n = O(b_n)$ if the ratio a_n/b_n is uniformly bounded in n and $a_n \sim b_n$ if $a_n/b_n \to 1, n \to \infty$. Similar notation will be used for functions and sequences of functions. Throughout, we shall say that a function $f : \mathbb{C}^n \to \mathbb{C}$ is of exponential type $a > 0$ at infinity, if

$$|f(z)| = O(\mathrm{e}^{a|z|}), \qquad |z| \to \infty.$$

Finally, given $f : \mathbb{C}^n \to \mathbb{C}$, by f_A, we will always mean the restriction of f to a proper subset A of $\mathbb{C}^n$. Given $x \in \mathbb{R}^n$, by the vector $\mathbf{x}$, we will mean the point x with a given direction from the origin. Similar notation will be used for complex vectors. Associated with the Euclidean metric on $\mathbb{R}^n$ and $\mathbb{C}^n$, we have the usual inner product and, in what follows, $\mathbf{v}_1 \cdot \mathbf{v}_2$ will denote the usual inner product of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^n$ or $\mathbb{C}^n$ and $|.|$ will denote the usual Euclidean norm induced by this inner product. We will also use the convention of denoting the two real polar axes in a Cartesian coordinate system by x_1 and x_2. Throughout, $L_2(\mathbb{R}^n)$ will denote the class of all square integrable functions $F : \mathbb{R}^n \to \mathbb{C}$.

The remainder of this paper is organized as follows. In section 3, we discuss classical Paley theorems. In section 4, we state our local Paley–Wiener theorems and problems with local inversion, and section 5 is devoted to the proofs of our results.

2. The Paley–Wiener theorems in $\mathbb{R}^n$

The relationship of the growth of an entire function with the properties of its Fourier transform is embodied in the well-known Paley–Wiener theorems. For functions of complex variable, we have

Theorem 1 [PW1]. *A function $f : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type with its restriction $f_{\mathbb{R}} \in L_2(\mathbb{R})$ iff*

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_a^b F(x)\, \mathrm{e}^{-\mathrm{i}xz}\, \mathrm{d}x, \qquad z \in \mathbb{C}.$$

for some $F \in L_2(\mathbb{R})$ with support in $[a, b]$. Moreover, $[a, b]$ is the smallest set containing the support of F and

$$a = -\limsup_{y\to\infty} \frac{\log|f(-\mathrm{i}y)|}{y}, \qquad b = \limsup_{y\to\infty} \frac{\log|f(\mathrm{i}y)|}{y}.$$

Analogues of [PW1], which we denote for simplicity by [PW2], exist for $n \geqslant 1$ complex variables. See for example [7, p 181, theorem 7.3.1] for a clear exposition of these results.

In what follows, we will need the notion of a *support function* of the smallest convex set outside of which a compactly supported function $F : \mathbb{R}^n \to \mathbb{C}$ vanishes. This function is defined by

$$\rho(\mathbf{u}) := \sup_{\mathbf{x}\in\tau_F} (\mathbf{u}\cdot\mathbf{x})$$

where τ_F is the support of F and $\mathbf{u} \in \mathbb{R}^n$ is a given unit vector. The function $\rho(\mathbf{u})$ is used to define a convex region $\tau_{c,F} \supset \tau_F$ which is formed from tangent planes to τ_F having normal vectors $\mathbf{u}$ and located at a distance $\rho(\mathbf{u}, \tau_F)$ from the origin.

It is clear that the function ρ depends on F via its support τ_F but since F is always fixed, for ease of notation, we will drop this dependence henceforth.

3. Main results: local Paley–Weiner theorems, comparisons and problems with local inversion

In this section, we state our main results which constitute generalizations of [PW] for $n = 2, 3$. Following is our first result:

Theorem 2 [LPW1]. *Let $n = 2, 3$ and let $f : \mathbb{C}^n \to \mathbb{C}$ be a function whose restriction to S^{n-1} is analytic. Let k, a be positive numbers. Suppose that for any fixed real vector $\mathbf{u} \in S^{n-1}$,*

$$|f(\mathbf{z})| = O(\mathrm{e}^{ka(\mathbf{u}\cdot\mathrm{Im}(\mathbf{z}))}), \qquad \mathbf{u}\cdot\mathrm{Im}(\mathbf{z}) \to \infty, \qquad \mathbf{z} \in \mathbb{C}^n.$$

Then,

$$f(\mathbf{z}) = \int_{B_a} F(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}k(\mathbf{z}\cdot\mathbf{x})}\,\mathrm{d}^n x, \qquad \mathbf{z} \in S^{n-1}$$

where F is a bounded function supported in the closed ball $B_a \in \mathbb{R}^n$ with centre 0 and radius a.

Remark 1.

(a) Note that in the statement of [LPW1], compared to that of [PW1], we use smoothness properties of f only on S^{n-1} to establish our result and square integrability and analyticity of the restriction of f iff the given sphere is not required nor used in our results. The function F obtained is both bounded and compactly supported in $\mathbb{R}^n$. As $\mathbf{z} \in \mathbb{C}^n$ and $\mathbf{u} \in \mathbb{R}^n$, the inner product $\mathrm{Im}\,\mathbf{z}\cdot\mathbf{u}$ is well defined.

(b) [LPW1] is similar to [9, theorems 9, 12, section 4] which were established earlier. In these later results, the authors consider a wide class of distribution functions and study bounds on their coefficients under similar smoothness assumptions to ours. The method of proof in [9] uses a combination of clever and sophisticated machinery of Bessel and spherical harmonics and is of independent interest. In particular, we mention that Bessel and spherical harmonics allow for extensions from $n = 2$ to $n = 3$. As our proofs will show, we are able to establish [LPW1] using different techniques which involve an interplay between plane wave expansions and geometrical arguments. These later techniques also provide a natural but different method to move from $n = 2$ to $n = 3$.

