

RATIONAL ENDOMORPHISMS OF FANO HYPERSURFACES

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The goal of this paper is to use specialization to characteristic p techniques to give restrictions on the degrees of rational endomorphisms of complex Fano hypersurfaces.

Kollár [11] showed that Fano varieties that are p -fold cyclic covers in characteristic p can carry holomorphic forms. The positivity of these forms is a powerful tool for studying these varieties, and by specializing to characteristic p one can prove results about Fano varieties in characteristic 0. Let $X \subset \mathbb{P}_{\mathbb{C}}^{n+1}$ be a very general hypersurface of degree d . Kollár used specialization mod p to prove that if $d \geq 2\lceil(n+3)/3\rceil$ then X is not rational (or even ruled). If $d \geq 2\lceil(n+2)/3\rceil$ then Totaro [14] used specialization mod p to show that X is not even stably rational. In [3], the authors used specialization mod p to prove that the degree of irrationality for Fano hypersurfaces can be arbitrarily large.

Let $\text{RatEnd}(X)$ denote the set of rational endomorphisms of X with degree at least 1, and let $\text{Bir}(X) \subset \text{RatEnd}(X)$ denote the subset of birational automorphisms. If X is of general type then it follows from [9] that $\text{Bir}(X) = \text{RatEnd}(X)$ (in other words, there are no rational endomorphisms of degree ≥ 2). Thus, studying degree ≥ 2 endomorphisms on smooth hypersurfaces is only interesting in the Fano or Calabi-Yau range. On the other hand, every rational variety admits rational endomorphisms of every degrees, so one can view our main result as strengthening the nonrationality of the varieties involved.

Theorem A. *Let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface over \mathbb{C} of degree d and dimension $n \geq 3$. Let p be a prime number such that*

$$d \geq p \left\lceil \frac{n+3}{p+1} \right\rceil.$$

If ϕ is any rational endomorphism of X of degree λ then $\lambda \equiv 0$ or $1 \pmod{p}$.

Very little is known about rational endomorphisms of nonrational varieties. Dedieu ([4]) has given constraints on degrees of rational endomorphisms of K3 surfaces. These are the first constraints on rational endomorphisms of nonrational hypersurfaces.

As above, let $X \subset \mathbb{P}_{\mathbb{C}}^{n+1}$ be very general of degree d and dimension n with a rational endomorphism of degree λ . In low dimensions, we have the following constraints on λ :

(n, d)	(3,5)	(4,6)	(5,6)	(5,7)
type	Calabi-Yau	Calabi-Yau	Fano	Calabi-Yau
$\lambda \equiv 0$ or 1	$(\text{mod } 5)$	$(\text{mod } 3)$	$(\text{mod } 3)$	$(\text{mod } 3)$ and $(\text{mod } 7)$

The first author's research is partially supported by the National Science Foundation under the Stony Brook/Simons Center for Geometry and Physics RTG grant DMS-1547145.

Unirational varieties are sources of varieties with many rational endomorphisms. If X is unirational, precomposing the unirational parametrization with any map $X \dashrightarrow \mathbb{P}^n$ gives a rational endomorphism:

$$X \dashrightarrow \mathbb{P}^n \dashrightarrow X.$$

For example, any smooth cubic hypersurface of dimension ≥ 2 is unirational, and therefore admits many rational endomorphisms. Moreover, Beheshti and Riedl [2] proved that if $X \subset \mathbb{P}_{\mathbb{C}}^{n+1}$ is smooth and $n \geq 2^{d!}$ then X is unirational. So for any $d > 0$ there exist degree smooth d hypersurfaces of large dimension with many rational endomorphisms. On the other hand, any rational endomorphism constructed this way will have degree divisible by the degree of the unirational parametrization. Kollár showed (by specializing mod p) that if $X \subset \mathbb{P}_{\mathbb{C}}^{n+1}$ is very general of degree $d \geq p \lceil (n+3)/(p+1) \rceil$ then any unirational parametrization (in fact, any parametrization by a uniruled variety) has degree divisible by p . Indeed, it is natural to conjecture there are Fano hypersurfaces with no rational endomorphisms of degree ≥ 2 . Such a variety would provide an answer to a famous open problem in that it would be an example of a rationally connected variety which is not unirational.

Another source of varieties with rational endomorphisms are varieties that birationally admit *elliptic fibrations*, i.e. a variety X with a rational map:

$$X \dashrightarrow B$$

such that $X/\mathbb{C}(B)$ is birational to a $\mathbb{C}(B)$ -elliptic curve $E_{\mathbb{C}(B)}$ (with a $\mathbb{C}(B)$ -point). For example, any cubic hypersurface is birational to an elliptic fibration. Note that for any integer m , an elliptic curve $E_{\mathbb{C}(B)}$ has a multiplication by n map:

$$m: E_{\mathbb{C}(B)} \rightarrow E_{\mathbb{C}(B)},$$

which has degree n^2 . If p is a prime and $n^2 \equiv 0$ or $1 \pmod{p}$ for all $n \geq 2$, then p is equal to 2 or 3. Thus the degree restrictions in Theorem A are enough to prove the following:

Corollary B. *Let $X \subset \mathbb{P}_{\mathbb{C}}^{n+1}$ be very general of degree d . If $d \geq 5 \lceil (n+3)/6 \rceil$ then X is not birational to an elliptic fibration.*

For example, we see that a very general Calabi-Yau quintic threefold is not birational to an elliptic fibration. For Calabi-Yau hypersurfaces this was already known by work of Grassi and Wen (see [5, Thm. 1.8] and [6]). Similarly, if X is birational to a genus 1 fibration without a section but with a line bundle of relative degree δ then X admits rational endomorphisms of degree $(m\delta + 1)^2$ for all $m \geq 0$. Likewise, if X is birational to a fibration by g -dimensional abelian varieties then X admits rational endomorphisms of degree m^{2g} for all $m \geq 0$. Theorem A can be applied to give results in these settings as well.

The idea behind this paper is to spread out rational endomorphisms in characteristic p . The main technical difficulty that we encounter is proving that certain cyclic covers in mixed characteristic have “mild singularities” after allowing for the possibility of an arbitrary finite base change. The technical result that we prove is:

Theorem C. *Let \mathbb{Z}_p^{sh} be a strict Henselization of the p -adic integers with field of fractions \mathbb{Q}_p^{sh} and residue field $\overline{\mathbb{F}}_p$. Let X be a smooth scheme over \mathbb{Z}_p^{sh} . Let $D \subset X$ be a divisor which is smooth over \mathbb{Z}_p^{sh} such that $\mathcal{O}(D) \cong L^p$ for some line bundle L on X . Let Y be the p -cyclic cover of X branched at D . Suppose the natural section $s \in H^0(X_{\overline{\mathbb{F}}_p}, \mathcal{O}_{X_{\overline{\mathbb{F}}_p}}(D))$ has nondegenerate critical points [10, 17.3]. Then Y has sustained separably uniruled modifications (see Definition 1.1).*

Roughly speaking Y/\mathbb{Z}_p^{sh} has sustained separably uniruled modifications if for every finite base change $\mathbb{Z}_p^{sh} \subset R$, every exceptional divisor over X_R is separably uniruled. The proof of Theorem C involves resolving these mixed characteristic cyclic covers by a sequence of weighted blowups.

Sketch of proof of Theorem A. Consider a scheme Y over \mathbb{Z}_p^{sh} with mild singularities. We may spread out a rational endomorphism on the generic fiber by considering the (normalization of the closure of the) graph Γ in $Y \times_{\mathbb{Z}_p^{sh}} Y$, with projections π_1, π_2 . The central fiber of Γ possibly breaks up into a union of components

$$\Gamma_0 \cup E_1 \cup \cdots \cup E_n,$$

where Γ_0 is the unique component birational to Y_0 under π_1 and the E_i are exceptional divisors over Y . The second projection then gives a rational endomorphism

$$\phi_0: Y_0 \simeq_{\text{bir.}} \Gamma_0 \xrightarrow{-\pi_2} Y_0$$

of the central fiber. The assumption on mild singularities implies that all exceptional divisors E_i are separably uniruled, but Y_0 is not separably uniruled. Therefore, we see that each map $\pi_2|_{E_i}: E_i \rightarrow Y_0$ must have degree divisible by p . Comparing the degree of the original rational endomorphism on the generic fiber to the degree of ϕ_0 we see they are congruent mod p . Finally, we use the positivity properties of forms in characteristic p to show that Y_0 has no separable rational endomorphisms of degree ≥ 2 .

Acknowledgments. We would like to thank Claire Voisin for raising this question. We thank Dan Abramovich for explaining weighted blowups in mixed characteristic. We would also like to thank Kenneth Ascher, Nguyen-Bac Dang, Kristin DeVleming, François Greer, Kiran Kedlaya, János Kollár, Robert Lazarsfeld, John Lesieutre, Daniel Litt, James McKernan, John Ottem, and Burt Totaro for helpful conversations.

