Visualising integers, distance and groups on number lines




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Visualising positive and negative numbers on a number line is helpful for exploring problems involving operations with positive and negative numbers. This is because number lines lend themselves to exploring problems involving continuous linear contexts such as travelling distances and temperature. Teachers in a professional development program explored how to make sense of operations with positive and negative integers by visualising a context on a number line.

Understanding why a negative integer times a negative integer results in a positive integer is a challenge. The middle school teachers in our professional development project reported that they end up teaching rules for integers through memorisation because they do not understand why some of these rules work. One teacher said, "I know the rules, but have difficulty visualising the reasons behind why this works."

According to the *Numeracy Learning Continuum*, students at the end of Year 8 should be able to understand and use numbers in context by being able to "compare, order and use positive and negative numbers to solve every day problems" (ACARA, 2010) and according to the *Common Core Standards for Mathematics* in the United States (CCSSM 2010), seventh-grade students should learn how to multiply and divide signed numbers (e.g., (−1)(−1)=1). Interestingly, Kilhamn (2009) found that one-third of college students were unable to solve the following simple subtraction problem "(−3)−(−8)." Gullick and Wolford (2014) specify that learning algebra and more sophisticated maths involves being able to work with negative numbers and operations.

Furthermore, the middle school teachers in our project found it challenging to connect the rules for integers to a real-world context. When students learn rules without meaning, it is hard to make sense of why things work and know what context to use these rules. Wessman-Enzinger and Moone (2014) explain that contexts are important for making sense of operations with integers. Contexts from multiple disciplines such as timelines, temperature, and deep-sea diving ~~to~~ help students visualise operations with integers on a number line (Van de Walle, Karp, & Bay-Williams 2016). Number lines are very helpful for thinking about linear distance. Therefore, this article specifically focuses on the linear contexts that lend themselves to exploring integers on a number line (Van de Walle et al.).

Distance and absolute value with positive integers on a number line

Focusing attention on the distance from zero is helpful for developing conceptual understanding of absolute value (Taylor & Mittag, 2015). Students can think about the distance from zero involving both directions on the number line. The distance moved on the number line regardless of directionality involves thinking about the concept of absolute value. Thinking about movement on the number line on opposite sides of the zero involves thinking about directionality and the meaning of positive and negative numbers.

We initially wanted teachers to think about the integers on the positive side of the number line. Therefore, we presented teachers with a context of a robotic book retrieval system in a library. The robotic arm grabs a book and puts it in a bin. The retrieval bin starts at zero and moves back and forth to retrieve books. We asked the teachers "If the robotic arm started at aisle two and moved to aisle six, what distance did it move?"

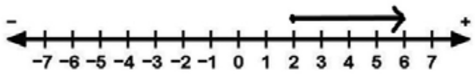

Figure 1. Number line that shows an example with the context of a robotic arm at a library.

The teachers quickly answered "four". Then they were asked to represent this problem and solution as a number sentence. They responded as follows:

2+4=6 Counted from 2 to 6. (Moved up on the number line. Distance moved from aisle 2 to aisle 6 is the absolute value 4)

6−4=2 Started at 6 and counted down to 2 (moved down on the number line. Absolute value is 4)

2−6=−4 Where it started and where it stopped (The absolute value of −4 is the distance.)

6−2=4 Subtract the starting point from the ending point.

The teachers explained that the answer was four because it represented the distance travelled and movement on the number line. In other words, it represented the absolute value 4. Wessman-Enzinger and Mooney (2014) call this conceptual model a translation. This is when integers are treated as vectors or directed numbers.

Absolute value and distance involving subtraction of integers

Movement to the left side of the number line involves subtraction. This process involves thinking about the meaning of the operation, the value of the number, and the distance moved on the number line. The next question we posed to extend thinking with negative integers: "How far would the robotic arm move from 6 to −2?"

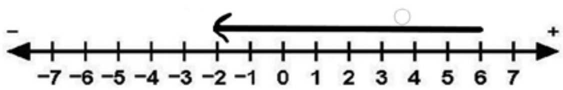

Figure 2. A number line that shows the distance from 6 to -2 is 8.

Teachers made the following responses:

6−(−2)=8 "It is the absolute value of 6 plus two." (Moved 8 spaces backwards and took the absolute value)

"It is absolute value of 6 – 0 plus the absolute value of 0 – 2" (Thought of it as 6 hops on the number line backward. And 2 more hops from zero.)

When the professional development facilitator asked the teachers to reflect on what was meant by "distance plus". "Why plus?" A teacher pointed out that it did start at 6 and end at −2. This means the distance from 6 to 0 was 6. And the distance from 0 to –2 was 2. So that would mean that the robotic traveled 8 rows to the left. This means students can figure out the distance travelled by 6 to −2 by setting up the following equation 6−x=−2 and solving for *x* which would be 8. The important point here is not to confuse the operation with the value of the number. The number line is a helpful tool to visualise that the operations (addition, subtraction, and multiplication) require different actions by the problem solver.