(c) It is easy to establish a partial converse of [LPW1] which is the following: suppose

$$f(\mathbf{z}) = \int_{B_a} F(\mathbf{x})\, \mathrm{e}^{-\mathrm{i}k(\mathbf{z}\cdot\mathbf{x})}\, \mathrm{d}^n x, \qquad \mathbf{z} \in \mathbb{C}^n$$

where F is a bounded function supported in the closed ball $B_a \in \mathbb{R}^n$ with centre 0 and radius a. Then,

$$|f(\mathbf{z})| = O(\mathrm{e}^{ka(\mathbf{u}\cdot\mathrm{Im}(\mathbf{z}))}), \qquad \mathbf{u}\cdot\mathrm{Im}(\mathbf{z}) \to \infty.$$

(d) [LPW1] stated above does not, in general, yield the smallest support volume for the function F. However, the smallest *convex* support volume for this function can be obtained in analogy with [PW2]. This is contained in theorem 3.

Theorem 3 [LPW2]. *Let $n = 2, 3$, k be a positive constant and $f : \mathbb{C}^n \to \mathbb{C}$ be a function whose restriction $f_{S^{n-1}}$ is analytic. Suppose that for any fixed real vector $\mathbf{u} \in S^{n-1}$,*

$$|f(\mathbf{z})| = O(\mathrm{e}^{k\rho(\mathbf{u})(\mathbf{u}\cdot\mathrm{Im}(\mathbf{z}))}), \qquad \mathbf{u}\cdot\mathrm{Im}(\mathbf{z}) \to \infty, \qquad \mathbf{z} \in \mathbb{C}^n.$$

Then,

$$f(\mathbf{z}) = \int_{\tau_c} F(\mathbf{x})\, \mathrm{e}^{-\mathrm{i}k(\mathbf{z}\cdot\mathbf{x})}\, \mathrm{d}^n x, \qquad \mathbf{z} \in S^{n-1}$$

where F is a bounded function supported in the convex region τ_c having support function $\rho(\mathbf{u}, \tau_c)$ for any vector $\mathbf{u} \in S^{n-1}$.

Remark 2. [LPW2] is similar to [9, corollary 4.7]. The proof in this former paper uses, much in the spirit of [9, theorems 9, 12], Bessel and spherical harmonics whereas our proof uses a different method of proof which relies on geometry and wave expansions. We believe both methods of proof to be of independent interest. It is also easy to see that, much as in [LPW1], we have the following: suppose that

$$f(\mathbf{z}) = \int_{\tau_c} F(\mathbf{x})\, \mathrm{e}^{-\mathrm{i}k(\mathbf{z}\cdot\mathbf{x})}\, \mathrm{d}^n x, \qquad \mathbf{z} \in \mathbb{C}^n$$

where F is a bounded function supported in the convex region τ_c having support function $\rho(\mathbf{u}, \tau_c)$ for any vector $\mathbf{u} \in S^{n-1}$. Then,

$$|f(\mathbf{z})| = O(\mathrm{e}^{k\rho(\mathbf{u})(\mathbf{u}\cdot\mathrm{Im}(\mathbf{z}))}), \qquad \mathbf{u}\cdot\mathrm{Im}(\mathbf{z}) \to \infty.$$

3.1. Problems with local inversion

The new theorems [LPW1] and [LPW2] stated here are inherently different from the conventional Paley–Wiener theorems [PW1] and [PW2]. For example, consider the case where the function $f(\mathbf{z})$ is the boundary value of an entire function $G(\omega)$ which satisfies the conditions of the conventional Paley–Wiener theorem, i.e.,

$$f(\mathbf{z}) = G(\omega)|_{\mathbf{z}=k\omega}.$$

While each theorem guarantees that the associated function is the transform of a compactly supported function in R^n, the supports for these two functions will, in general, be different. Indeed, $f(\mathbf{z})$ is totally independent of components of $G(\omega)$ which vanish on the sphere $\mathbf{z} = k\omega$ so that while these components contribute to the overall support associated with $G(\omega)$ they do not contribute to the support associated with $f(\mathbf{z})$. Such components, which are known as *non-radiating sources*, are well known to play an important role in inverse source and scattering problems.

The difference between the conventional and generalized Paley–Wiener theorems is also apparent from the fact that the two functions f and F are reciprocally related via a Fourier transform pair in the conventional Paley–Wiener theorem. On the other hand, in the new theorems f can be computed from F via the boundary value of a Fourier transform but a unique inverse mapping does not exist. Indeed, any non-radiating source supported within the support of F can be added to F without changing f.

4. Proofs: local inversion via plane wave expansions and geometry

In this section, we present the proofs of our results. Our proofs are new and use a beautiful interplay between plane wave expansions and geometrical arguments. In what follows, we will often need to continue onto complex angles and so when doing this we assume that S^n is extended to the complex sphere defined through the inner product $\langle\cdot,\cdot\rangle = 1$ rather than the branch $\langle\cdot,\cdot\rangle^{1/2} = 1$. As is well known, the first definition allows for analytic continuations onto complex angles.

4.1. Construction of plane wave approximant

In this subsection, we construct a plane wave approximant which will prove important in what follows. Before we do this, we find it convenient to briefly recall some elementary facts which we will often use.

