MATH 416, PROBLEM SET 4

Comments about homework.

- Solutions to homework should be written clearly, with justification, in complete sentences. Your solution should resemble something you'd write to teach another student in the class how to solve the problem.
- You are encouraged to work with other 416 students on the homework, but solutions must be written independently. Include a list of your collaborators at the top of your homework.
- You should submit your homework on Gradescope, indicating to Gradescope where the various pieces of your solutions are. The easiest (and recommended) way to do this is to start a new page for each problem.
- Attempting and struggling with problems is **critical** to learning mathematics. Do not search for published solutions to problems. I don't have to tell you that doing so constitutes academic dishonesty; it's also a terrible way to get better at math.

  If you get stuck, ask someone else for a hint. Better yet, go for a walk.

Warning. These are not necessarily *model solutions*; they are meant to help you understand the problems you didn't totally solve and maybe to give you alternative solutions. Sometimes I will give less or more detail here than I would expect from you.

Problem 1. Let a and b be constants. Describe completely and concisely the execution of our polynomial-multiplication algorithm on the inputs

$$A(x) = B(x) = a + bx.$$

Solution. The algorithm begins by padding the coefficient sequences with 0s to obtain $(a, b, 0, 0)$ and $(a, b, 0, 0)$. We are working in the case $n = 4$ and $\zeta = i$. Next, the algorithm evaluates $A(1)$, $A(-i)$, $A(-1)$ and $A(i)$. To achieve this, it calls $\mathrm{FFT}((a, b, 0, 0), -i)$, which runs as follows.

$$(s_0, s_1) = \mathrm{FFT}((a, 0), -1) = (a + 0, a - 0) = (a, a);$$
$$(t_0, t_1) = \mathrm{FFT}((b, 0), -1) = (b + 0, b - 0) = (b, b).$$

Next:

$$r_0 = a + (-i)^0 b = a + b$$
$$r_1 = a + (-i)b = a - ib$$
$$r_2 = a - (-i)^0 b = a - b$$
$$r_3 = a - (-i)b = a + ib.$$

This subroutine returns

$$\mathrm{FFT}((a,b,0,0),-i) = (a+b, a-ib, a-b, a+ib).$$

(Notice that this is evidently the sequence $(A(1), A(-i), A(-1), A(i))$.) Next, the algorithm computes the products of the polynomials in point–value form:

$$\begin{aligned} A(1)B(1) &= (a+b)(a+b) &&= a^2+2ab+b^2 \\ A(-i)B(-i) &= (a-ib)(a-ib) &&= a^2-2iab-b^2 \\ A(-1)B(-1) &= (a-b)(a-b) &&= a^2-2ab+b^2 \\ A(i)B(i) &= (a+ib)(a+ib) &&= a^2+2iab-b^2. \end{aligned}$$

Finally, we interpolate by computing

$$\begin{aligned} &\mathrm{IFT}(a^2+2ab+b^2, a^2-2iab-b^2, a^2-2ab+b^2, a^2+2iab-b^2) \\ &= \tfrac{1}{4}\,\mathrm{FFT}((a^2+2ab+b^2, a^2-2iab-b^2, a^2-2ab+b^2, a^2+2iab-b^2), i). \end{aligned}$$

To compute this, the algorithm calls

$$\begin{aligned} (s_0,s_1) &= \mathrm{FFT}((a^2+2ab+b^2, a^2-2ab+b^2),-1) = (2a^2+2b^2, 4ab) \\ (t_0,t_1) &= \mathrm{FFT}((a^2-2iab-b^2, a^2+2iab-b^2),-1) = (2a^2-2b^2, -4iab) \end{aligned}$$

and finally returns

$$\begin{aligned} r_0 &= s_0 + i^0 t_0 = 4a^2 \\ r_1 &= s_1 + i t_1 \;= 4ab + 4ab = 8ab \\ r_2 &= s_0 - i^0 t_0 = 4b^2 \\ r_3 &= s_1 - i t_1 \;= 4ab - 4ab = 0, \end{aligned}$$

giving

$$A(x)B(x) = (a+bx)^2 = \tfrac{1}{4}(4a^2+8abx+4b^2x^2) = a^2+2abx+b^2x^2,$$

as expected. $\square$

Problem 2. Let $\zeta = e^{i\cdot 2\pi/8} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ and consider the sequence $a = (0,1,2,3,4,3,2,1)$.