Multiplication of integers on a number line

Multiplying integers on a number line requires thinking about the iteration of a number of groups. The size of the group and the number of groups must be specified. We wanted the teachers to extend their understanding of adding and subtracting integers to multiplication of integers. Therefore, we posed the following question: "How can we interpret 2|6−4| in terms of distance for the robotic arm?" Teachers as a group immediately responded that the answer to the question was 4.

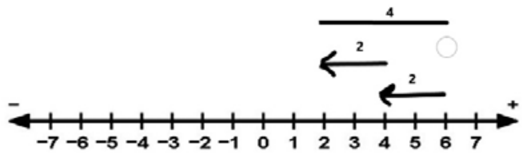

Figure 3. Distance is |2• (-2)| = 4

A teacher commented that it was like the previous problem where the robotic arm started at 6 and moved to 4 which is the distance of 2. If it goes twice the distance of 2 then it would be 4 or, 2|6−4|=4. This problem built on absolute value and distance involving subtraction of integers.

Some middle school teachers reported that their students got confused using number lines. As we explored this further, it was determined that some teachers did not think about the number line as the measure of distance and some certainly did not connect it to multiplying numbers on the number line. For example, when using the number line, it is important to think about the unit 2 as the distance from 0 to 2 that make up two units. A segment in the number line represents a unit that has equal distance.

Exploring zero on number line involving positive and negative integers

We then explored the relationship of zero on a number line in relation to positive and negative integers. We first started with the positive integers and asked teachers to explore how they could get to zero from any positive number. In other words, we had them explore the meaning of additive inverse.

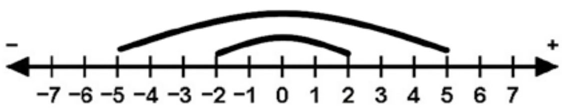

Figure 4. Moving between 2 and -2, and 5 and -5 on a number line.

One teacher reported that if you start with 2, you have to add –2 to get to zero. (2 + (–2)) or subtract 2 – 2. A negative integer can be added, or a positive integer can be subtracted. When subtracting, it involves an operation of removing 2 or taking 2 away. Some teachers reported that they had taught students strategies on how to use the number line as the "opposite of right was left" and students became confused as they did not have any meaning attached to the strategies. Students used it as a meaningless rule.

A teacher came up with the following example. If you walk two meters from zero on the number line, you have to walk in the other direction to get back to zero. This involves adding the inverse to get to zero. Another teacher said, "for example, you wake up in the morning and its zero degrees and then it goes up to 20 degrees, then to get the temperature back to zero you go down 20 degrees." These examples work with the number line because they are continuous context problems which lend themselves to a number line: the temperature gradually increases and drops, and walking distance requires moving from one location to the other.

Another teacher pointed out that you could use an example of positive and negative charged atoms that cancel each other out. However, this context does not lend itself to thinking about positive and negative numbers on a number line. Other tools such as algebra tiles work well to cancel positive and negative integers.

Exploring additive identity on the number line involves a counterbalance model where students think about using positive and negative integers to balance and cancel each other out (Weissman-Enzinger & Mooney, 2014). In relation to the number line, a way to get to zero from a positive integer is to negate it with a negative integer or subtract the number, which moves the distance back to zero.

The zero represents a relative position on the number line when thinking about numbers as a movement across the number line (Wessman-Enzinger & Mooney, 2014). In other words, the zero is a reference point on the number line and is significant for measuring distance. Positive numbers and negative numbers show distance from zero. According to the Common Core Standards (CCSSM, 2010), sixth-grade students should recognise that the opposite of zero is zero.

Negative numbers: Continuity vs. discrete objects (distance as continuous)

When thinking about positive and negative integers as opposites, you are thinking about two discrete quantities. If you think about a continuous number such as going up and down stairs, you are looking at distance or movement. The continuous contexts lend themselves to the number line. For example, you are starting on the second floor and moving to zero. This idea of continuous quantity is important for working with linear functions and other functions in higher level mathematics.

Multiplication of negative and positive integers

Positive times a negative integer

Teachers were asked to think about the following problem in relation to the robotic arm problem context and the negative part of the number line. "What is 2 • (–3)?" A teacher replied it is means 2 groups of −3 so it is −6. The robotic arm is moving 6 spots to the left.

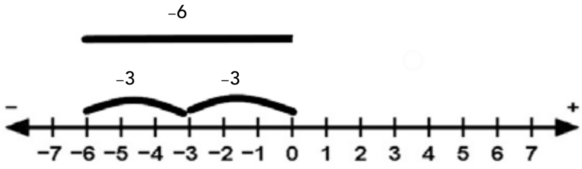

Figure 5. Multiplying negative integers of two groups of negative 3.