Recall $z \in S^1$ iff $z := \cos(\alpha)\hat{x}_1 + \sin(\alpha)\hat{x}_2$, $\alpha \in \mathbb{R}$, where $\hat{x}_1$ and $\hat{x}_2$ are orthogonal and of unit length. Here, $|z|^2 = z \cdot z = \cos(\alpha)^2 + \sin(\alpha)^2 = 1$ because of orthogonality. Similarly, $z \in S^2$ iff $z = \sin(\alpha)\cos(\beta)\hat{x}_1 + \sin(\alpha)\sin(\beta)\hat{x}_2 + \cos(\alpha)\hat{x}_3$, $\alpha, \beta \in \mathbb{R}$, where $\hat{x}_1$, $\hat{x}_2$ and $\hat{x}_3$ are again orthogonal and of unit length. Now let $f : \mathbb{C}^n \to \mathbb{C}$ be a function. Then, as is well known, see [6], the restriction $f_{S^{n-1}}$ is analytic on S^{n-1}, iff it is analytic in the polar angles α and β, respectively[3]. In what follows, see [3, pp 14–5], we will also need to employ the idea of plane wave expansions of smooth solutions to the homogeneous Helmhotz equation

$$(\nabla^2 + k^2)(.)(\mathbf{x}) = 0, \qquad x \in \mathbb{R}^n \tag{4.1}$$

where ∇^2 is the Laplacian operator in R^n and k is a real positive parameter called the 'wavenumber', see [3]. Let now $\mathbf{u} \in S^{n-1}$ be a unit vector and $a > 0$. In what follows, we will consider a class of solutions to (4.1) in a half-space:

$$U_{\mathbf{u},a} := \{\mathbf{x} \in \mathbb{R}^n : x_2 = \mathbf{u} \cdot \mathbf{x} > a\}$$

4.2. LPW1: the case $n = 2$

In this subsection, we prove theorem 3 for $n = 2$. We begin with our first

Lemma 1 (plane wave expansion in $\mathbb{R}^2$). *Let $\mathbf{u} \in \mathbb{R}^2$ be a real unit vector and $k, a > 0$ positive numbers. Also let $f : \mathbb{C}^2 \to \mathbb{C}$ be a function whose restriction f_{S^1} is analytic and suppose that f satisfies the condition*

$$|f(\mathbf{z})| = O(\mathrm{e}^{ka(\mathbf{u}\cdot \mathrm{Im}\,\mathbf{z})}), \qquad \mathbf{u}\cdot \mathrm{Im}\,\mathbf{z} \to \infty, \quad \mathbf{z} \in \mathbb{C}^2. \tag{4.2}$$

Now define a plane wave expansion as follows: let

$$\mathbf{z} := \hat{\mathbf{x}}_1 \sin\alpha + \hat{\mathbf{x}}_2 \cos\alpha, \qquad \alpha \in \mathbb{R}$$

[3] We note that when S^n is extended to be complex, we follow the standard definitions of $\mathbb{C}^2$ and $\mathbb{C}^3$ as used, for example, in [10, p 1533].

for some orthogonal $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ of unit length in a Cartesian coordinate system where the polar x_2 axis is directed along the $\mathbf{u}$ direction. Set

$$\phi_{\mathbf{u}}(\mathbf{x}) := \int_{-\frac{\pi}{2}+\mathrm{i}\infty}^{\frac{\pi}{2}-\mathrm{i}\infty} f(\mathbf{z})\,\mathrm{e}^{\mathrm{i}k(\mathbf{z}\cdot\mathbf{x})}\,\mathrm{d}\alpha, \qquad \mathbf{x} \in \mathbb{R}^2. \tag{4.3}$$

Then (4.3) converges uniformly throughout the half-space $U_{\mathbf{u},a}$, satisfies the homogeneous Helmholtz equation (4.1) there and the following asymptotic condition:

$$\phi_{\mathbf{u}}(|\mathbf{x}|\mathbf{z}) \sim -\frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}{\sqrt{8\pi k}} f(\mathbf{z}) \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}} \tag{4.4}$$

as $k|\mathbf{x}| \to \infty$ in $U_{\mathbf{u},a}$.

Proof. Because of (4.2), we readily have

$$\begin{aligned} |f(\mathbf{z})\,\mathrm{e}^{\mathrm{i}k(\mathbf{z}\cdot\mathbf{x})}| &= O(\mathrm{e}^{-k(\mathbf{x}\cdot\mathbf{u}-a)\,\mathrm{Im}\,\mathbf{z}\cdot\mathbf{u}}), \qquad &\mathbf{u}\cdot\mathrm{Im}\,\mathbf{z} \to \infty \\ &= O(\mathrm{e}^{-k(\mathbf{x}\cdot\mathbf{u}-a)\mathrm{Im}\cos\alpha}), \qquad &\mathbf{u}\cdot\mathrm{Im}\,\mathbf{z} \to \infty. \end{aligned}$$

We now note that $\mathrm{Im}\cos\alpha \to \infty$ when $\alpha \to \pm\pi/2 \mp \mathrm{i}\infty$ so that

$$\lim_{\alpha\to\pm\pi/2\mp\mathrm{i}\infty} |f(\mathbf{z})\,\mathrm{e}^{\mathrm{i}k\mathbf{z}\cdot\mathbf{x}}| = 0. \tag{4.5}$$