(a) Describe completely and concisely the execution of the Fast Fourier Transform that computes $\mathrm{FFT}(a,\zeta)$. [1] (In particular, you should describe the values of all local variables at all stages of the iteration.)

(b) Conclude by giving the unique polynomial of degree ≤ 7 that interpolates the eight points

$$(1,0),\quad (\tfrac{1}{\sqrt{2}} - i\tfrac{1}{\sqrt{2}}, 1),\quad (-i,2),\quad (-\tfrac{1}{\sqrt{2}} - i\tfrac{1}{\sqrt{2}}, 3),$$
$$(-1,4),\quad (-\tfrac{1}{\sqrt{2}} + i\tfrac{1}{\sqrt{2}}, 3),\quad (i,2),\quad (\tfrac{1}{\sqrt{2}} + i\tfrac{1}{\sqrt{2}}, 1).$$

Solution. In this case $n = 8$ and $\zeta = e^{i\cdot 2\pi/8} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$. Notice that $\zeta^2 = i$ and $\zeta^4 = -1$.

The algorithm begins by setting

$$(s_0, s_1, s_2, s_3) = (0,2,4,2)$$
$$(t_0, t_1, t_2, t_3) = (1,3,3,1).$$

Then it computes $\mathrm{FFT}((0,2,4,2),\zeta^2)$ and $\mathrm{FFT}((1,3,3,1),\zeta^2)$, first by computing

$$\mathrm{FFT}((0,4),\zeta^4) = \mathrm{FFT}((0,4),-1) = (4,-4),$$
$$\mathrm{FFT}((2,2),\zeta^4) = \mathrm{FFT}((2,2),-1) = (4,0),$$
$$\mathrm{FFT}((1,3),\zeta^4) = \mathrm{FFT}((1,3),-1) = (4,-2),$$
$$\mathrm{FFT}((3,1),\zeta^4) = \mathrm{FFT}((3,1),-1) = (4,2).$$

(The computation of FFT for input of length 2 was considered at the beginning of Problem 1.)

1. To compute $\mathrm{FFT}((0,2,4,2),i)$ with

$$(s_0, s_1) = \mathrm{FFT}((0,4),-1) = (4,-4)$$
$$(t_0, t_1) = \mathrm{FFT}((2,2),-1) = (4,0),$$

we compute:

$$\begin{aligned} r_0 &= s_0 + i^0 t_0 = 4 + 4 &&= 8 \\ r_1 &= s_1 + i t_1 = -4 + i\cdot 0 &&= -4 \\ r_2 &= s_0 - i^0 t_0 = 4 - 4 &&= 0 \\ r_3 &= s_1 - i t_1 = -4 - i\cdot 0 &&= -4. \end{aligned}$$

2. To compute $\mathrm{FFT}((1,3,3,1),i)$ with

$$(s_0, s_1) = \mathrm{FFT}((1,3),-1) = (4,-2)$$
$$(t_0, t_1) = \mathrm{FFT}((3,1),-1) = (4,2),$$

[1] Recall that we defined the DFT in such a way that $\mathrm{DFT}(a) = \mathrm{FFT}(a,\zeta^{-1})$, so the algorithm should produce $n \cdot \mathrm{IFT}(0,1,2,3,4,3,2,1)$.

we compute:

$$\begin{aligned} r_0 &= s_0 + i^0 t_0 = 4 + 4 = 8 \\ r_1 &= s_1 + i t_1 \qquad\quad\; = -2 + 2i \\ r_2 &= s_0 - i^0 t_0 = 4 - 4 = 0 \\ r_3 &= s_1 - i t_1 \qquad\quad\; = -2 - 2i. \end{aligned}$$

