Math 416

Worksheet 13. Graph search

We turn toward the algorithmic problem of connectivity: given vertices s and t in a graph G, is there a path from s to t? Can we find it algorithmically?

Breadth-first search The idea: start at s, explore all neighbors one 'layer' at a time; at the end can check whether t is in one of the layers. In this way, BFS produces a rooted tree with root s.

Breadth-first search (BFS) takes as input a graph G and a vertex s in G. It produces a rooted tree with root s and stratifies the graph into layers $L_0, L_1, \ldots$ as follows.

$$
\begin{aligned}
L_0 &= \{s\} \\
L_1 &= \{\text{neighbors of } s\} \\
L_2 &= \{v \notin L_0 \cup L_1 \text{ that are adjacent to a vertex in } L_1\} \\
\vdots \quad &\quad \vdots \\
L_{i+1} &= \{v \notin L_0 \cup \cdots \cup L_i \text{ that are adjacent to a vertex in } L_i\}.
\end{aligned}
$$

The **distance** between two vertices x and y in a graph is the length of the shortest path from x to y (if there is no path we say the distance is ∞). We can give a different definition of the layers of BFS by setting

$$L_i = \{\text{vertices of distance } i \text{ from } s\}.$$

BFS is an instance of the following underspecified algorithm for exploring a graph from a vertex s: Starting with $R = \{s\}$, iteratively grow R by adding new vertices adjacent to a vertex in R, until you run out of such vertices. One way to implement this is to grow R in 'layers' as BFS does.

Depth-first search. Another such algorithm is suggested by a natural approach to exploring a maze: explore until you reach a dead-end and then backtrack.

Algorithm 1: the explore subroutine

```
explore(v):
    visited(v) = true;
    // (previsit work)
    foreach neighbor u of v do
        if not visited(u) then
            explore (u);
    // (postvisit work)
```

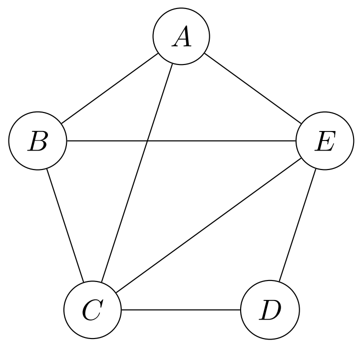
Algorithm 2: Depth-first search

```
DFS(G, s, t):
    Input: a graph G = (V, E), a start vertex s and a target vertex t
    Output: whether s is connected to t
    // initialize
    foreach v ∈ V do
        set visited(v) = false.
    explore (s);
    if visited(t) then
        return true
```

If you visit w in virtue of its being a neighbor of v, then you can also add the edge (v, w) to a tree. At the end, you will have a *spanning tree* of the connected component rooted at the start vertex s. This is called the DFS tree. (The algorithm is underspecified; you should use an ordering on the vertices to break ties.)

Problem 1.

(a) Perform BFS on the graph in (1) starting at any vertex. Identify the layers L_i.
(b) Perform DFS on the graph in (1) starting at any vertex.



(1)

Problem 2. Prove the following.

(a) If x and y are adjacent vertices and $x \in L_i$ and $y \in L_j$, then $|i - j| \leq 1$.
(b) The set of vertices visited by BFS is exactly the connected component containing s.
(c) Explain how edges can be kept track of in the execution of BFS in such a way that BFS produces a **subtree** of G that **spans** the connected component containing s. (Hint: give the vertices an ordering.)

Item (a) says that edges not in the BFS tree pass between vertices in the same layer or between vertices in adjacent layers.

Definition. Recall that a **rooted tree** is a tree (connected, acyclic graph) with a specified vertex r, called the **root**.

Recall that a graph T is a tree iff between any two vertices of T there is a *unique* path. Thus in a rooted tree every vertex has a unique path to the root.

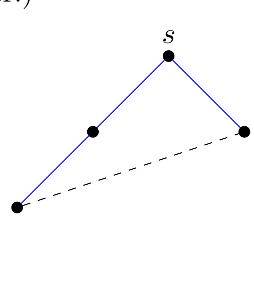
Definition. If x and y are vertices in a rooted tree (T, r), then we say...

(a) x is a **descendant** of y (or y is an **ancestor** of x) if y occurs on the unique path from x to the root r.
(b) x is a **child** of y (or y is a **parent** of x) if x is a descendant of y and a neighbor of y.

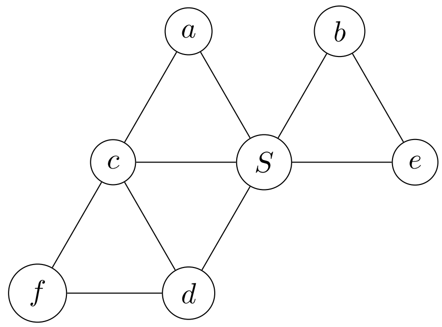
Problem 3. Draw a picture explaining the definition.