As this convergence is exponential, it is easy to conclude that as long as $\mathbf{u}\cdot\mathbf{x} > a$, the integral (4.3) converges uniformly throughout the half-space $U_{\mathbf{u},a}$. Moreover, the function $\phi_{\mathbf{u}}(\mathbf{x})$ satisfies the homogeneous Helmholtz equation (4.1) throughout $U_{\mathbf{u},a}$ since the plane waves $\exp(\mathrm{i}k(\mathbf{z}\cdot\mathbf{x}))$ satisfy (4.1). Next, we note that the integrand can be written in the form $\mathrm{i}|x|(c\cos(\alpha) + s\sin(\alpha))$ where the real vector $\mathbf{x} := (|x|c, |x|s)$ and $c^2 + s^2 = 1$. Then, the critical points in the complex α plane occur where

$$-c\sin(\alpha) + s\cos(\alpha) = 0$$

or in other words $\alpha = \phi + n\pi$ where ϕ is an angle with $\cos(\phi) = c$, $\sin(\phi) = s$ and n is an arbitrary integer. So, there are an infinite number of critical points equally spaced along the real axis in the complex α plane. To get an asymptotic expansion under the analytic condition on f, we deform the path of integration to pass over the unique critical point in the real interval $-\pi/2 < \alpha < \pi/2$ with some expected different behaviour possibly if $\phi = \pi/2 \bmod \pi$ and we apply the method of steepest descent, see figure 1. This gives the complete asymptotic expansion in exponentials times descending powers of $|x|$ as required. This completes the proof of the lemma. □

Remark 3. The integrand in (4.3) is an analytic function of the integration variable α as well as the real space coordinates x_1, x_2. It follows from this, and the fact that the integral converges uniformly throughout the half-space $U_{\mathbf{u},a}$, that the function $\phi_{\mathbf{u}}(\mathbf{x})$ is an analytic function of $\mathbf{x}$, regular throughout this half-space[4]. Moreover, due to the analyticity of the integrand, the α contour of integration in (4.3) can be deformed between the end points $-\pi/2 + \mathrm{i}\infty$ and $\pi/2 - \mathrm{i}\infty$ as shown. More precisely, if this contour is taken to lie along the complex contour from $-\pi/2 + \mathrm{i}\infty$ to $-\pi/2$ and then along the real α axis from $-\pi/2$ to $\pi/2$ and then finally down the complex contour from $\pi/2$ to $\pi/2 - \mathrm{i}\infty$ as illustrated in figure 1, the integration is over real-valued unit vectors z that span the unit semi-circle $\mathbf{z}\cdot\mathbf{u} > 0$ and complex unit vectors such that $\mathrm{Re}\,\mathbf{z}\cdot\mathbf{u} = 0$ and $0 \leqslant \mathrm{Im}\,\mathbf{z}\cdot\mathbf{u} < \infty$. The plane waves $\exp(\mathrm{i}k(\mathbf{z}\cdot\mathbf{x}))$ comprising the

[4] In fact, given $\epsilon > 0$, it can be shown that the integral converges uniformly for all complex $\mathbf{z}$ in the polycylinder $\mathbf{u}\cdot\mathrm{Re}\,\mathbf{z} > a + \epsilon$, $|\mathrm{Im}\,\mathbf{z}| < \epsilon$. See [2] and [22] for a complete discussion of the properties of analytic functions of several variables over real environments.

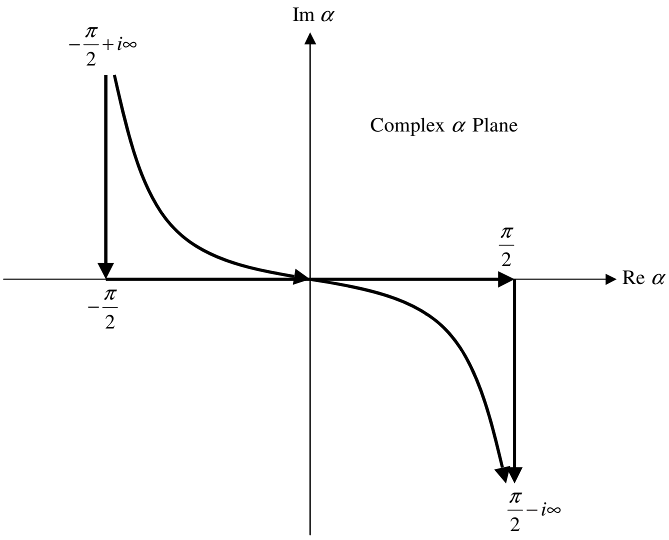

Figure 1. The contour of integration used in (4.3). The contour can be deformed to lie along the lines $\mathrm{Re}\,\alpha = \pm\pi/2$ and along the real axis between $-\pi/2$ and $\pi/2$ as illustrated in the figure.

expansion in (4.3) correspondingly separate into the two classes of *homogeneous plane waves* (corresponding to real unit propagation vectors) and *evanescent plane waves* (corresponding to complex unit propagation vectors) which decay exponentially with increasing x_2.

Next we have

Lemma 2. *Let* $\mathbf{u}, \mathbf{v} \in S^1$ *be directed along the positive* x_2 *and* x_2' *axes of Cartesian axes* x_1, x_2 *and* x_1', x_2'*, respectively, and satisfying that* $\mathbf{u} \cdot \mathbf{v} = \cos\theta$ *with* $0 \leqslant \theta < \pi/2$*. Then, the functions* $\phi_{\mathbf{u}}$ *and* $\phi_{\mathbf{v}}$ *defined by the plane wave expansions (4.3) are equal in* $U_{\mathbf{u},a} \cap U_{\mathbf{v},a}$*.*

