HOMEWORK 3 – MATH 632.

1. (Counterexample to Bertini over finite fields) Show that
$$X = (xw^q - x^q w + yz^q - y^q z = 0) \subset \mathbb{P}^3_{\overline{\mathbb{F}}_q}$$
is a smooth surface. Further show that for any plane
$$H = (ax + by + cz + dw = 0) \subset \mathbb{P}^3_{\overline{\mathbb{F}}_q}$$
with coefficients $a, b, c, d \in \mathbb{F}_q$, the intersection $H \cap X$ is singular.

2. (The projective dual curve) Let $k = \overline{k}$ and let $C \subset \mathbb{P}^2_k$ be a smooth curve. Let $p \in C$ and let $\mathbb{T}_p(C) \in (\mathbb{P}^2)^\vee_k$ be the projective tangent space to C at p (thought of as a point in the dual projective space). Show that the map:
$$\mathbb{T}\colon C \to (\mathbb{P}^2)^\vee_k \text{ given by } p \mapsto \mathbb{T}_p C$$
is an algebraic map. As a challenge, compute the degree of the image curve.

3. (Adjunction for divisors) Let X be a smooth variety over an algebraically closed field.

(A) If $D \subset X$ is a smooth variety, prove that $\omega_{D/k} \cong (\omega_{X/k} \otimes \mathcal{O}_X(D))|_D$.
(B) Show that if $X = \mathbb{P}^1 \times \mathbb{P}^1$ then $\mathcal{O}_X(a, b)$ is very ample if and only if $a, b > 0$.
(C) If $k = \mathbb{C}$, use adjunction to compute the genus of a smooth curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$.

4. (The Atiyah flop) Consider the Segre embedding:
$$\mathbb{P}^1 \times \mathbb{P}^1 \to Q = (xw - yz = 0) \subset \mathbb{CP}^3$$
given by $[a : b] \times [c : d] \mapsto [ac : ad : bc : bd]$. Q has two families of lines (the image of $[a : b] \times \mathbb{P}^1$ or $\mathbb{P}^1 \times [c : d]$; e.g. $L_1 = (z = w = 0)$ is the image of $[1 : 0] \times \mathbb{P}^1$ and $L_2 = (x = z = 0)$ is the image of $\mathbb{P}^1 \times [0 : 1]$). Let $X \subset \mathbb{CP}^4$ be the cone over Q. This is a singular threefold. Consider:
$$\pi_i\colon X_i \to X$$
the blow-up of X at the the planes P_i that are the cones over L_i. Prove:

(A) X_1 and X_2 are smooth.
(B) The maps π_i are isomorphisms away from the cone point $P = [0 : 0 : 0 : 0 : 1] \in X$.
(C) $\pi_i^{-1}(P) \cong \mathbb{P}^1$. (Conclude the planes P_i are not Cartier divisors in X.)
(D) Show that the varieties X_i are not isomorphic over X.

5. Give an example (with proof!) of a smooth projective complex curve that is not isomorphic to a plane curve.

6. Let C be a smooth projective curve over $k = \overline{k}$. Show that a line bundle $\mathcal{L}$ on C is very ample $\iff$ for every pair of (possibly nondistinct) point $P, Q \in C$:
$$\dim_k \Gamma(C, \mathcal{L}(-P - Q)) = \dim_k \Gamma(C, \mathcal{L}) - 2.$$

7. A curve C is called *hyperelliptic* if there is a degree two map $\pi\colon C \to \mathbb{P}^1_k$.

(A) Show that there are hyperelliptic curves of any genus $g \geq 0$.
(B) Use the previous exercise to prove that if C is hyperelliptic then Ω_C is not very ample.