1. SPECIALIZING RATIONAL ENDOMORPHISMS

The purpose of this section is to explain the relationship between uniruling degrees and specializations of endomorphisms for schemes with mild singularities. Throughout, let R be a DVR, let $T = \text{Spec}(R)$ with closed point $0 \in T$, let $\eta = \text{Frac}(R)$ be the field of fractions, and let κ be the residue field. Let $\overline{\eta}$ and $\overline{\kappa}$ be their algebraic closures.

Many of the concepts below are discussed in [11, IV.1.1 and VI.1.6].

Definition 1.1. Let X be a normal scheme. We say that X has *ruled modifications* if every exceptional divisor of every normal birational modification $f: Y \rightarrow X$ is ruled. Likewise, we

say that X has *separably uniruled modifications* if every exceptional divisor of every normal birational modification $f: Y \rightarrow X$ is separably uniruled.

Example 1.2. A regular scheme X admits ruled modifications (see [1], or [11, Thm. VI.1.2]).

Remark 1.3. Having ruled modifications can be checked locally. To show a scheme X admits (separably uni-)ruled modifications it suffices to find a proper birational map from a regular scheme:

$$\nu: \bar{X} \rightarrow X$$

such that all exceptional divisors are (separably uni-)ruled [11, Thm. VI.1.7].

We believe the following is well known to experts.

Lemma 1.4. *If X is a complex variety that admits an étale map to a toric variety:*

$$\psi: X \rightarrow \mathbb{T}$$

then X admits ruled modifications.

Proof. Take a toric resolution of \mathbb{T} ; by this, we mean a morphism of toric varieties:

$$\mu: \bar{\mathbb{T}} \rightarrow \mathbb{T}$$

such that $\bar{\mathbb{T}}$ is smooth, μ is proper, and the morphism is defined by a refinement of the fan of \mathbb{T} . Then (1) every exceptional divisor E of μ is a toric variety, and (2) the map

$$\mu|_E: E \rightarrow B := \mu(E)$$

is a toric map. If $\mathbb{C}(B)$ is the fraction field of B then $E_{\mathbb{C}(B)}$ is a rational $\mathbb{C}(B)$ -variety.

Now consider the following fiber square:

$$\begin{array}{ccc} \bar{X} & \longrightarrow & \bar{\mathbb{T}} \\ \downarrow \nu & & \downarrow \mu \\ X & \longrightarrow & \mathbb{T} \end{array}$$

Then \bar{X} is smooth as it is étale over $\bar{\mathbb{T}}$. Let $D \subset \bar{X}$ be an irreducible exceptional divisor of ν with image $E \subset \bar{\mathbb{T}}$ an exceptional divisor of μ and let $B' \subset X$ be the image of D and $B \subset \mathbb{T}$ the image of E . Then we have $D = E \times_B B'$ and as a result $D_{\mathbb{C}(B')} = E_{\mathbb{C}(B')}$ is rational over $\mathbb{C}(B')$. So D is ruled. \square

For applications it will be necessary for our families X_R to have ruled modifications after arbitrary finite base change $R \subset R'$. This will allow us to specialize properties from the geometric generic fiber $X_{\bar{\eta}}$ to the special fiber.

Definition 1.5. Let X_R be a normal R -scheme. We say X_R has *sustained ruled modifications* if for every extension of DVRs $R \subset R'$ such that $\dim_{\eta}(\text{Frac}(R'))$ is finite, there is a further extension $R' \subset R''$ with $\dim_{\eta}(\text{Frac}(R''))$ finite such that $X_{R''}$ is a normal scheme with ruled modifications. We define *sustained separably uniruled modifications* with appropriate substitutions.

Example 1.6. If X_R is a smooth R -scheme then X_R is regular and for every extension $R \subset R'$ of DVRs we have $X_{R'}$ is a smooth R' -scheme, and is therefore regular. Thus by Example 1.2, X_R admits sustained ruled modifications.

Example 1.7. Let $X_C \rightarrow C$ be a family of complex varieties over a smooth complex curve such that the total space X_C is smooth. Let R be the local ring at any point $0 \in C$. Assume the central fiber $X_0 \subset X_R$ is reduced with simple normal crossing. Then X_R has sustained ruled modifications. Indeed let $R \subset R'$ be an extension of DVRs with uniformizers t and t' respectively. We can assume that $t = u(t')^k$ for some k and unit $u \in R'$. After possibly extending $R' \subset R''$ to include the k -th root of u we can assume that $t = (t'')^k$. By the simple normal crossing assumption, every point in $(X_{R''})_0$ has a neighborhood of the form

$$U := \left(t''^k = x_1 \cdots x_m \right) \subset X \times \mathbb{A}^1$$

for some subsequence x_1, \dots, x_m of a regular sequence on X_R . This gives (after possibly shrinking further) an étale map:

$$U \rightarrow \mathbb{T} = \left(x_1 \cdots x_m = (t'')^k \right) \subset \mathbb{A}^{n+1}$$

to a toric variety. By Lemma 1.4, X has sustained ruled modifications.

Definition 1.8. Let X be a variety. A *uniruling degree* of X is the degree of a dominant and generically finite rational map $V \times \mathbb{P}^1 \dashrightarrow X$.

Theorem 1.9. *Let X_R be a normal R -scheme, assume $\kappa = \bar{\kappa}$ is algebraically closed, and assume η is perfect.*

- (1) *Suppose that X_κ is reduced and irreducible. If ℓ divides every uniruling degree of X_κ and X_R has ruled modifications then for any rational endomorphism ϕ_η of the generic fiber X_η , there is an endomorphism ϕ_κ of X_κ such that*

$$\deg(\phi_\eta) \equiv \deg(\phi_\kappa) \pmod{\ell}.$$

- (2) *Let us now assume that*

$$X_\kappa = D_1 \cup \cdots \cup D_m \cup X'_\kappa$$

is a reduced divisor and all the D_i are ruled divisors. If ℓ divides every uniruling degree of X'_κ and X_R admits sustained ruled modifications, then for every rational endomorphism $\phi_{\bar{\eta}}$ of $X_{\bar{\eta}}$ there is a rational endomorphism ϕ_κ of X'_κ such that

$$\deg(\phi_{\bar{\eta}}) \equiv \deg(\phi_\kappa) \pmod{\ell}.$$

- (3) *Suppose that $\text{char } \kappa = p$ and X_κ is reduced, irreducible, and not separably uniruled. If X_R admits sustained separably uniruled modifications, then for every rational endomorphism $\phi_{\bar{\eta}}$ of $X_{\bar{\eta}}$, there is a rational endomorphism ϕ_κ of X_κ such that*

$$\deg(\phi_\kappa) \equiv \deg(\phi_\kappa) \pmod{p}.$$

Proof. For (1), consider the normalization of the closure of the graph of ϕ_η ,

$$\bar{\Gamma}_R \rightarrow X_R \times_R X_R.$$

Then $\bar{\Gamma}_R$ is flat over T and the central fiber Γ_κ has 2 projections π_1 and π_2 . By flatness, the projection map π_1 (resp. π_2) has degree 1 (resp. $\deg(\phi_\eta)$). Thus π_1 maps a unique component of Γ_κ birationally to X_κ . We will show that $\phi_\kappa := \pi_2 \circ (\pi_1|_{X'}^{-1})$ has the required degree.

As X_R has ruled modifications, every other component of Γ_κ is ruled. Thus we may write

$$\Gamma_\kappa = X'_\kappa \cup E_1 \cup \cdots \cup E_r$$

where X'_κ is the component mapping birationally to X_κ and the reduced exceptional divisors E_i^{red} are all ruled. The degrees of the maps

$$E_i \rightarrow X_\kappa$$

under the second projection are all either 0 or uniruling degrees. Therefore,

$$\begin{aligned} \deg(\phi_\eta) &= \deg(\pi_2) \\ &= \deg(\pi_2|_{X'_\kappa}) + \sum_i \deg(\pi_2|_{E_i}) \\ &\equiv \deg(\pi_2|_{X'_\kappa}) \pmod{\ell} \end{aligned}$$