Proof. To prove the lemma we let $\mathbf{u} = \hat{\mathbf{x}}_2$ be directed along the positive x_2 axis and $\mathbf{v} = \hat{\mathbf{x}}_2'$ be directed along the positive x_2' axis of two Cartesian coordinate systems x_1, x_2 and x_1', x_2' rotated by the angle $\theta < \pi/2$ about the origin relative to each other as illustrated in figure 2. We then construct the two functions $\phi_{\mathbf{u}}$ and $\phi_{\mathbf{v}}$ according to (4.3). The two functions are analytic in the respective half-spaces of convergence $U_{\mathbf{u},a}$ and $U_{\mathbf{v},a}$ of their defining plane wave expansions and both plane wave expansions converge throughout the intersection of these two half-spaces as illustrated in figure 2. We may take the α and α' contours of integration in the plane wave expansions of $\phi_{\mathbf{u}}$ and $\phi_{\mathbf{v}}$ to lie along the real axis between $-\pi/2$ to $\pi/2$ and along the lines $\mathrm{Re}\,\alpha\,\mathrm{Re}\,\alpha'$ equal to $\pm\pi/2$ as illustrated in figure 1. Then, we write both expansions in terms of the α and α' integration variables as follows:

$$\phi_{\mathbf{u}}(\mathbf{x}) = \int_{-\frac{\pi}{2}+\mathrm{i}\infty}^{\frac{\pi}{2}-\mathrm{i}\infty} f(\mathbf{z})\,\mathrm{e}^{\mathrm{i}k(\mathbf{z}\cdot\mathbf{x})}\,\mathrm{d}\alpha, \qquad \mathbf{x} \in U_{\mathbf{u},a},$$

$$\phi_{\mathbf{v}}(\mathbf{x}) = \int_{-\frac{\pi}{2}+\theta+\mathrm{i}\infty}^{\frac{\pi}{2}+\theta-\mathrm{i}\infty} f(\mathbf{z})\,\mathrm{e}^{\mathrm{i}k(\mathbf{z}\cdot\mathbf{x})}\,\mathrm{d}\alpha', \qquad \mathbf{x} \in U_{\mathbf{v},a},$$

where the α contour in the first integral is along the real axis from $-\pi/2$ to $\pi/2$ and the α' contour in the second integral is along the real axis from $-\pi/2+\theta$ to $\pi/2+\theta$ in the second integral. Now for $\mathbf{x} \in U_{\mathbf{u},a} \cap U_{\mathbf{v},a}$, the two plane wave expansions then differ by $\delta\phi(\mathbf{x})$ which,

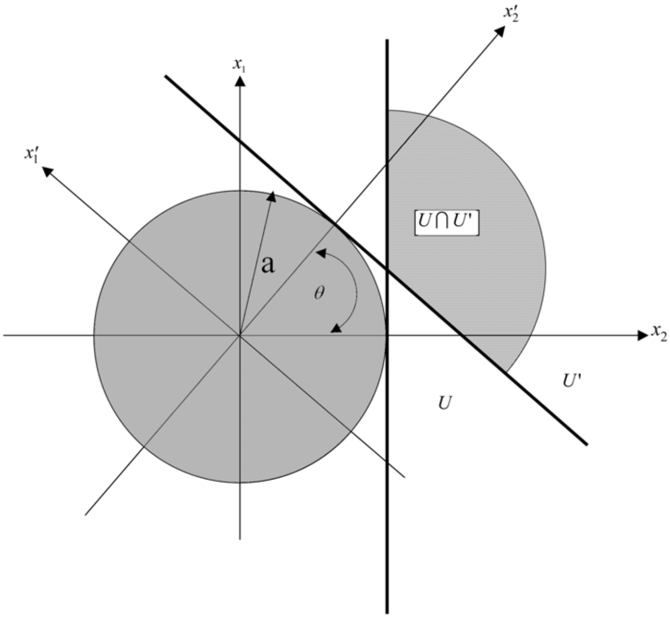

Figure 2. The two functions $\phi_{\hat{\mathbf{x}}_2}(\mathbf{x})$ and $\phi_{\hat{\mathbf{x}}_2'}(\mathbf{x})$ are analytic functions in the respective half-spaces $U = \{\mathbf{x} : \mathbf{x_2} > \mathbf{a}\}$ and $U' = \{\mathbf{x} : \mathbf{x}_2' > \mathbf{a}\}$ including the common region $U \bigcap U'$. The cross-hatched region in the figure depicts a circular subset of the common region.

we observe, can be expressed in the form

$$\delta\phi(\mathbf{x}) := \phi_{\mathbf{u}}(\mathbf{x}) - \phi_{\mathbf{v}}(\mathbf{x}) = \int_{-\frac{\pi}{2}+\mathrm{i}\infty}^{-\frac{\pi}{2}+\theta+\mathrm{i}\infty} f(\mathbf{z})\,\mathrm{e}^{\mathrm{i}k(\mathbf{z}\cdot\mathbf{x})}\,\mathrm{d}\alpha - \int_{\frac{\pi}{2}-\mathrm{i}\infty}^{\frac{\pi}{2}+\theta-\mathrm{i}\infty} f(\mathbf{z})\,\mathrm{e}^{\mathrm{i}k(\mathbf{z}\cdot\mathbf{x})}\,\mathrm{d}\alpha \tag{4.6}$$

The integrands of the above two integrals are analytic functions of α that tend to zero at the end points of the α contours as $\alpha \to \pm\pi/2 \mp \mathrm{i}\infty$ and as $\alpha \to \pm\pi/2 + \theta \mp \mathrm{i}\infty$ because of (4.5) and the fact that $\mathbf{u} \cdot \mathbf{v} = \cos(\theta)$.