Negative times a positive integer

We asked, "what is meant by −2 groups of 3 or (−2 x 3)?" This required further thinking because it was different than the previous question where the teachers had to think about adding two sets of negative numbers. Now they had to think about taking away two groups of positive numbers. The meaning of this context was a challenge to figure out.

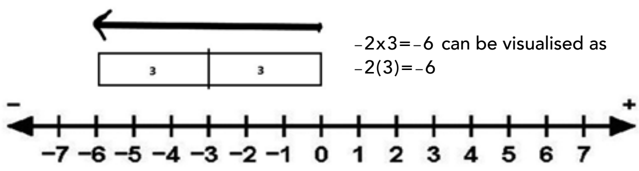

Figure 6. Negative 2 times 3.

Multiplying two negative integers

We now posed the question: what does −2 times −3 mean and how can we make sense of it on the number line? For example, you were in an elevator going to the underground parking lot. A teacher explained that you can think of it as the same problem with 2 times −3. This would be −6 and then you take negative of that. This would be −(−6). This means you add six. The problem could be thought of "negative 2 sets" of "negative 3." The equation is −(2)(−3)=6.

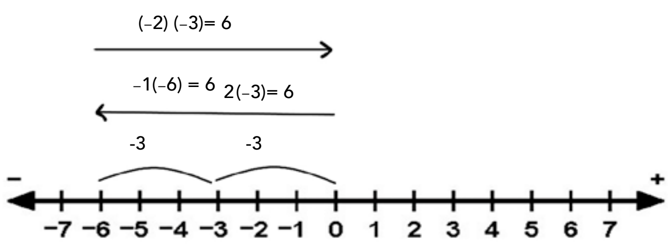

Figure 7. Two sets of negative 3.

The distributive property helps make sense of multiplying two negative numbers.

Conclusion

Working with integers (positive integers, negative integers and zero) on the number line requires linear continuous contexts. It allows students to visualise operations. In order to perform operations with positive and negative integers, students need to understand the distinction between negative numbers versus subtraction. Subtraction is an operation whereas a negative whole number is an integer. Supporting students to develop an understanding of integers involves the fundamental idea of a mathematical group related to group theory. one particular group is the set of integers along with an operation. This understanding lays the foundation for students to develop a more sophisticated understanding of algebra including:

- $(-a)+a=0$
- $a(-a)=-a2$
- $-a(-a)=a2$

The distance between integers is always non-negative; hence the absolute value of 2 is the distance between 2 and zero which is 2. The absolute value of −2 is the distance between −2 and zero which is 2. We found the following progression of problems that can be solved using a number line helpful for visualising operations with integers involving linear distance.

We first limited exploration of the number line to positive integers. Students can explore moving right and left and measuring distance and understanding the meaning of absolute value. A progression from addition and subtraction to multiplication can be used. Once students have explored addition, subtraction, and multiplication on the right-hand side of the number line, adding questions that involved negative integers provided more complexity.

We found that carefully choosing problems and having teachers think about the meaning of the integers and operations within a problem context were helpful. This allowed teachers to explore integers and operations by visualising them. Carefully selecting problem types to support student learning is helpful for making connections (Bishop, Lamb, Phillip, Whitacre, & Schappelle, 2016). Van de Walle et al. (2016) suggests that a teacher can provide students with a set of problems where students can come up with generalisations about the signs. Bishop et al. (2016) point out the importance of having students explore relationships and justifying their answers by posing questions to make them think. The chart below is helpful for supporting students to explore number lines and operations with positive and negative integers on the number line.

Table 1. Visualising Operations with Positive and Negative Integers on a Number line.

Concept	Key Ideas	Number Line Tasks
Distance from zero and absolute value	Understand exploring distance on the number line. Movement forward on the number line to indicate distance. Can start at zero and move to any number on the number line. The distance is the measure between two points on the number line.	On the number line, explore positive numbers moving to the right **Problem:** If you started your journey at 2 km from your house and travelled 2 more km what distance did you travel?
Absolute value and distance involving subtraction of integers	Moving to the left of the number line involves subtraction of distance. To figure out the distance traveled, the starting point needs to be subtracted from the ending point and then find absolute value.	On the number line, explore moving towards zero or moving left on number line **Problem:** If there was 6 m of water in a dam and now there is 2 m left. How much water was drained?
Multiplication of integers on a number line	Exploring multiplication on the number line involves examining the number of units multiplied by the size of the unit.	On the number line, explore multiplication on both sides of zero **Problem:** If you ran a race starting at distance marker 3 and ran for three km and doubled the amount of km you ran, how many km would you have run and what would be the distance you would have run from km marker 3? **Problem:** Elaina started at her house and ran 10 km. She ran back toward her house 2 km. She stopped to get a drink and ran two times as many km back towards her house. How far is she from her house and what is the distance she ran back to her house?

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