3. Now the algorithm merges the two sequences. With

$$\begin{aligned} (s_0, s_1, s_2, s_3) &= \mathrm{FFT}((8, -4, 0, -4), i) \\ (t_0, t_1, t_2, t_3) &= \mathrm{FFT}((8, -2 + 2i, 0, -2 - 2i), i), \end{aligned}$$

we compute:

$$\begin{aligned} r_0 &= s_0 + \zeta^0 t_0 = 16 \\ r_1 &= s_1 + \zeta t_1 \;\; = -4 + \zeta(-2 + 2i) \\ r_2 &= s_2 + \zeta^2 t_2 = 0 \\ r_3 &= s_3 + \zeta^3 t_3 = -4 + \zeta^3(-2 - 2i) \\ r_4 &= s_0 - \zeta^0 t_0 = 0 \\ r_5 &= s_1 - \zeta t_1 \;\; = -4 - \zeta(-2 + 2i) \\ r_6 &= s_2 - \zeta^2 t_2 = 0 \\ r_7 &= s_3 - \zeta^3 t_3 = -4 - \zeta^3(-2 - 2i). \end{aligned}$$

Using the observation $(-1 + i)\zeta = (1 + i)\zeta = -\sqrt{2}$, one simplifies this result to

$$(16, -4 - 2\sqrt{2}, 0, -4 + 2\sqrt{2}, 0, -4 + 2\sqrt{2}, 0, -4 - 2\sqrt{2}).$$

The x-coordinates of the points given are (in order) $1, \zeta^{-1}, \zeta^{-2}, \ldots, \zeta^{-7}$. And the sequence of y-coordinates is $(0, 1, 2, 3, 4, 3, 2, 1)$. Therefore, the unique degree-7 polynomial interpolating the points has coefficient sequence

$$\begin{aligned} \mathrm{IFT}(0, 1, 2, 3, 4, 3, 2, 1) &= \tfrac{1}{8}\,\mathrm{FFT}((0, 1, 2, 3, 4, 3, 2, 1), \zeta) \\ &= (2, -\tfrac{1}{2} - \tfrac{1}{8}\sqrt{2}, 0, -\tfrac{1}{2} + \tfrac{1}{8}\sqrt{2}, 0, -\tfrac{1}{2} + \tfrac{1}{8}\sqrt{2}, 0, -\tfrac{1}{2} - \tfrac{1}{8}\sqrt{2}). \end{aligned}$$

That is, the polynomial is

$$2 + (-\tfrac{1}{2} - \tfrac{1}{8}\sqrt{2})x + (-\tfrac{1}{2} + \tfrac{1}{8}\sqrt{2})x^3 + (-\tfrac{1}{2} + \tfrac{1}{8}\sqrt{2})x^5 + (-\tfrac{1}{2} - \tfrac{1}{8}\sqrt{2})x^7.$$

It can easily be verified that this polynomial interpolates the given points. □

Notice that the coefficients of this polynomial are all *real* numbers. This won't necessarily be true, even if we start with all real y-values (as in this problem). You should think about what property of the starting sequence ensures this result.

Problem 3. For this problem, fix $n \in \mathbb{N}$ and $\zeta = e^{i\cdot 2\pi/n}$.

(a) Compute $1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1}$.

(b) Compute $1 \cdot \zeta \cdot \zeta^2 \cdot \cdots \cdot \zeta^{n-1}$.

(*Hint:* Your answer might depend on whether n is even or odd.)

(c) Use Euler's formula to prove

$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

(d) Use Euler's formula to prove that for all $x \in \mathbb{R}$ and all $n \in \mathbb{Z}$

$$(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx).$$

Solution.

(a) If $n = \zeta = 1$, then the sum is 1. Suppose $n > 1$. Use the geometric sum formula and remember that $\zeta^n = 1$:

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1} = \frac{1-\zeta^n}{1-\zeta} = 0.$$

(b) Suppose first that n is even. Pair up the factors ζ^k and ζ^{n-k} for $k \in [1, n/2)$ to get

$$\prod_{k=0}^{n-1} \zeta^k = 1 \cdot \prod_{k=1}^{n/2-1} \zeta^k \zeta^{n-k} \cdot \zeta^{n/2} = \prod 1 \cdot \zeta^{n/2} = -1.$$

Pair in the same way assuming that n is odd, except this time there is no leftover factor:

$$\prod_{k=0}^{n-1} \zeta^k = 1 \cdot \prod_{k=1}^{(n-1)/2} \zeta^k \zeta^{n-k} = 1.$$

(Alternatively, write $\zeta = e^{i\cdot 2\pi k/n}$ and compute.)