Moreover, because the asymptotic condition (4.2) applies for any fixed unit vector in $\mathbb{R}^2$, applying it to suitable vectors $\mathbf{v}'$ instead of $\mathbf{v}$ in the above applies that the integrands above also tend to zero at intermediate points $\alpha \to \pm\pi/2 + \chi \mp \mathrm{i}\infty$ with $0 \leqslant \chi \leqslant \theta$. Thus each integral is separately zero and it then follows that $\delta\phi(\mathbf{x}) = 0$. As $\mathbf{x}$ is arbitrary, $\delta\phi(\mathbf{x}) = 0$ at all points $\mathbf{x} \in U_{\mathbf{u},a} \cap U_{\mathbf{v},a}$ which establishes the lemma. □

We are ready for the

Proof of theorem 3 for *n* = 2. We first show the statement in remark 1(c). Indeed, we have

$$|f(\mathbf{z})| \leqslant \max_{\mathbf{x}\in B_a} |F(\mathbf{x})| \int_{B_a} \mathrm{e}^{k \mathrm{Im}\,\mathbf{z}\cdot\mathbf{x}}\,\mathrm{d}^2(x).$$

Now as $\mathrm{Im}\,\mathbf{z} \cdot \mathbf{x} \to x_2 = \mathrm{Im}\,\mathbf{z} \cdot \mathbf{u}$ where $x_2 = \mathbf{u} \cdot \mathbf{x}$, we see that indeed

$$\begin{aligned} |f(\mathbf{z})| &\leqslant \max_{\mathbf{x}\in B_a} |F(\mathbf{x})| \int_{B_a} \mathrm{e}^{k x_2\, \mathrm{Im}\,\mathbf{z}\cdot\mathbf{u}}\,\mathrm{d}^2(x) \\ &\leqslant 2a \max_{\mathbf{x}\in B_a} |F(\mathbf{x})| \int_{-a}^{a} \mathrm{e}^{k x_2\, \mathrm{Im}\,\mathbf{z}\cdot\mathbf{u}}\,\mathrm{d}x_2 \end{aligned}$$

$$\leqslant 2a \max_{\mathbf{x}\in B_a} |F(\mathbf{x})| \frac{e^{ka\,\mathrm{Im}\,\mathbf{z}\cdot\mathbf{u}} - e^{-ka\,\mathrm{Im}\,\mathbf{z}\cdot\mathbf{u}}}{k\,\mathrm{Im}\,\mathbf{z}\cdot\mathbf{u}}$$
$$= O(e^{ka\mathbf{u}\cdot\mathrm{Im}\,\mathbf{z}}), \qquad \mathbf{u}\cdot\mathrm{Im}(\mathbf{z}) \to \infty.$$

We now proceed with the proof of theorem 3. Let $\mathbf{z} \in S^1$. Given any $\mathbf{u} \in S^1$ and fixed $a > 0$, we know, by virtue of lemma 1, that we may construct a function $\phi_{\mathbf{u}}$, analytic in $U_{\mathbf{u},a}$, and that satisfies (4.1) and (4.4). Moreover, using lemma 2, we see that we may then construct a family of such functions, each of which is equal in a common region of convergence in B_a^c. But the vector $\mathbf{u}$ is arbitrary. So analytic continuation then allows us to define a unique function ϕ analytic in B_a^c satisfying (4.1) there and (4.4) as $k|x| \to \infty$ in every direction within this region. The function ϕ so constructed can be represented by the well-known integral representation [13]

$$\phi(\mathbf{x}) = \int_{\partial B_a} \left[\phi(\mathbf{x}')\frac{\partial}{\partial n'}G(\mathbf{x}-\mathbf{x}') - G(\mathbf{x}-\mathbf{x}')\frac{\partial}{\partial n'}\phi(\mathbf{x}')\right] \mathrm{d}S', \qquad \mathbf{x} \in B_a^c \tag{4.7}$$

where ∂B_a is the boundary of B_a, $\frac{\partial}{\partial n'}$ denotes differentiation with respect to the outward unit normal vector to ∂B_a, and

$$G(.) = -\frac{\mathrm{i}}{4}H_0^+(k|.|)$$

is the two-dimensional outgoing wave Green function to the Helmholtz equation (4.1) with H_0^+ being the zero-order Hankel function of the first kind. Note that the value of ϕ and its normal derivative appearing in (4.7) on ∂B_a are to be understood as appropriate limits of these quantities taken from B_a^c onto ∂B_a.

By use of Green's theorem, (4.7) may be transformed into the form

$$\phi(\mathbf{x}) = \int_{B_a} \left[F(\mathbf{x}')\nabla_{\mathbf{x}'}^2 G(\mathbf{x}-\mathbf{x}') - G(\mathbf{x}-\mathbf{x}')\nabla_{\mathbf{x}'}^2 F(\mathbf{x}')\right] \mathrm{d}^2x', \qquad \mathbf{x} \in B_a^c$$

where F is any continuous and twice differentiable function defined in B_a and satisfying the boundary conditions

$$F(\mathbf{x}) = \phi(\mathbf{x}), \qquad \mathbf{x} \in \partial B_a$$
$$\frac{\partial}{\partial n}F(\mathbf{x}) = \frac{\partial}{\partial n}\phi(\mathbf{x}), \qquad \mathbf{x} \in \partial B_a.$$