To prove (2), let $\eta \subset \eta'$ be a finite extension such that $\phi_{\bar{\eta}}$ is defined over η' . Moreover, assume there is a DVR $R' \subset \eta'$ such that $X_{R'}$ has ruled modifications and $\text{Frac}(R') = \eta'$. Note that the residue field of B is also κ (as κ is assumed to be algebraically closed). Consider the normalization of the closure of the graph of $\phi_{\eta'}$:

$$\bar{\Gamma} \rightarrow X_{R'} \times_{R'} X_{R'}.$$

The case when $\deg(\phi_{\bar{\eta}}) \equiv 0 \pmod{\ell}$ is uninteresting (by considering any constant map $X'_\kappa \rightarrow X'_\kappa$). So we may assume that $\deg(\phi_{\bar{\eta}}) \not\equiv 0 \pmod{\ell}$. As in the proof of (1), we have by flatness that the map π_1 (resp. π_2) has degree 1 (resp. $\deg(\phi_{\bar{\eta}})$). So we may write

$$\Gamma_\kappa = \bar{X}'_\kappa \cup D'_1 \cup \cdots \cup D'_m \cup E_1 \cup \cdots \cup E_r$$

where \bar{X}'_κ (resp. D'_i) is birational to \bar{X}_κ (resp. D_i) under π_1 , and the E_i are contracted by π_1 . Then all the E_i are ruled. The assumption that $\deg(\phi_{\bar{\eta}}) \not\equiv 0 \pmod{\ell}$ and that ℓ divides any uniruling degree of X_κ implies that \bar{X}'_κ maps dominantly and generically finitely onto \bar{X}_κ . Thus the composition

$$\phi_\kappa : \bar{X}_\kappa \simeq_{\text{bir}} \bar{X}'_\kappa \rightarrow \bar{X}_\kappa$$

is a rational endomorphism of \bar{X}'_κ and a similar computation to part (1) proves

$$\deg(\phi_\kappa) \equiv \deg(\phi_{\bar{\eta}}) \pmod{\ell}.$$

The proof for (3) follows the same strategy as (1). The added hypothesis that X_κ is not separably uniruled implies that every component E_i of Γ_κ other than the unique one birational to X_κ (via π_1) is separably uniruled. Therefore each map $E_i \rightarrow X_R$ has degree divisible by p . The proof then follows part (1). \square

Corollary 1.10. *Working over \mathbb{C} , let $X \rightarrow B$ be a smooth family over a complex variety B . Fix integers $\ell \geq 2$ and $\lambda \geq 1$. Then the locus*

$$B_{(\ell, \lambda)} := \left\{ b \in B \mid \begin{array}{l} \ell \text{ divides every uniruling degree of } X_b \text{ and } \nexists \text{ a rational} \\ \text{endomorphism } \varphi_b : X_b \dashrightarrow X_b \text{ with degree } \equiv \lambda \pmod{\ell} \end{array} \right\},$$

is given by the complement of a countable union of closed subvarieties of B .

Lemma 1.11. (1) *Suppose that $X \rightarrow B$ is a map of complex varieties. Let $\bar{\eta}$ be the geometric generic point of B . If a very general fiber $X_t \in X$ admits a rational endomorphism of degree λ then so does the geometric generic fiber $X_{\bar{\eta}}$.*

(2) *Suppose that $K \subset L$ is an extension of algebraically closed fields. If X_L admits a rational endomorphism of degree λ then so does X_K .*

Proof. For part (1), by [15, Lem. 2.1] there is a field isomorphism $\bar{\eta} \cong \mathbb{C}$ such that the fiber product $X_{\bar{\eta}} \times_{\bar{\eta}} \mathbb{C}$ is isomorphic to $X_{\mathbb{C}}$ as varieties over \mathbb{C} . Consider the rational endomorphism of schemes: $X_{\bar{\eta}} \cong X_t \dashrightarrow X_t \cong X_{\bar{\eta}}$. The first and last isomorphisms are isomorphisms of schemes, and the composition commutes with the map to $\bar{\eta}$. This gives the required endomorphism of $X_{\bar{\eta}}$.

For part (2), there is a finitely generated integral ring A with $K \subset A \subset L$ which contains all the coefficients defining the rational endomorphism. Then the rational map defined over L spreads out to a rational map $X_A \dashrightarrow X_A$. Choosing a general K -point of $\text{Spec}(A)$ gives the rational map $X_K \dashrightarrow X_K$ which has the same degree. \square

2. RESOLVING CYCLIC COVERS AFTER BASE CHANGE

The purpose of this section is to prove that p -cyclic covers admit sustained separably uniruled modifications in mixed characteristic p . To do this we need to resolve the typical singularities of these covers. Our resolutions involve certain weighted blowups, which at every step alternate between two types of isolated double point hypersurface singularities – up to some mild quotient singularities (the equations of the singularities are given by (2.1) in §2.3). We believe that an alternative approach to these results is through resolution techniques coming from log geometry (see e.g. [8]). The arguments in this section are mostly computational in nature.

2.1. Local equations for singularities of p -cyclic covers. To start we recall the definition of a cyclic cover. Let X be a scheme and \mathcal{L} a line bundle on X . Let $s \in H^0(X, \mathcal{L}^{\otimes m})$ be a section. Let

$$\mathbb{L} = \text{Spec}_{\mathcal{O}_X} (\oplus_{i \geq 0} \mathcal{L}^{-i} \cdot y^i)$$

(resp. $\mathbb{L}^{\otimes m}$) be the total space of the line bundle \mathcal{L} (resp. $\mathcal{L}^{\otimes m}$). Then s defines a section:

$$\begin{array}{ccc} \mathbb{L}^{\otimes m} & \longrightarrow & X \\ & \longleftarrow \text{ } \underset{s}{\text{ } \longrightarrow} & \end{array}$$

There is also an m th power map: $\mathbb{L} \xrightarrow{p_m} \mathbb{L}^{\otimes m}$ which is a μ_m -quotient.

Definition 2.1. The m -cyclic cover of s is $Y := \mathfrak{p}_m^{-1}(s(X))$. We say that the cyclic cover Y has *branch divisor* $(s = 0) \subset X$.

It follows that $Y \cong \text{Spec}_{\mathcal{O}_X}(\oplus_{i \geq 0} \mathcal{L}^{-i} \cdot y^i / (y^m - s))$.

Fix an odd prime \mathfrak{p} and let $X_{\mathbb{Z}_\mathfrak{p}^{sh}}$ be an integral scheme that is smooth over $\mathbb{Z}_\mathfrak{p}^{sh}$. Let

$$\sigma: Y_{\mathbb{Z}_\mathfrak{p}^{sh}} \rightarrow X_{\mathbb{Z}_\mathfrak{p}^{sh}}$$

be a \mathfrak{p} -cyclic cover with branch divisor $D = (s = 0)$. Assume that

- the branch divisor D is smooth over $\mathbb{Z}_\mathfrak{p}^{sh}$, and
- the section $s \in H^0(X_{\overline{\mathbb{F}}_\mathfrak{p}}, \mathcal{O}_{X_{\overline{\mathbb{F}}_\mathfrak{p}}}(D))$ has *nondegenerate critical points* [10, §20.3].

Let $q \in Y(\overline{\mathbb{F}}_\mathfrak{p})$ and $\sigma(q) \in X(\overline{\mathbb{F}}_\mathfrak{p})$ be points.

Now assume that

$$\mathfrak{p}, x_1, \dots, x_n$$

form a regular sequence for the regular local ring S of X at $\sigma(q)$. By the cyclic cover construction, Y has local equation:

$$0 = y^\mathfrak{p} + u + f_1 + f_2 + f_3 \subset S[y]$$

where y is the fiber coordinate, $u, f_1, f_2 \in \mathbb{Z}_\mathfrak{p}^{sh}[x_1, \dots, x_n]$ are constant, linear, and quadratic polynomials in the x_i , and

$$f_3 \in (x_1, \dots, x_n)^3 \subset S.$$

Note that $Y_{\overline{\mathbb{F}}_\mathfrak{p}}$ is singular at $q \iff f_1 = 0 \in S/\mathfrak{p}$. In addition, if $q \in Y_{\overline{\mathbb{F}}_\mathfrak{p}}$ is a singular point, then $u \in \mathbb{Z}_\mathfrak{p}^{sh}$ is a unit (as the branch divisor was assumed to be smooth).

If $q \in Y(\overline{\mathbb{F}}_\mathfrak{p})$ is a singular point, then as s has nondegenerate critical points, we have $f_2 \in \overline{\mathbb{F}}_\mathfrak{p}[x_1, \dots, x_n]$ is a non-degenerate quadratic form. Thus, an appropriate ($\text{GL}_n(\mathbb{Z}_\mathfrak{p}^{sh})$ -linear) change of the terms of the x -terms of the regular sequence of S will diagonalize f_2 modulo \mathfrak{p} , so we may assume that Y has an equation of the form

$$y^\mathfrak{p} + u + x_1^2 + \dots + x_n^2 + \mathfrak{p}(f_1 + f_2) + f_3 \in S[y],$$

where $u \in \mathbb{Z}_\mathfrak{p}^{sh}$ is a unit, $x_1, \dots, x_n \in S$ is a regular sequence at $\sigma(q)$, f_1 and $f_2 \in \mathbb{Z}_\mathfrak{p}^{sh}[x_1, \dots, x_n]$ are homogeneous linear and quadratic polynomials in the x_1, \dots, x_n and f_3 is in the ideal $(x_1, \dots, x_n)^3 \subset S$. The following lemma gives a local description of the singularities on Y :

Lemma 2.2. *There are elements $\epsilon_1, \dots, \epsilon_n \in \mathfrak{p}\mathbb{Z}_\mathfrak{p}^{sh}$ such that if we set $\bar{x}_i := x_i - \epsilon_i$, then the regular sequence*

$$\mathfrak{p}, \bar{x}_1, \dots, \bar{x}_n \in S$$

locally gives Y the form

$$0 = y^\mathfrak{p} + u + \bar{x}_1^2 + \dots + \bar{x}_n^2 + \mathfrak{p}f_2 + f_3 \in S[y]$$

where $u \in \mathbb{Z}_\mathfrak{p}^{sh}$ is a unit, $f_2 \in \mathbb{Z}_\mathfrak{p}^{sh}[\bar{x}_1, \dots, \bar{x}_n]$ is quadratic in the \bar{x}_i s, and $f_3 \in (\bar{x}_1, \dots, \bar{x}_n)^3 \subset S$.