(c) Use Euler's formula to expand $e^{i(x+y)}$ in two ways:

$$\begin{aligned} \cos(x+y) + i\sin(x+y) &= e^{i(x+y)} = e^{ix}e^{iy} \\ &= (\cos x + i\sin x)(\cos y + i\sin y) \\ &= (\cos x\cos y - \sin x\sin y) + i(\sin x\cos y + \cos x\sin y). \end{aligned}$$

Now identify real parts to get the requested formula for $\cos(x+y)$. We could also identify imaginary parts to get the usual formula for $\sin(x+y)$.

(d) This follows from Euler's formula and basic properties of exponentials:

$$(\cos x + i\sin x)^n = (e^{ix})^n = e^{i(nx)} = \cos(nx) + i\sin(nx).$$

Problem 4. For two sequences $a_\bullet = (a_0, \dots, a_{n-1})$ and $b_\bullet = (b_0, \dots, b_{n-1})$, each of length n, we define their *convolution* to be the sequence whose l^{th} term is

$$(a_\bullet * b_\bullet)_l = \sum_{j+k=l} a_j b_k.$$

(a) Explain the connection with polynomials.
(b) Prove that $\mathrm{DFT}(a_\bullet * b_\bullet)$ (a sequence of length $2n-1$) is the componentwise product of $\mathrm{DFT}(a_\bullet, 0, \dots, 0)$ and $\mathrm{DFT}(b_\bullet, 0, \dots, 0)$.

Solution. If a and b are the coefficient sequences (padded with 0s) of $A(x)$ and $B(x)$, respectively, then $(a * b)$ is the coefficient sequence of $A(x)$ and $B(x)$.

Let $c = \mathrm{DFT}(a)$ and $d = \mathrm{DFT}(b)$. (We should assume that a and b are padded with 0s to have length $2n$.) Let $\zeta = e^{i\cdot\pi/n}$. Use the formula for DFT, multiply, and collect like terms. Writing α for ζ^{-k}, we have:

$$\begin{aligned}
c_k d_k &= \left(\sum_{j=0}^{n-1} a_k \zeta^{-kj}\right)\left(\sum_{l=0}^{n-1} b_k \zeta^{-kl}\right) \\
&= \left(\sum_{j=0}^{n-1} a_j \alpha^j\right)\left(\sum_{l=0}^{n-1} b_l \alpha^l\right) \\
&= \sum_{m=0}^{2n-2}\left(\sum_{j=0}^{m} a_j b_{m-j}\right)\alpha^m \\
&= \sum_{m=0}^{2n-2}\left(\sum_{j=0}^{m} a_j b_{m-j}\right)\zeta^{-km}
\end{aligned}$$

This is exactly the formula for the k^{th} term of $\mathrm{DFT}(a * b)$. $\square$

Problem 5. Recall that we write DFT for the discrete Fourier transform and IFT for the inverse discrete Fourier transform.

(a) Compute $\mathrm{IFT}(1, 1, \ldots, 1)$.
(b) Compute $\mathrm{DFT}(1, \frac{1}{3}, \frac{1}{3^2}, \ldots, \frac{1}{3^{n-1}})$.
(c) Compute $\mathrm{DFT}\left(\binom{n-1}{0}, \binom{n-1}{1}, \ldots, \binom{n-1}{k}, \ldots, \binom{n-1}{n-1}\right)$.

Define two operations on sequences by

$$\begin{aligned} \mathrm{lshift}(z_0, \ldots, z_{n-1}) &:= (z_1, \ldots, z_{n-1}, z_0) \\ \mathrm{rshift}(z_0, \ldots, z_{n-1}) &:= (z_{n-1}, z_0, \ldots, z_{n-2}). \end{aligned}$$

(d) Find formulas for $\mathrm{DFT}(\mathrm{lshift}(z))$ and $\mathrm{DFT}(\mathrm{rshift}(z))$ in terms of $\mathrm{DFT}(z)$. Make a conclusion that would be intelligible to a linear algebra student.
(e) Use the previous part to find a general formula for the DFT of a cyclic permutation of a sequence in terms of its DFT.
(f) Use the previous parts to find a general formula for the IFT of a cyclic permutation of a sequence in terms of its IFT.

Solution.