Problem 4. (a) Explain how the DFS and the BFS algorithms can be "enriched" to keep track of the edges of a DFS tree and a BFS tree, both rooted trees with root s.
(b) With the graph in (1), which starting vertex gives the BFS tree with the minimum height?

Problem 5. Look at the graph below. If the blue (solid) portion is the DFS tree starting at s, why can't the graph have an edge indicated by the dashed line? (Certainly the DFS tree cannot have that edge, but the original graph can't either!)



Problem 6. Run DFS on the graph in (†), starting at S and then using the alphabetical ordering. Draw the DFS tree. Verify that every non-tree edge is incident to a vertex and its descendant.



(†)

Problem 7. Prove the following assertion: In a call to `explore`(v), all vertices marked 'visited' between calling and the end of the execution are descendants of v in the DFS tree T.

Explain why the following Corollary is true.

Corollary. Suppose that T is the DFS tree rooted at r in a graph G. If x and y are vertices of T and x and y are adjacent in G, then one of x and y is a descendant of the other in T.

Definition. H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A **spanning tree** of G is a subgraph H such that (1) H is a tree and (2) $V(H) = V(G)$.

Theorem. A graph is connected if and only if it has a spanning tree.

Problem 8. Prove the theorem. (*Hint:* What might this have to do with algorithms?)

On implementation and running time In practice, it is often better to represent graphs by **adjacency lists** rather than adjacency matrices. In the adjacency list representation, to each vertex v is associated a list $L[v]$ of the neighbors of v.

Problem 9. While the adjacency-matrix representation takes $O(n^2)$ space for an n-vertex graph, the adjacency-list representation requires $O(m+n)$ space, which for sparse graphs (i.e., graphs for which m is small relative to n) is less. Explain.

Problem 10. Explain why BFS runs in $O(m+n)$ time on a graph with n vertices and m edges, assuming the graph is given by its adjacency list representation.

Problem 11. By consulting one of the CS majors in your group, complete and explain the following analogy.

BFS : queue :: DFS : $\boxed{}$

Problem 12. Give a (vague but convincing) explanation for why DFS runs in linear time, using adjacency lists.

Testing bipartiteness with BFS (Assume that G is connected; if not, we can work component-by-component.)

Definition. A graph G is **bipartite** iff there is a partition $V = R \sqcup B$ of its vertices into two (disjoint) pieces R and B so that every edge is incident to one vertex in R and one in B.

This presents an algorithmic problem: determine whether a graph G is bipartite and, if it is, find a bipartition.

Theorem. G is bipartite if and only if G has no odd cycle.

Problem 13. We're about to prove the harder direction $\Leftarrow$; why is the other direction true?

(*Hint:* Suppose that you have an odd cycle and that the vertices are partitioned into R and B. How many R–B edges can appear in the cycle?)

Here is the idea: start with s and declare $s \in B$. Put all neighbors of s into R. Color their neighbors Blue. Continue. Hope for the best. This will either produce a bipartition or should yield a blue edge or a red edge.

Do BFS starting at s and obtain layers $L_0, L_1, \ldots$.

$$B = L_0 \cup L_2 \cup L_4 \cup \cdots$$
$$R = L_1 \cup L_3 \cup L_5 \cup \cdots \qquad (*)$$

At the end review the m edges to check whether there's an R–R or B–B edge.

Proposition. Let G be a connected graph and $L_0, L_1, \ldots$ the layers produced by BFS. Then exactly one of the following holds.

(i) There is no edge (x, y) of G with $x, y \in L_i$.
This implies that G is bipartite with bipartition $(*)$.

(ii) There is an edge (x, y) of G with $x, y \in L_i$.
This implies that G has an odd cycle and is therefore not bipartite.

Problem 14.

(a) Why can't both conditions hold?

(b) Suppose that $x, y \in L_i$ are adjacent, as in condition (ii). The vertices x and y certainly have a common ancestor in the BFS tree, namely the root r. Explain why there may not be a cycle including x, y, and r, though. So we choose a common ancestor z of x and y in L_j for j maximal.

(c) There are unique paths from x to z and from y to z in T. How long are they?

(d) You should have produced an odd cycle. Verify that you have.

(e) Be sure that you have completed the proof of the Proposition.

Corollary. If G has no odd cycle, then G is bipartite.

Problem 15. Explain why the Corollary follows from the Proposition.

Problem 16. By running BFS starting at vertex A in each case, determine whether each of these graphs is bipartite and if not which odd cycle the algorithm exhibits.