Proof. We define the elements ϵ_i by a convergent series:

$$\epsilon_i = \sum_{m=1}^{\infty} \epsilon_{i,m} \in \mathbb{Z}_p^{sh}$$

for $\epsilon_{i,m} \in \mathbb{Z}_p^{sh}$. The idea is to iteratively complete the square for functions of the form:

$$p^m f_1 + x_1^2 + \cdots + x_n^2.$$

A technical problem is that \mathbb{Z}_p^{sh} is not p -adically complete so one should be careful that the series converges inside \mathbb{Z}_p^{sh} . There are two solutions to this problem. The first is that every coefficient is actually defined over some ring that is a finite étale extension of the p -adics which is p -adically complete. The second, is to instead work with a p -adic completion of the strict Henselization of \mathbb{Z}_p^{sh} .

Consider an element of the form:

$$F = u + x_1^2 + \cdots + x_n^2 + p^m f_1 + p f_2 + f_3 \in S[y].$$

Assume that $f_1 = a_1 x_1 + \cdots + a_n x_n$, and set $\epsilon_{i,m} := -p^m a_i / 2$. Then the change of coordinates:

$$\bar{x}_i + \epsilon_{i,m} = x_i$$

gives

$$\begin{aligned} & y^3 + u + (\bar{x}_1 - p^m a_1 / 2)(\bar{x}_1 + p^m a_1 / 2) + \cdots + (\bar{x}_n - p^m a_n / 2)(\bar{x}_n + p^m a_n / 2) + \\ & 3f_2(\bar{x}_1 - p^m a_1 / 2, \dots, \bar{x}_n - p^m a_n / 2) + f_3(\bar{x}_1 - p^m a_1 / 2, \dots, \bar{x}_n - p^m a_n / 2) \in S[y]. \end{aligned}$$

Expanding the terms $p f_2$ and f_3 into constant, linear, and quadratic terms in the \bar{x}_i gives:

$$y^3 + u' + \bar{x}_1^2 + \cdots + \bar{x}_n^2 + p^{m+1} f_1' + p f_2' + f_3' \in S[y].$$

where $u' \in \mathbb{Z}_p^{sh}$ is a unit, f_1' (resp. f_2') $\in \mathbb{Z}_p^{sh}[\bar{x}_1, \dots, \bar{x}_n]$ is linear (resp. quadratic), and f_3' is in the ideal $(\bar{x}_1, \dots, \bar{x}_n)^3 \subset S$. As $p^m | \epsilon_{i,m}$ we see that $\epsilon_i := \sum \epsilon_{i,m} \in \mathbb{Z}_p^{sh}$ converges. Moreover, the p -adic norm of the linear term goes to 0. \square

In order to prove Theorem C, we need to show that after any extension of DVRs $\mathbb{Z}_p^{sh} \subset R'$ (with $\dim_{\mathbb{Q}_p^{sh}}(\text{Frac}(R'))$ finite) there is a further extension $R' \subset R$ (with $\dim_{\mathbb{Q}_p^{sh}}(\text{Frac}(R))$ finite) such that Y_R admits ruled modifications. Let π be a uniformizer of R . We know that there are finitely many singular points $q_i \in Y_{\bar{\mathbb{F}}_p}$, each of which (by Lemma 2.2) has local equation

$$y^p + u_i + x_1^2 + \cdots + x_n^2 + p f_2 + f_3 = 0$$

where $u_i \in \mathbb{Z}_p^{sh}$ are units. We may assume that R contains the p -th roots of all the u_i and also assume that $k = \text{ord}_{\pi}(p)$ is divisible by $2(p-1)$.

The regular local ring

$$S_R := S \otimes_{\mathbb{Z}_p^{sh}} R,$$

comes with a regular sequence π, x_1, \dots, x_n . Given a singularity of Y_R with local equation above and unit term u , we may assume that $-\delta \in R$ is a p -th root of u . After changing the y -coordinate by δ , we see that Y_R is locally given by the equation

$$0 = y^p + py^2(\delta^2 y^{p-3} + \dots + (p-1)\delta^{p-2}/2) + py\delta^{p-1} + x_1^2 + \dots + x_n^2 + pf_2 + f_3 \in \mathcal{S}_R[y].$$

We have now reduced to resolving singularities of this form.

2.2. Charts for two types of weighted blowups. To resolve the singularities of Y_R , we alternate between two types of weighted blowups. Here we will define and give affine charts for these weighted blowups. First we define these blowups for affine space.

The affine space \mathbb{A}_R^{n+1} has a regular sequence x_1, \dots, x_n, y, π at the origin. For \mathbb{A}_R^{n+1} we consider the weighted blowups with weights: $((p-1)/2, \dots, (p-1)/2, 1, 1)$ and $((p+1)/2, \dots, (p+1)/2, 1, 1)$. We refer to these weighted blowups as

$$\nu^- : \text{WB}^- \rightarrow \mathbb{A}_R^{n+1}, \text{ and } \nu^+ : \text{WB}^+ \rightarrow \mathbb{A}_R^{n+1}.$$

These charts will be very similar, so rather than describing them separately, we will just write

$$\nu^\pm : \text{WB}^\pm \rightarrow \mathbb{A}_R^{n+1}.$$

To define WB^\pm , we first consider the affine scheme:

$$Z^\pm := \text{Spec}(R[x'_1, \dots, x'_n, y', \omega, T]/(\pi = \omega T)).$$

There is a map:

$$Z^\pm \rightarrow \mathbb{A}_R^{n+1}$$

induced by $(x_1, \dots, x_n, y) \mapsto (x'_1 T^{(p\pm 1)/2}, \dots, x'_n T^{(p\pm 1)/2}, y' T)$. There is a $(\mathbb{G}_m)_R$ -action on Z^\pm defined by the following weights: x'_i has weight $(p \pm 1)/2$, y' has weight 1, ω has weight 1, and T has weight -1 . The map $Z^\pm \rightarrow \mathbb{A}_R^{n+1}$ is \mathbb{G}_m -equivariant. We define the weighted blowup as

$$\text{WB}^\pm := (Z^\pm \setminus (x'_1 = \dots = x'_n = y' = \omega = 0))/\mathbb{G}_m,$$

which comes with a natural projective morphism:

$$\nu^\pm : \text{WB}^\pm \rightarrow \mathbb{A}_R^{n+1}.$$

For each coordinate $x'_1, \dots, x'_n, y', \omega$, there is an affine chart of WB^\pm defined by the \mathbb{G}_m -quotient of the complement of the associated divisor in Z^\pm .

The $x'_1 \neq 0$ chart, denoted $\text{WB}_{x'_1}^\pm$ (similarly for the x'_i charts) is computed by taking the spectrum of the ring of invariants:

$$\text{WB}_{x'_1}^\pm := (R[x'_1, 1/x'_1, x'_2, \dots, x'_n, y', \omega, T]/(\pi = \omega T))^{\mathbb{G}_m}.$$

As $x'_1 \neq 0$, if we let z be a $(p \pm 1)/2$ -th root of x'_1 this defines an étale cyclic cover:

$$\text{Spec}R[z, 1/z, x'_2, \dots, x'_n, y', \omega, T]/(\pi = \omega T) \rightarrow \text{WB}_{x'_1}^\pm.$$

There is a \mathbb{G}_m -action on the new ring where z has weight 1, and this étale cover descends to a finite (but no longer étale) degree $(p \pm 1)/2$ cyclic cover of the quotients:

$$(R[x'_1, 1/x'_1, x'_2, \dots, x'_n, y', \omega, T]/(\pi = \omega T))^{\mathbb{G}_m} \subset (R[z, 1/z, x'_2, \dots, x'_n, y', \omega, T]/(\pi = \omega T))^{\mathbb{G}_m}.$$

The ring of invariants of the finite cover can be computed to be

$$R[\alpha_2, \dots, \alpha_n, \beta, \gamma, \zeta]/(\pi = \gamma\zeta)$$

where

$$\alpha_i = x'_i/z^{(p\pm 1)/2}, \beta = y'/z, \gamma = \omega/z, \text{ and } \zeta = zT.$$

Define $\widehat{\text{WB}}_{x_1}^\pm$ to be the spectrum of this ring of invariants. The ramification locus of

$$\widehat{\text{WB}}_{x_1}^\pm \rightarrow \text{WB}_{x_1}^\pm$$

is the set of points in $\beta = \gamma = \zeta = 0 \subset \widehat{\text{WB}}_{x_1}^\pm$. The map:

$$\hat{\nu}^\pm: \widehat{\text{WB}}_{x_1}^\pm \rightarrow \mathbb{A}_R^{n+1}$$

is given in coordinates by $(\zeta^{(p\pm 1)/2}, \alpha_2 \zeta^{(p\pm 1)/2}, \dots, \alpha_n \zeta^{(p\pm 1)/2}, \beta \zeta)$. The preimage of the exceptional divisor of ν^\pm in $\widehat{\text{WB}}_{x_1}^\pm$ is given by $\zeta = 0$. The preimage of the strict transform of the central fiber is given by $\gamma = 0$.