(a) Let $\zeta = e^{i \cdot 2\pi/n}$ be a primitive n^{th} root of unity and let $c = \mathrm{DFT}(1, 1, \ldots, 1)$. Notice that the sum

$$1 + \zeta^k + \zeta^{2k} + \cdots + \zeta^{(n-1)k} \tag{0.1}$$

is $\frac{1-(\zeta^k)^n}{1-\zeta^k} = 0$ unless $k = 0$. If $k = 0$, then each term in (**??**) is 1, so the sum is n. Therefore $\mathrm{IFT}(1, 1, \ldots, 1) = (1, 0, 0, \ldots, 0)$.

Alternatively, think about polynomials: IFT returns the coefficients of the (unique) polynomial of degree $\leq n-1$ interpolating the points

$$(\zeta^{-1}, 1), (\zeta^{-2}, 1), \ldots, (1, 1).$$

One such polynomial is the constant polynomial $1 + 0z + 0z^2 + \cdots$. By uniqueness this is the answer.

(b) Let $c = (c_0, \ldots, c_{n-1})$ be $\mathrm{DFT}(1, \frac{1}{3}, \frac{1}{3^2}, \ldots, \frac{1}{3^{n-1}})$. Using the geometric sum formula, we have

$$c_k = \sum_{l=0}^{n-1} \frac{1}{3^l} \zeta^{-kl} = \frac{1 - (\frac{1}{3})^n}{1 - \frac{1}{3}\zeta^{-k}} = \frac{1}{3^n} \cdot \frac{3^n - 1}{1 - \frac{1}{3}\zeta^{-k}}.$$

(E.g. for $n = 4$ this gives

$$\begin{aligned} \mathrm{DFT}(1, \tfrac{1}{3}, \tfrac{1}{9}, \tfrac{1}{27}) &= \tfrac{80}{81}(\tfrac{3}{2}, (1 + \tfrac{1}{3}i)^{-1}, \tfrac{3}{4}, (1 - \tfrac{1}{3}i)^{-1}) \\ &= \tfrac{80}{81}(\tfrac{3}{2}, \tfrac{9}{10} - \tfrac{3}{10}i, \tfrac{3}{4}, \tfrac{9}{10} + \tfrac{3}{10}i) \\ &= (\tfrac{40}{27}, \tfrac{8}{9} - \tfrac{8}{27}i, \tfrac{20}{27}, \tfrac{8}{9} + \tfrac{8}{27}i), \end{aligned}$$

in case you are interested.)

(c) Let $c = (c_0, \dots, c_{n-1})$ be DFT$\left(\binom{n-1}{0}, \binom{n-1}{1}, \dots, \binom{n-1}{k}, \dots, \binom{n-1}{n-1}\right)$. Use the Binomial Theorem to see that

$$c_k = \sum_{l=0}^{n-1} z_l \zeta^{-kl} = (\zeta^{-k} + 1)^{n-1}.$$

(d) Let c be DFT(z) and write c^* for DFT(lshift(z)).

$c^*_k = z_1 + z_2\zeta^{-k} + z_3\zeta^{-2k} + \cdots + z_{n-1}\zeta^{-(n-2)k} + z_0\zeta^{-(n-1)k}$

Multiply and divide by ζ^k, and remember that $\zeta^n = 1$:

$= \zeta^k(z_1\zeta^{-k} + z_2\zeta^{-2k} + z_3\zeta^{-3k} + \cdots + z_{n-1}\zeta^{-(n-1)k} + z_0^{-nk})$

$= \zeta^k(z_0 + z_1\zeta^{-k} + z_2\zeta^{-2k} + \cdots + z_{n-1}\zeta^{-(n-1)k})$

$= \zeta^k c_k.$

In summary, we have DFT(lshift(z)) $= (c_0, \zeta c_1, \zeta^2 c_2, \dots, \zeta^{(n-1)} c_{n-1})$. By similar reasoning, DFT(rshift(z)) $= (c_0, \zeta^{-1} c_1, \zeta^{-2} c_2, \dots, \zeta^{-(n-1)} c_{n-1})$.

The discrete Fourier transform diagonalizes the linear transformations lshift and rshift.

(e) A cyclic permutation is just a power of lshift or rshift, so we just iteratively apply the result from the previous part. For example,

$$\mathrm{DFT}(\mathrm{lshift}^m(z)) = (c_0, \zeta^m c_1, \zeta^{2m} c_2, \dots, \zeta^{m(n-1)} c_{n-1}).$$

(Notice that e.g. lshift$^{(n-1)}$ = rshift, and $\zeta^{n-1} = \zeta^{-1}$.)