By making use of the fact that G is a Green function, we may further reduce the above equation to the form

$$\phi(\mathbf{x}) = \int_{B_a} F(\mathbf{x}')G(\mathbf{x}-\mathbf{x}')\,\mathrm{d}^2x', \qquad \mathbf{x} \in B_a^c \tag{4.8}$$

and where $F(\mathbf{x}') = -(\nabla^2 + k^2)F(\mathbf{x}')$, $\mathbf{x}' \in \partial B_a$. As a final step, we note that

$$G(|\mathbf{x}|\mathbf{z} - \mathbf{x}') \sim -\frac{e^{\mathrm{i}\frac{\pi}{4}}}{\sqrt{8\pi k}} e^{-\mathrm{i}k(\mathbf{z}\cdot\mathbf{x}')} \frac{e^{\mathrm{i}k|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}}, \qquad \mathbf{x}' \in B_a$$

as $k|\mathbf{x}| \to \infty$, $\mathbf{x} \in \mathbf{B_a^c}$ in the direction of the unit vector $\mathbf{z}$. Making use of the above result in (4.8), we obtain

$$\phi(|\mathbf{x}|\mathbf{z}) \sim -\frac{e^{\mathrm{i}\frac{\pi}{4}}}{\sqrt{8\pi k}} \tilde{F}(k\mathbf{z}) \frac{e^{\mathrm{i}k|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}}, \qquad k|\mathbf{x}| \to \infty, \quad \mathbf{x} \in \mathbf{B_a^c} \tag{4.9}$$

where

$$\tilde{F}(k\mathbf{z}) = \int_{B_a} F(\mathbf{x}')\,e^{-\mathrm{i}k(\mathbf{z}\cdot\mathbf{x}')}\,\mathrm{d}^2(x') \tag{4.10}$$

is the spatial Fourier transform of the function F. It remains to compare (4.4) and (4.9). This completes the proof. □

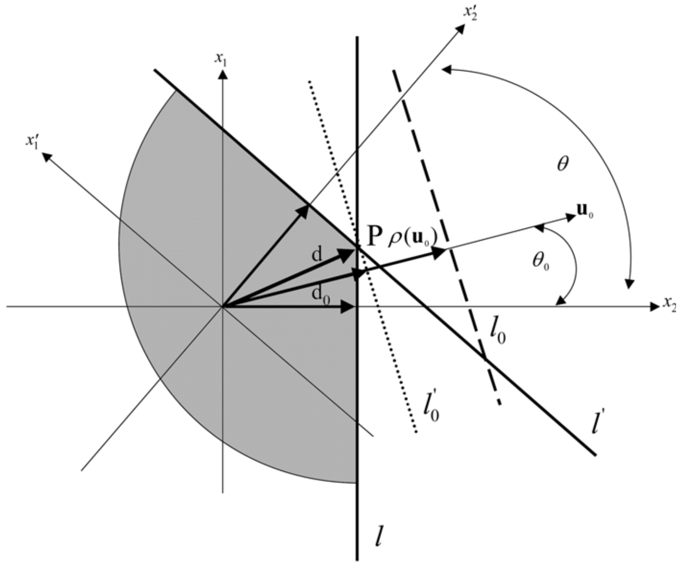

Figure 3. The lines l and l' having unit normals along the directions of the x_2 and x_2' axes, respectively, intersect at the point labelled P which is at distance d from the origin. A third line l_0 having unit normal $\mathbf{u}_0$ lying between the x_2 and x_2' axes at distance $\rho(\mathbf{u}_0)$ from the origin must intersect the two lines l and l' to the left of the point P in order for all three lines to intersect the convex region formed by the intersection of the interiors of the three lines (shown dotted in the figure). The line l_0' is at the maximum distance $\rho(\mathbf{u}_0)$ for this to occur.

4.3. LPW2: the case $n = 2$

We now prove theorem 4 in the case when $n = 2$.

Proof of theorem [LPW2, $n = 2$]. For the proof, we encourage the reader to look at figure 3. Firstly it is clear that exactly the same proof as in theorem 3 shows that we have the desired integral representation over the ball B_a with $a = \rho(\mathbf{u})$ for any $\mathbf{u} \in S^1$. Thus, the essence of theorem 3 is to show that for such $\mathbf{u}$, we may replace $B_{\rho(\mathbf{u})}$ with the convex set τ_c having support function $\rho(\mathbf{u})$, i.e. we need to show that each line

$$U^*_{\mathbf{u},\rho(\mathbf{u})} := \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \cdot \mathbf{u} = \rho(\mathbf{u})\}$$

must intersect τ_c which establishes that $\rho(\mathbf{u})$ must be the support function of τ_c. We will now set out to show this by showing that the convex volume defined by the support function $\rho(\mathbf{u})$ is equal to the convex volume formed from the intersection of all interiors $\overline{U_{\mathbf{u},\rho(\mathbf{u})}}^c$ with the lines $U^*_{\mathbf{u},\rho(\mathbf{u})}$ bounding the half-spaces $\overline{U_{\mathbf{u},\rho(\mathbf{u})}}^c$.

That this is so is not trivial as can be seen from figure 3 which shows three straight lines labelled l, l', and l_0 whose intersection of interiors forms the convex region shown dotted in the figure. Clearly, the perpendicular distance $\rho(\mathbf{u}_0)$ of the line l_0 from the origin is *not* the support function along the direction $\mathbf{u}_0$. In order for $\rho(\mathbf{u}_0)$ to be the support function it is necessary that the line l_0 should intersect or lie to the left of the intersection point P of the two lines l and l'. The limiting case where the line just intersects the point P is shown in the figure and labelled line l_0'. We conclude that $\rho(\mathbf{u}_0)$ will be the support function of τ_c if and only if $\rho(\mathbf{u}_0) \leqslant d_0$ where d_0 is the projection of the distance d to the intersection point P onto the $\mathbf{u}_0$ direction.