The $y' \neq 0$ chart, denoted WB_y^\pm , is given by

$$\text{WB}_y^\pm := \text{Spec}(R[\alpha_1, \dots, \alpha_n, y, \gamma]/(\pi = y\gamma))$$

where $\alpha_i = x'_i/(y'^{(p\pm 1)/2})$, $y = y'T$, $\gamma = \omega/y'$. The map

$$\nu_y^\pm: \text{WB}_y^\pm \rightarrow \mathbb{A}_R^{n+1}$$

is given in coordinates by $(\alpha_1 y^{(p\pm 1)/2}, \dots, \alpha_n y^{(p\pm 1)/2}, y)$. The exceptional divisor is $(y = 0)$. The strict transform of the central fiber is $\gamma = 0$.

The $\omega \neq 0$ chart, denoted WB_ω^\pm , is given by

$$\text{WB}_\omega^\pm := \text{Spec}(R[\alpha_1, \dots, \alpha_n, \beta])$$

The map

$$\nu_\omega^\pm: \text{WB}_\omega^\pm \rightarrow \mathbb{A}_R^{n+1}$$

is given in coordinates by $(\alpha_1 \pi^{(p\pm 1)/2}, \dots, \alpha_n \pi^{(p\pm 1)/2}, \beta \pi)$. The exceptional divisor of ν_ω^\pm is given by $\pi = 0$ and the strict transform of the central fiber does not meet this chart.

Lastly, we need to define the weighted blowup of a smooth affine R -scheme. Assume A is a finitely generated R -algebra and the map

$$\text{Spec}(A) \rightarrow \text{Spec}(R)$$

is smooth. Let $\mathfrak{q} \in \text{Spec}(A)$ be an $\overline{\mathbb{F}}_p$ -point with maximal ideal defined by a regular sequence: $\mathfrak{m} = (\pi, x_1, \dots, x_n, y)$. Then (after possibly restricting to an open set) the map:

$$\text{Spec}(A) \rightarrow \mathbb{A}_R^{n+1}$$

given in coordinates by (x_1, \dots, x_n, y) is étale and there is only one point over 0. The weighted blowup of $\text{Spec}(A)$ with weights $((p \pm 1)/2, \dots, (p \pm 1)/2, 1, 1)$ at the maximal ideal \mathfrak{m} with

respect to the regular sequence $(\pi, x_1, \dots, x_n, y)$ is defined to be the fiber product:

$$\begin{array}{ccc} \mathrm{WB}^\pm(A) & \longrightarrow & \mathrm{WB}^\pm \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathbb{A}_R^{n+1}. \end{array}$$

The map $\mathrm{WB}^\pm(A) \rightarrow \mathrm{WB}^\pm$ is étale and the above charts give rise to charts for the weighted blowup of $\mathrm{Spec}(A)$ by base change.

2.3. Resolving in mixed characteristic p . The goal is to show that a hypersurface singularity of the form

$$0 = y^p + py^2(\delta^2 y^{p-3} + \dots + (p-1)\delta^{p-2}/2) + py\delta^{p-1} + x_1^2 + \dots + x_n^2 + pf_2 + f_3 \in S_R[y]$$

admits separably uniruled modifications. We will alternate between the types of weighted blowups defined in the previous section. Recall that $k = \mathrm{ord}_\pi(p)$, thus $p/\pi^i \in R$ for $i \leq k$. Assume that $(2p-2)s < k$. We first define two series of equations:

$$(2.1) \quad \begin{aligned} F_s &= y^p + y^2 g(y) p / \pi^{(2p-4)s} + \delta^{p-1} y p / \pi^{(2p-2)s} + x_1^2 + \dots + x_n^2 + pf_2 + f_3. \\ G_s &= \pi y^p + y^2 g(y) p / \pi^{(2p-4)s+p-3} + \delta^{p-1} p y / \pi^{(2p-2)s+p-2} + x_1^2 + \dots + x_n^2 + pf_2 + f_3. \end{aligned}$$

We say that a scheme has a singularity with equation F_s (resp. G_s) if (a) locally it can be embedded in a smooth affine R -scheme $\mathrm{Spec}(A)$; (b) there is an $\overline{\mathbb{F}}_p$ point in $\mathrm{Spec}(A)$ and a regular sequence $(\pi, x_1, \dots, x_n, y)$ such that the scheme is defined by $F_s = 0$ (resp. $G_s = 0$) where $\delta \in R$ is a unit, $g(y) \in R[y]$ is a polynomial in y , f_2 is a quadratic homogeneous polynomial in the x_i , and $f_3 \in (x_1, \dots, x_n)^3 \subset A$.

Proposition 2.3. *Assume that $(2p-2)s < k$.*

- (1) *The weighted blowup WB^- of a singularity with equation F_s has 2 types of singularities: (a) a (non-isolated) cyclic quotient singularity with cyclic group of order $(p-1)/2$ and (b) an isolated singularity with equation G_s .*
- (2) *If $(2p-2)(s+1) < k$ then the weighted blowup WB^+ of a singularity with equation G_s has 2 types of singularities: (a) a (non-isolated) cyclic quotient singularity with cyclic group of order $(p+1)/2$ and (b) an isolated singularity with equation F_{s+1} .*
- (3) *If $(2p-2)(s+1) = k$ then the weighted blowup WB^+ of a singularity with equation G_s has only cyclic quotient singularities with cyclic group of order $(p+1)/2$.*
- (4) *All the exceptional divisors are ruled.*

Proof. We check this in our affine charts. The isolated singularity will show up in the affine chart WB_ω^\pm , the cyclic singularities will show up in the $\mathrm{WB}_{x_i}^\pm$ -charts, and ruledness of exceptional divisors will be checked in the charts WB_y^\pm .

We start by looking at the WB_ω^\pm -charts. By definition we are looking at a hypersurface in the affine scheme:

$$\mathrm{WB}_\omega^\pm(A) = \mathrm{Spec}(A \otimes_{R[x_1, \dots, x_n, y]} R[\alpha_1, \dots, \alpha_n, \beta]).$$

This is an étale cover of $\text{Spec}(R[\alpha_1, \dots, \alpha_n, \beta]) \cong \mathbb{A}_R^{n+1}$. We compute the pullbacks of F_s .

$$\begin{aligned} (v^-)^*(F_s) &= (\beta\pi)^p + (\beta\pi)^2 g(\pi\beta) p / \pi^{(2p-4)s} + \beta\pi\delta^{p-1} p / \pi^{(2p-2)s} \\ &\quad + \pi^{p-1}(\alpha_1^2 + \dots + \alpha_n^2) + p\pi^{p-1} f_2 + \pi^{3(p-1)/2} f_3'. \end{aligned}$$

where $f_3' \in (\alpha_1, \dots, \alpha_n)^3$. The strict transform of $F_s = 0$ is given by dividing by $1/\pi^{p-1}$. This gives the equation:

$$\begin{aligned} (v^-)^*(F_s)/\pi^{p-1} &= \pi\beta^p + \beta^2 g(\pi\beta) p / \pi^{(2p-4)s+p-3} + \beta\delta^{p-1} p / \pi^{(2p-2)s+p-2} \\ &\quad + \alpha_1^2 + \dots + \alpha_n^2 + p f_2(\alpha_1, \dots, \alpha_n) + \pi^{(p-1)/2} f_3'. \end{aligned}$$

which is a singularity with equation G_s . Now we check that $(\pi, \alpha_1, \dots, \alpha_n, \beta)$ is the only singular point in this chart. We start by restricting to the exceptional divisor ($\pi = 0$). This gives the equation:

$$\alpha_1^2 + \dots + \alpha_n^2 = 0 \subset \mathbb{A}_{\mathbb{F}_p}^{n+1}.$$

This is smooth away from points of the form $\mathfrak{m} = (\pi, \alpha_1, \dots, \alpha_n, \beta + \tau)$ (for some $\tau \in R$). Now we need to check regularity when τ is a unit in R . As τ is a unit:

$$(v^-)^*(F_s)/\pi^{p-1} = \pi(-\tau)^p \neq 0 \in \mathfrak{m}/\mathfrak{m}^2$$

so $(v^-)^*(F_s)/\pi^{p-1}$ is regular away from the closed point $(\pi, \alpha_1, \dots, \alpha_n, \beta)$.