(f) Writing Δ_ζ for the diagonal transformation

$$(c_0, \dots, c_{n-1}) \mapsto (c_0, \zeta c_1, \dots, \zeta^{n-1} c_{n-1}),$$

we have by the previous part that DFT $\cdot$ lshift$^m = \Delta_{\zeta^m} \cdot$ DFT. Invert and rearrange to get

$$\mathrm{IFT} \cdot \mathrm{rshift}^m = \Delta_{\zeta^{-m}}\, \mathrm{IFT},$$

i.e., IFT(rshift$^m(c)$) $= (z_0, \zeta^{-m} z_1, \dots, \zeta^{-m(n-1)} z_{n-1})$. Every cyclic permutation is a power of rshift, so this is enough. $\square$

Problem 6. There are n kindergarteners seated in a circle who have each brought in rocks for show and tell. Different kids bring in different numbers of rocks, so let $z = (z_0, \ldots, z_{n-1})$ be the sequence of amounts of rocks brought by the n kids. To prevent jealousy, their teacher has them periodically redistribute the rocks as follows: at regular intervals, each kid hands $1/3$ of their rocks to the kid on their left and the remaining $2/3$ of them to the right. The circular symmetry of the problem suggests that the Discrete Fourier Transform might be a useful tool to analyze what happens to the rocks in the long term.

The redistribution procedure starting with the initial sequence z gives a list of new sequences $z, z^{(1)}, z^{(2)}, \ldots$ satisfying

$$z^{(t+1)} = \tfrac{1}{3}\,\mathrm{lshift}(z^{(t)}) + \tfrac{2}{3}\,\mathrm{rshift}(z^{(t)}).$$

Say $\mathrm{DFT}(z) = (c_0, c_1, \ldots, c_{n-1})$.

(a) Prove that $c_k^{(t)} = (\frac{1}{3}\zeta^k + \frac{2}{3}\zeta^{-k})^t c_k$, where $c_k^{(t)}$ is the k^{th} term of $\mathrm{DFT}(z^{(t)})$.
(*Hint:* DFT is linear. (Why?))

(b) What happens to the rocks as $t \to \infty$?
(*Hint:* Consider separately the cases when n is even and n is odd.)

(c) What would change if for some fixed $p \in (0, 1)$ other than $1/3$, the kindergarteners passed their rocks in proportion p to the left and $1 - p$ to the right?

Solution. (a) DFT is given by multiplication by the matrix $M_n(\zeta^{-1})$ (as we called it in class) and is therefore linear. We use our result from Problem 5(d), linearity, and induction on m.

$$\begin{aligned} c^{(t+1)} &= \mathrm{DFT}(z^{(t)}) = \mathrm{DFT}(\tfrac{1}{3}\,\mathrm{lshift}(z^{(t)}) + \tfrac{2}{3}\,\mathrm{rshift}(z^{(t)})) \\ &= \tfrac{1}{3}\,\mathrm{DFT}(\mathrm{lshift}(z^{(t)})) + \tfrac{2}{3}\,\mathrm{DFT}(\mathrm{rshift}(z^{(t)})) \end{aligned}$$

and so, by Problem 5(d),

$$c_k^{(t+1)} = \tfrac{1}{3}\zeta^k c_k^{(t)} + \tfrac{2}{3}\zeta^{-k} c_k^{(t)}.$$

The base case $t = 0$ is trivial, so, assuming inductively that our formula is correct for t, we obtain it for $t + 1$:

$$\begin{aligned} c_k^{(t+1)} &= (\tfrac{1}{3}\zeta^k + \tfrac{2}{3}\zeta^{-k})c_k^{(t)} = (\tfrac{1}{3}\zeta^k + \tfrac{2}{3}\zeta^{-k})(\tfrac{1}{3}\zeta^k + \tfrac{2}{3}\zeta^{-k})^t c_k \\ &= (\tfrac{1}{2}\zeta^k + \tfrac{2}{3}\zeta^{-k})^{t+1} c_k. \end{aligned}$$

(b) For convenience, write α_k for $\frac{1}{3}\zeta^k + \frac{2}{3}\zeta^{-k}$, so that $c_k^{(t)} = \alpha_k^t c_k$. Suppose first that n (the number of kids) is odd. Notice that $\alpha_0 = 1$, and

Claim. if $k \neq 0$ then $|\alpha_k| < 1$.