Thus, it suffices to prove that $\rho(\mathbf{u}_0) \leqslant d_0$. To do this, we construct, much as in lemma 1, the functions

$$\delta\phi_{\pm}(.) = \int_{C_{\pm}} f(\mathbf{z})\, \mathrm{e}^{\mathrm{i}k(\mathbf{z}\cdot(.))}\, \mathrm{d}\alpha \tag{4.11}$$

where the contour C_+ extends from $-\pi/2+\mathrm{i}\infty$ to $-\pi/2+\theta+\mathrm{i}\infty$ and the contour C_- extends from $\pi/2-\mathrm{i}\infty$ to $\pi/2+\theta-\mathrm{i}\infty$ and where θ is the angle between the positive x_2 and x_2' axes as shown in figure 3. Now, using an argument identical to that employed in the proof of lemma 2, it can be shown that the functions $\delta\phi_{\pm}$ each must vanish in $\overline{B_a^c}$ where

$$a = \mathrm{Max}\{\rho(\mathbf{u}) : \mathbf{u} \in S^1\}.$$

Moreover, it follows from (a) that the integrands of the above integrals vanish at the end points of the contours $C_{\pm}$ and thus converge uniformly when $\mathbf{x}$ lies in the intersection of the two half-spaces

$$\overline{U_{\mathbf{x}_2,\rho(\mathbf{x}_2)}} \cap U^*_{\mathbf{x}'_2,\rho(\mathbf{x}'_2)}.$$

Since the integrands in (4.11) are analytic functions of the integration variable α as well as $\mathbf{x}$, it then follows that the functions must also vanish throughout the region of analyticity which includes the real environment formed by the intersection

$$U_{\mathbf{x}_2,\rho(\mathbf{x}_2)} \cap U_{\mathbf{x}'_2,\rho(\mathbf{x}'_2)}.$$

If we now distort the contour C_+ to run at constant Imα from $-\pi/2+\mathrm{i}\infty$ to $-\pi/2+\theta+\mathrm{i}\infty$ and the contour C_- to run at constant Imα from $\pi/2-\mathrm{i}\infty$ to $\pi/2+\theta-\mathrm{i}\infty$ and make use of the mean value theorem [1], we conclude that there must exist at least one point along each contour at which the integrands must vanish, i.e.,

$$\lim_{\alpha\to\mp\pi/2+\theta_0\pm\mathrm{i}\infty} |f(\mathbf{z})\, \mathrm{e}^{\mathrm{i}k(\mathbf{z}\cdot\mathbf{x})}| \to 0 \tag{4.12}$$

for some $\theta_0 < \theta$ where, in particular, θ_0 is the angle formed between the $\mathbf{u}_0$ and the $\mathbf{x}_2$ directions in figure 3. Now (4.12) must hold at all points in the intersection $U_{\mathbf{x}_2,\rho(\mathbf{x}_2)} \cap U_{\mathbf{x}'_2,\rho(\mathbf{x}'_2)}$. Thus if we take, amongst all such points, the intersection point P as indicated in figure 3 and make use of (a), we learn that (4.12) yields

$$\lim_{\alpha\to\mp\pi/2+\theta_0\pm\mathrm{i}\infty} |f(\mathbf{z})\, \mathrm{e}^{\mathrm{i}k\mathbf{z}\cdot\mathbf{x}}| = \lim_{\mathrm{Im}\,\mathbf{u}_0\cdot\mathbf{z}\to\infty} \exp(-k(d_0-\rho(\mathbf{u}_0))\mathrm{Im}\,\mathbf{u}_0\cdot\mathbf{z}) = 0$$

where d_0 is the projection of the distance d onto the direction $\mathbf{u}_0$. We must then conclude from this that $\rho(\mathbf{u}_0) \leqslant d_0$ which has to be proved. □

4.4. Remaining proofs

In this last subsection, we explain how theorems 2 and 3 hold in the case $n = 3$. As it turns out, both proofs follow in a similar way to the case $n = 2$ and so we only provide the analogue of lemma 1 in this case which is the essential ingredient. As there are no new ideas to the proof of this later lemma, we choose to state it without proof.

We have

Lemma 3 (plane wave expansion in $\mathbb{R}^3$). *Let $\mathbf{u} \in \mathbb{R}^3$ be a real unit vector and $k, a > 0$ be real positive numbers. Also let $f : \mathbb{C}^3 \to \mathbb{C}$ be a function whose restriction to S^2 is analytic. Suppose we have*

$$|f(\mathbf{z})| = O(\mathrm{e}^{ka(\mathbf{u}\cdot\mathrm{Im}\,\mathbf{z})}), \qquad \mathbf{u}\cdot\mathrm{Im}\,\mathbf{z} \to \infty, \quad \mathbf{z} \in \mathbb{C}^2.$$

Let

$$\mathbf{z} = \hat{\mathbf{x}}_1 \sin\alpha\cos\beta + \hat{\mathbf{x}}_2 \sin\alpha\sin\beta + \hat{\mathbf{x}}_3\cos\alpha, \qquad \alpha, \beta \in \mathbb{R}$$

for some orthogonal $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_2$ of unit length in a Cartesian coordinate system where the polar x_3 axis is directed along the $\mathbf{u}$ direction. Define

$$\phi_{\mathbf{u}}(\mathbf{x}) = \int_{-\pi}^{\pi} \mathrm{d}\beta \int_{0}^{\pi/2-\mathrm{i}\infty} \sin\alpha f(\mathbf{z})\,\mathrm{e}^{\mathrm{i}k\mathbf{z}\cdot\mathbf{x}}\,\mathrm{d}\alpha. \tag{4.13}$$

Then the plane wave expansion (4.13) converges uniformly throughout the half-space $U_{\mathbf{u},a}$ and satisfies the homogeneous Helmholtz equation (4.1) there. Also we have the asymptotic condition

$$\phi_{\mathbf{u}}(|\mathbf{x}|\mathbf{z}) \sim -\frac{1}{4\pi} f(\mathbf{z}) \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}|}}{|\mathbf{x}|}$$

as $k|\mathbf{x}| \to \infty$ in $U_{\mathbf{u},a}$.

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