A similar calculation (changing appropriate $-$ signs to $+$ signs) applies for part (2). In part (3), the strict transform has equation:

$$\begin{aligned} (v^+)^*(G_s)/\pi^{p+1} &= \beta^p + p\beta^2 g(\pi\beta) / \pi^{(2p-4)k} + p\beta\delta^{p-1} / \pi^{(2p-2)(s+1)} \\ &\quad + \alpha_1^2 + \dots + \alpha_n^2 + p f_2(\alpha_1, \dots, \alpha_n) + \pi^{(p+1)/2} f_3'. \end{aligned}$$

but now $\tau = p\delta^{p-1} / \pi^{(2p-2)(s+1)}$ is a unit in R . The exceptional divisor of the strict transform has equation:

$$\beta^p + \tau\beta + \alpha_1^2 + \dots + \alpha_n^2 = 0 \subset \mathbb{A}_{\mathbb{F}_p}^{n+1}.$$

The derivative $\partial/\partial\beta$ is never 0 (as τ is a unit). Thus the exceptional divisor is a smooth Cartier divisor and the scheme is regular.

Now we examine the WB_y^\pm -chart. This is defined as

$$\text{WB}_y^\pm(A) = \text{Spec}(A \otimes_{R[x_1, \dots, x_n, y]} R[\alpha_1, \dots, \alpha_n, y, \gamma] / (\pi = y\gamma)),$$

and there is an étale map $\text{WB}_y^\pm(A) \rightarrow \text{WB}_y^\pm$. The strict transform of F_s is

$$\begin{aligned} (v_y^-)^*(F_s)/y^{p-1} &= y + g(y) p / (y\gamma)^{(2p-4)s} y^{p-3} + p\delta^{p-1} / (y\gamma)^{(2p-2)s} y^{p-2} \\ &\quad + \alpha_1^2 + \dots + \alpha_n^2 + p f_2(\alpha_1, \dots, \alpha_n) + y^{(p-1)/2} f_3'. \end{aligned}$$

Restricting to the exceptional divisor $y = 0$ gives the equation:

$$\alpha_1^2 + \dots + \alpha_n^2 = 0 \subset \mathbb{A}_{\mathbb{F}_p}^{n+1}.$$

So the only place we need to check regularity is at points with maximal ideal

$$\mathfrak{m} = (\pi, \alpha_1, \dots, \alpha_n, y, \gamma + \tau) \subset A \otimes_{R[x_1, \dots, x_n, y]} R[\alpha_1, \dots, \alpha_n, y, \gamma].$$

To check regularity, we need to show that $(\nu_y^-)^*(F_s)/y^{p-1}$ and $\pi - y\gamma$ are $\overline{\mathbb{F}}_p$ -independent in $\mathfrak{m}/\mathfrak{m}^2$. This is straightforward as $(\nu_y^-)^*(F_s)/y^{p-1} \equiv y \pmod{\mathfrak{m}^2}$.

A similar calculation (changing appropriate $-$ signs to $+$ signs) applies for part (2). In part (3), the strict transform has equation:

$$\begin{aligned} (\nu_y^+)^*(F_s)/y^{p+1} &= \gamma + g(y)\mathfrak{p}/(y\gamma)^{(2p-2)s+p-2}y^{\mathfrak{p}} + \mathfrak{p}\delta^{p-1}/(y\gamma)^{(2p-2)s+p-2}y^{\mathfrak{p}} \\ &\quad + \alpha_1^2 + \cdots + \alpha_n^2 + \mathfrak{p}f_2(\alpha_1, \dots, \alpha_n) + y^{(p+1)/2}f_3'. \end{aligned}$$

Observe that

$$\mathfrak{p}\delta^{p-1}/y^{(2p-2)(s+1)}\gamma^{(2p-2)s+p-2} = \tau\gamma^{\mathfrak{p}} \in R[\alpha_1, \dots, \alpha_n, y, \gamma]/(y\gamma = \pi)$$

for some unit $\tau \in R$. Restricting to the exceptional divisor gives:

$$\gamma + \tau\gamma^{\mathfrak{p}} + \alpha_1^2 + \cdots + \alpha_n^2 = 0 \subset \mathbb{A}_{\overline{\mathbb{F}}_p}^{n+1}.$$

This is smooth, thus the total space is regular.

Now we want to check ruledness. In part (1) the exceptional divisor has equation:

$$\alpha_1^2 + \cdots + \alpha_n^2 = 0 \subset \mathbb{A}_{\overline{\mathbb{F}}_p}^{n+1}$$

which is rational (as it is a quadric). In part (2) the exceptional divisor has equation:

$$\gamma + \alpha_1^2 + \cdots + \alpha_n^2 = 0 \subset \mathbb{A}_{\overline{\mathbb{F}}_p}^{n+1}$$

which is also rational. Finally in part (3) the exceptional divisor has equation:

$$\gamma + \tau\gamma^{\mathfrak{p}} + \alpha_1^2 + \cdots + \alpha_n^2 = 0 \subset \mathbb{A}_{\overline{\mathbb{F}}_p}^{n+1}.$$

This is birational to a hypersurface with homogeneous equation:

$$(\gamma\theta^{p-1} + \tau\gamma^{\mathfrak{p}} + \theta^{p-2}(\alpha_1^2 + \cdots + \alpha_n^2) = 0) \subset \mathbb{P}_{\overline{\mathbb{F}}_p}^{n+1}$$

Note that this is a degree \mathfrak{p} hypersurface with points of multiplicity $\mathfrak{p} - 1$ (any point where $\gamma = \theta = \alpha_1^2 + \cdots + \alpha_n^2 = 0$). Projecting from any such point gives a degree 1 map onto $\mathbb{P}_{\overline{\mathbb{F}}_p}^{n+1}$, thus these exceptional divisors are rational as well.

It just remains to study the chart $\text{WB}_{x_1}^{\pm}$. There is a (branched) cyclic cover

$$\widehat{\text{WB}}_{x_1}^{\pm}(A) \rightarrow \text{WB}_{x_1}^{\pm}(A).$$

with degree $(\mathfrak{p} \pm 1)/2$. By definition we have

$$\widehat{\text{WB}}_{x_1}^{\pm}(A) := \text{Spec}(A \otimes_{R[x_1, \dots, x_n, y]} R[\alpha_2, \dots, \alpha_n, \beta, \gamma, \zeta]/(\pi = \gamma\zeta)).$$

The preimage of the strict transform of F_s in $\widehat{\text{WB}}_{x_1}^{\pm}(A)$ is

$$\begin{aligned} (\hat{\nu}_{x_1}^-)^*F_s/\zeta^{p-1} &= \zeta\beta^{p-1} + \beta^2g(\zeta\beta)\mathfrak{p}/(\gamma\zeta)^{(2p-4)s}\zeta^{p-3} + \delta^{p-1}\beta\mathfrak{p}/(\gamma\zeta)^{(2p-2)s}\zeta^{p-2} \\ &\quad 1 + \alpha_2^2 + \cdots + \alpha_n^2 + \mathfrak{p}f(1, \alpha_2, \dots, \alpha_n) + \zeta^{(p-1)/2}f_3'. \end{aligned}$$

We want to show this equation defines a regular scheme in $\widehat{\text{WB}}_{x_1}^-(A)$. It suffices to show the intersection with the preimage of the exceptional divisor ($\zeta = 0$) is smooth. This restriction has equation:

$$1 + \alpha_2^2 + \cdots + \alpha_n^2 = 0 \subset \mathbb{A}_{\overline{\mathbb{F}}_p}^{n+1}$$

which is smooth. It follows that the strict transform of $F_s = 0$ has cyclic quotient singularities in the chart $\text{WB}_{x_1}^-$. Part (2) and (3) follow by a similar calculation. \square

Lemma 2.4. *Suppose that V and W are normal R -schemes of finite type and there is a finite map $V \rightarrow W$ of degree d . Suppose that V has ruled modifications. Then for any birational map $W' \rightarrow W$ from a normal R -scheme W' , every exceptional divisor has a uniruling of degree $\leq d$.*

Proof. Let V' be the normalization of the closure of the graph of $V \dashrightarrow W'$. There is some exceptional divisor of $V' \rightarrow V$ (that is ruled) which dominates E by a degree $\leq d$ map. \square

Proof of Theorem C in char. $p > 2$. We need to show that Y_R admits separably uniruled modifications. By Proposition 2.3, there is a birational model $Y'_R \rightarrow R$ with ruled exceptional divisors such that Y'_R only has cyclic quotient singularities by cyclic groups of order $\leq (p+1)/2$. Then by Lemma 2.4 we are done, as any exceptional divisor over Y_R is either exceptional over Y'_R (which has separably uniruled modifications) or is exceptional for the map $Y'_R \rightarrow Y_R$. \square