Proof of Claim. The point is that $\zeta^k = \cos(2k\pi/n) + i\sin(2k\pi/n)$ and $\zeta^{-k} = \cos(2k\pi/n) - i\sin(2k\pi/n)$ are complex conjugates (i.e., reflections of each other across the real axis), and so α_k is just the point 2/3 of the way along the line segment from ζ^k to ζ^{-k}, so it lies in the disk $|z| < 1$. (Draw a picture!)

You can give a careful argument by observing that $\alpha_k = \cos(2k\pi/n) - \frac{1}{3}i\sin(2k\pi/n)$, so that

$$|\alpha_k| = \sqrt{\cos(2k\pi/n)^2 + \tfrac{1}{9}\sin(2k\pi/n)^2} < \sqrt{\cos(-)^2 + \sin(-)^2} = 1.$$

At least this is true *unless* $\zeta^k = \zeta^{-k}$, which happens if and only if $\zeta^k = \pm 1$, i.e., if -1 is an n^{th} root of unity. Since we've assumed that n is odd, -1 is not an n^{th} root of unity, so the only problematic case is when $\zeta^k = 1$, i.e., when $k = 0$. $\square$

We observe as a consequence of the claim that $\lim\limits_{t\to\infty} \alpha_k^t = 0$ if $k \neq 0$, so

$$\lim_{t\to\infty} c^{(t)} = (c_0, 0, 0, \dots, 0).$$

That $c_0 = z_0 + z_1 + \cdots + z_{n-1}$ follows directly from the definition of DFT, so, using the linearity (hence continuity!) of IFT, we have

$$\begin{aligned}\lim_{t\to\infty} z^{(t)} &= \lim_{t\to\infty} \mathrm{IFT}(c^{(t)}) \\ &= \mathrm{IFT}(\lim_{t\to\infty} c^{(t)}) \\ &= \mathrm{IFT}(z_0 + z_1 + \cdots + z_{n-1}, 0, 0, \dots, 0).\end{aligned}$$

For a constant c, the sequence $(c, 0, 0, \dots, 0)$ has IFT $(c/n, c/n, \dots, c/n)]$, so in the long run the rocks tend to be equally distributed.

But there is another case! Suppose that n is even. Now we have to worry about the issue that $\zeta^{n/2} = -1$. So $|\alpha_k| < 1$ unless $k \in \{0, n/2\}$, and $\alpha_0 = 1$ and $\alpha_{n/2} = -1$. Thus $c_0^{(t)} = c_0$ for all t (as in the odd case), and $c_{n/2}^{(t)} = (-1)^t c_{n/2}$ for all t.

As t increases, $c^{(t)}$ alternates between the sequences $(c_0, \dots, c_{n/2}, \dots)$ and $(c_0, \dots, -c_{n/2}, \dots)$ where the $\dots$ conceal entries that are approaching 0 exponentially fast. For the purposes of understanding long-term behavior, we'd might as well treat those entries as 0.

As before we have $c_0 = z_0 + \cdots + z_{n-1}$, and now we compute

$$c_{n/2} = z_0 - z_1 + z_2 - \cdots + z_{n-2} - z_{n-1}.$$

A standard computation in the style of Problem 5(a) shows that

$$\mathrm{IFT}(c_0, 0, \dots, 0, c_{n/2}, 0, \dots, 0)$$

is the sequence whose even terms each equal the average of $z_0, z_2, \dots, z_{n-2}$ and whose odd terms each equal the average of $z_1, z_3, \dots, z_{n-1}$. On the other hand, $\mathrm{IFT}(c_0, 0, \dots, 0, -c_{n/2}, 0, \dots, 0)$ is the same but with *even* and *odd* switched.

So, after many exchanges, the odd kids' rocks will be roughly equally distributed and the even kids' rocks will be roughly equally distributed, but the even kids and the odd kids will swap collections at each stage.

(c) Nothing would change, even if we chose $p = 1 - p = 1/2$. If you go back and look at our arguments for the previous parts, at no point do we use anything special about $\frac{1}{3}$. $\square$