Proof of Theorem C in char. 2. In the central fiber at a singular point of the cyclic cover we have the equation:

$$0 = y^2 + u + q + f_3 \in (S/2)[y]$$

where S is a dimension $n+1$ regular local ring which is smooth over \mathbb{Z}_2^{sh} (and $2, x_1, \dots, x_n$ is a regular sequence), $u \in \overline{\mathbb{F}}_2$ is a unit, $q \in (S/2)[x_1, \dots, x_n]$ is a quadratic form with nondegenerate Hessian, and f_3 vanishes to order at least 3. As the characteristic is 2, q having a nondegenerate Hessian implies that n is even. By abuse of notation we lift this to an equation

$$0 = y^2 + u + q + 2f_1 + f_3 \in S[y]$$

where f_1 (resp. q) is a linear (resp. quadratic) form in $\mathbb{Z}_2^{sh}[x_1, \dots, x_n]$, $q/2$ has a nondegenerate Hessian, and $u \in \mathbb{Z}_2^{sh}$ is a unit. Now base change along a finite extension $\mathbb{Z}_2^{sh} \subset R$. After possibly extending so that u has a square root $-\delta \in R$ we can make a change of coordinates to get the equation:

$$0 = y^2 + (\delta y + f_1)2 + q + f_3 \in S_R[y].$$

where $\text{ord}_\pi(2) = k$ for a uniformizer $\pi \in R$. Now consider ordinary blowup at the singular point $(\pi, x_1, \dots, x_n, y)$. The exceptional divisor of this blowup is given by

$$y^2 + q = 0 \subset \mathbb{P}_{\overline{\mathbb{F}}_2}^{n+1}$$

where the projective coordinates are $[\pi : x_1 : \cdots : x_n : y]$. This is only singular at the point $[1 : 0 : \cdots : 0]$, and it follows that the blowup has a unique singularity of the form

$$0 = y^2 + (\delta y + f_1)2/\pi + q + f_3 \in S_R[y]$$

and the exceptional divisor is a quadric and thus ruled. Iterating the ordinary blowup brings us to the case that the exceptional divisor has the form:

$$y^2 + \delta y^2/\pi^{k-1} + f_1^2/\pi^{k-1} + q = 0 \subset \mathbb{P}_{\overline{\mathbb{F}}_2}^{n+1}.$$

As $\delta y^2/\pi^{k-1} = \tau y\pi$ for some unit $\tau \in \overline{\mathbb{F}}_2$ the whole quadric has nondegenerate Hessian and is therefore smooth. So we see that iterating ordinary blowups of points resolves the singularities and the exceptional divisors are all quadrics in projective space and thus are ruled. \square

3. ENDOMORPHISMS IN CHARACTERISTIC p

The purpose of this section is to prove the nonexistence of separable rational endomorphisms of degree at least 2 for p -cyclic covers in characteristic p . We start by defining pull backs for rational endomorphisms.

Definition 3.1. Let $\Phi: Y \dashrightarrow Y$ be a rational endomorphism of a proper, normal, \mathbb{Q} -factorial variety over an algebraically closed field. Let Γ_Φ be the normalization of the closure of Φ in $Y \times Y$ with projections p_1 and p_2 of degree 1 and $\deg(\Phi)$ respectively. We define the pullback along Φ :

$$\Phi^*: \text{Pic}(Y) \otimes \mathbb{Q} \rightarrow \text{Pic}(Y) \otimes \mathbb{Q}$$

to be the composition:

$$\text{Pic}(Y) \otimes \mathbb{Q} \xrightarrow{p_2^*} \text{Pic}(\Gamma_\Phi) \otimes \mathbb{Q} \hookrightarrow \text{Cl}(\Gamma_\Phi) \otimes \mathbb{Q} \xrightarrow{p_1^*} \text{Cl}(Y) \otimes \mathbb{Q} \cong \text{Pic}(Y) \otimes \mathbb{Q}.$$

We would like to thank Nguyen-Bac Dang and John Lesieutre for suggesting the use of the Khovanskii-Teissier inequalities to prove the following proposition.

Proposition 3.2. *Let*

$$\phi: Y \dashrightarrow Y$$

be a rational endomorphism of degree $\lambda \geq 1$ of a normal \mathbb{Q} -factorial projective variety Y of dimension n over an algebraically closed field of arbitrary characteristic. If $\text{Pic}(Y) \otimes \mathbb{Q} \cong \mathbb{Q}$ then

$$\phi^*: \text{Pic}(Y) \otimes \mathbb{Q} \rightarrow \text{Pic}(Y) \otimes \mathbb{Q}$$

is multiplication by a where $a \geq \sqrt[n]{\lambda}$.

Proof. Recall that for any 2 nef divisor classes H_1 and H_2 on a variety Y we have the inequalities

$$(H_1^i \cdot H_2^{n-i})^n \geq (H_1^n)^i \cdot (H_2^n)^{n-i}$$

(see Corollary 1.6.3(ii) and Remark 1.6.5 of [12] – this is closely related to the log concavity of dynamical degrees). Let $\Gamma \rightarrow Y \times Y$ be the graph of ϕ with projections $p_1: \Gamma \rightarrow Y$ and $p_2: \Gamma \rightarrow Y$ of degrees 1 and λ respectively. Let H be an ample generator for $\text{Pic}(Y)$. Set $H_i = p_i^*H$. Applying the above inequality for $i = n - 1$ gives

$$(H_1^{n-1} \cdot H_2)^n \geq (H_1^n)^{n-1} \cdot (H_2^n).$$

Applying the projection formula to the left hand side gives

$$(aH^n)^n \geq (H^n)^{n-1} \cdot \lambda(H^n),$$

which reduces to $a \geq \sqrt[n]{\lambda}$. \square

Now we recall Kollár's result on the existence of holomorphic forms on cyclic covers in characteristic p . Let $X \subset \mathbb{P}_{\mathbb{F}_p}^{n+1}$ be a degree e hypersurface and let $D = (s = 0) \subset X$ be divisor for some $s \in H^0(X, \mathcal{O}_X(pe))$. Assume that s has nondegenerate critical points ([11, 17.3]). Then the cyclic cover of X branched at D admits a resolution

$$(3.1) \quad \begin{array}{ccccc} Z & \xrightarrow{\mu} & Y & \xrightarrow{\sigma} & X, \\ & & \searrow & \nearrow & \\ & & & & v \end{array}$$

obtained by blowing up finitely many isolated singular points. When $n \geq 3$ there is an injection:

$$\mathcal{M} := \nu^*(\mathcal{O}_X(pe + e - n - 2)) \hookrightarrow \Omega_Z^{n-1}.$$

Remark 3.3. If we write $d = pe + f$ where $0 \leq f \leq p - 1$, then $d \geq p[(n + 3)/(p + 1)]$ if and only if $pe + e - n - 2 > 0$. Indeed, we have

$$e + \frac{f}{p} \geq \left\lceil \frac{n+3}{p+1} \right\rceil \iff e \geq \left\lceil \frac{n+3}{p+1} \right\rceil \iff e \geq \frac{n+3}{p+1} \iff pe + e - n - 2 > 0.$$

Lemma 3.4. For $n \geq 3$ we have $\text{Pic}(Y) \otimes \mathbb{Q} \cong \text{Cl}(Y) \otimes \mathbb{Q} \cong \mathbb{Q}$; in particular Y is \mathbb{Q} -factorial.

Proof. By [7, XII Cor 3.7] the hypersurface $X \subset \mathbb{P}_{\mathbb{F}_p}^{n+1}$ satisfies $\text{Pic}(X) \cong \mathbb{Z}$. The cyclic cover construction gives

$$Y \subset \mathbb{O}(e)$$

(the total space of $\mathcal{O}_X(e)$), and the composition

$$\text{Pic}(Y) \xrightarrow{\sigma^*} \text{Pic}(X) \cong \mathbb{Z} \xrightarrow{\sigma^*} \text{Pic}(Y)$$

is multiplication by p . It follows that $\text{Pic}(Y) \otimes \mathbb{Q} \cong \text{Pic}(X) \otimes \mathbb{Q} \cong \mathbb{Q}$. \square

Proposition 3.5. If $pe + e - n - 2 > 0$ (i.e. if \mathcal{M} is big and nef on Z) then any separable rational endomorphism $\phi: Y \dashrightarrow Y$ has degree 0 or 1.

Proof. The idea is that if $\deg(\phi) \geq 2$ then pulling back \mathcal{M} under a large iterate $\phi^{\circ k}$ gives an arbitrarily positive line bundle inside $\wedge^{n-1}\Omega_Z$, which is impossible.

Now let H_Y be an ample line bundle on Y and let $H_Z := \mu^*H_Y$. We consider the following invariant:

$$S := \sup \{ (\mathcal{L} \cdot H_Z^{n-1}) \mid \mathcal{L} \text{ is a line bundle on } Z \text{ and } \mathcal{L} \hookrightarrow \wedge^{n-1}\Omega_Z \} \in \mathbb{Z}.$$

As $\mathcal{M} \hookrightarrow \wedge^{n-1}\Omega_Z$ we have $S > 0$. We also claim that S is bounded. Indeed, for any line bundle $\mathcal{L} \hookrightarrow \wedge^{n-1}\Omega_Z$ we have

$$(\mu_*\mathcal{L})^{\vee\vee} \hookrightarrow (\mu_*\wedge^{n-1}\Omega_Z)^{\vee\vee}.$$

The sheaf $(\mu_*\mathcal{L})^{\vee\vee}$ is a rank one reflexive sheaf on Y . By ampleness there exists some $N_1 > 0$, $N_2 > 0$ such that

$$(\mu_*\wedge^{n-1}\Omega_Z)^{\vee\vee} \hookrightarrow ((H_Y)^{\otimes N_1})^{\oplus N_2}.$$

It follows that $(\mu_*\mathcal{L})^{\vee\vee}$ injects into one of the factors $H_Y^{\otimes N_1}$ which bounds the degree $(\mu_*\mathcal{L})^{\vee\vee} \cdot H_Y^{n-1}$. Lastly, applying projection formula gives:

$$(\mu_*\mathcal{L})^{\vee\vee} \cdot H_Y^{n-1} = \mathcal{L} \cdot H_Z^{n-1}.$$

Therefore, S is bounded.

Now assume for contradiction that there is a separable rational endomorphism ϕ of Y with $\deg(\phi) = \lambda \geq 2$. Let Γ_Y (resp. Γ_Z) be the normalization of the graph of $\phi^{o k} : Y \dashrightarrow Y$. Then we have a diagram:

$$\begin{array}{ccccc} & & \Gamma_Z & & \\ & \swarrow \psi_1 & \downarrow \gamma & \searrow \psi_2 & \\ Z & & \Gamma_Y & & Z \\ \downarrow \mu & & \downarrow \mu & & \downarrow \mu \\ Y & \xleftarrow{\pi_1} & & \xrightarrow{\pi_2} & Y \\ & \xleftarrow{\phi^{o k}} & & \xrightarrow{\phi^{o k}} & \end{array}$$

Let $\phi_Z^{o k}$ denote the rational endomorphism of Z . This gives rise to a diagram of the associated Picard groups (and divisor class groups).

$$(3.2) \quad \begin{array}{ccccccc} \mathrm{Pic}(Y) \otimes \mathbb{Q} & \xrightarrow{\pi_2^*} & \mathrm{Pic}(\Gamma_Y) \otimes \mathbb{Q} & \hookrightarrow & \mathrm{Cl}(\Gamma_Y) \otimes \mathbb{Q} & \xrightarrow{\pi_{1*}} & \mathrm{Pic}(Y) \otimes \mathbb{Q} \\ \downarrow \mu^* & & \downarrow \gamma^* & & \uparrow \gamma^* & & \uparrow \mu^* \\ \mathrm{Pic}(Z) \otimes \mathbb{Q} & \xrightarrow{\psi_2^*} & \mathrm{Pic}(\Gamma_Z) \otimes \mathbb{Q} & \hookrightarrow & \mathrm{Cl}(\Gamma_Z) \otimes \mathbb{Q} & \xrightarrow{\psi_{1*}} & \mathrm{Pic}(Z) \otimes \mathbb{Q} \end{array}$$

This diagram is commutative (the left and right squares commute by functoriality and the central square commutes as γ is birational). The composition of the top row is $(\phi^{o k})^*$ and the composition of the bottom row is $(\phi_Z^{o k})^*$.

Let Φ be any positive degree separable rational endomorphism $\Phi : Z \dashrightarrow Z$ of Z and let Γ_Φ be the normalization of the graph of Φ in $Z \times Z$. For any line bundle $\mathcal{L} \subset \wedge^{n-1}\Omega_Z$ there are injections:

$$p_2^*\mathcal{L} \hookrightarrow p_2^*\wedge^{n-1}\Omega_Z \hookrightarrow \wedge^{n-1}\Omega_{\Gamma_\Phi}.$$

and the composition is injective. Pushing forward this uniquely defines an injection:

$$(p_{1*}(p_2^*\mathcal{L}))^{\vee\vee} \hookrightarrow \left(p_{2*} \left(\wedge^{n-1}\Omega_{\Gamma_\Phi} \right) \right)^{\vee\vee} \cong \wedge^{n-1}\Omega_Z.$$

(As both sheaves are \mathcal{S}_2 the last isomorphism can be checked away from the codimension ≥ 2 locus where p_1^{-1} is undefined.) We also have that $(p_{1*}(p_2^*\mathcal{L}))^{\vee\vee} = \Phi^*(\mathcal{L}) \in \mathrm{Pic}(W) \otimes \mathbb{Q}$.

Thus for each $k > 0$ there is an injection

$$(\phi_Z^{o k})^* \mathcal{M} \hookrightarrow \wedge^{n-1}\Omega_Z,$$

and we know $\mathcal{M} = \mu^* \mathcal{N}$ for some ample line bundle \mathcal{N} on Y . By commutativity of the outer arrows of diagram 3.2,

$$\begin{aligned} S &\geq (\phi_Z^{\circ k})^* \mathcal{M} \cdot H_Z^{n-1} = \mu_* \left((\phi_Z^{\circ k})^* \mu^* \mathcal{N} \right) \cdot H_Y^{n-1} \\ &= (\phi_Y^{\circ k})^* \mathcal{N} \cdot H_Y^{n-1} \\ &= a \mathcal{N} \cdot H_Y^{n-1} \end{aligned}$$

for some constant $a \geq \sqrt[n]{\lambda^k}$ (by Proposition 3.2 and Lemma 3.4). For $k \gg 0$, this is arbitrarily large, which is a contradiction. \square

4. DEGREES OF ENDOMORPHISMS OF COMPLEX HYPERSURFACES

Let $X_{\mathbb{Z}_p^{sh}} \subset \mathbb{P}_{\mathbb{Z}_p^{sh}}^{n+1}$ be a degree e hypersurface that is smooth over \mathbb{Z}_p^{sh} with $n \geq 3$. Let

$$s \in H^0(X_{\mathbb{Z}_p^{sh}}, \mathcal{O}_{X_{\mathbb{Z}_p^{sh}}}(pe))$$

be a section such that $D_{\mathbb{Z}_p^{sh}} = (s = 0) \subset X_{\mathbb{Z}_p^{sh}}$ is smooth over \mathbb{Z}_p^{sh} and such that

$$s|_{X_{\mathbb{F}_p}} \in H^0(X_{\mathbb{F}_p}, \mathcal{O}_{X_{\mathbb{F}_p}}(pe))$$

has nondegenerate critical points.

Proposition 4.1. *Let $Y_{\mathbb{C}}$ be the p -cyclic cover of $X_{\mathbb{C}}$ branched at $D_{\mathbb{C}}$. If*

$$pe \geq p \lceil (n+3)/(p+1) \rceil$$

and $\phi: Y_{\mathbb{C}} \dashrightarrow Y_{\mathbb{C}}$ is any rational endomorphism of $Y_{\mathbb{C}}$ of degree λ then

$$\lambda \equiv 0 \text{ or } 1 \pmod{p},$$

and every uniruling degree of $Y_{\mathbb{C}}$ is divisible by p .

Proof. By Lemma 1.11, as $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ are defined over \mathbb{Q}_p^{sh} , there is an endomorphism ϕ of $Y_{\mathbb{C}}$ of degree λ which is defined over a finite extension of \mathbb{Q}_p^{sh} . Theorem C implies that $Y_{\mathbb{Z}_p^{sh}}$ has sustained separably uniruled modifications. By Proposition 3.5 any rational endomorphism of $Y_{\mathbb{F}_p}$ of degree greater than 1 is inseparable. Thus the degree of any rational endomorphism of $Y_{\mathbb{F}_p}$ is 0 or 1 (mod p). By [10, Lem. 7], $Y_{\mathbb{F}_p}$ is not separably uniruled. Thus by Theorem 1.9 we have $\lambda \equiv 0$ or 1 (mod p). \square

Proof of Theorem A. To start we prove the theorem when $d = pe$. By Mori's construction [13] (or see [11, §5]) there is a smooth family of complex varieties which contains all smooth hypersurfaces of degree pe in $\mathbb{P}_{\mathbb{C}}^{n+1}$ and all smooth degree p cyclic covers of smooth degree e hypersurfaces in $\mathbb{P}_{\mathbb{C}}^{n+1}$. If $d \geq p \lceil (n+3)/(p+1) \rceil$ then by Proposition 4.1 there are p -cyclic covers of degree e hypersurfaces such that every rational endomorphism has degree $\equiv 0$ or 1 (mod p) and every uniruling degree is divisible by p . Thus, if X is a very general hypersurface of degree pe , then every rational endomorphism has degree $\equiv 0$ or 1 (mod p).

Now we consider the general case with $d = pe + f$. Consider a pencil of hypersurfaces of degree d such that the general member is smooth and the central fiber is a union of hyperplanes and a very general hypersurface of degree pe meeting with simple normal crossings. By the previous paragraph and Remark 3.3 every rational endomorphism of the degree pe hypersurface has degree $\equiv 0$ or $1 \pmod{p}$. By [11, Lem. V.5.14.5] every uniruling degree of such a degree pe hypersurface is divisible by p . Therefore, by Theorem 1.9(2) and Lemma 1.11 every rational endomorphism of a very general hypersurface of degree d has degree $\equiv 0$ or $1 \pmod{p}$. \square

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