HOMEWORK 1 – MATH 632.

1. Let $k = \overline{k}$ be an algebraically closed field and let

$$f \in k[x_0, \cdots, x_n]$$

be a degree d homogeneous polynomial. Show that the complement $\mathbb{P}^n_k \setminus (f = 0)$ is affine.

2. Let X be a projective scheme over a field $k = \overline{k}$. Show that if X is affine then X is 0-dimensional.

3. Read Proposition I.7.1 and Theorem I.7.2 in Hartshorne (along with the proofs). No need to write anything.

4. Hartshorne Exercise II.7.3.

5. Let $Q = (xw - yz = 0) \subset \mathbb{P}^3_k$ be a quadric surface. Find a 1-dimensional, base-point free, linear system on Q; i.e. a map:

$$\phi \colon Q \to \mathbb{P}^1_k.$$

Conclude $Q \not\cong \mathbb{P}^2_k$.

6. Hartshorne Exercise II.7.1.

7. Hartshorne Exercise II.7.8.

8. Hartshorne Exercise II.7.9.

9. The blow-up:

$$\mu \colon X \to = \mathbb{A}^n_k$$

of $\mathbb{A}^n_k$ at the ideal $(x_1, \cdots, x_n)$ lives inside $\mathbb{A}^n_k \times \mathbb{P}^{n-1}_k$ and has *n standard affine charts* each isomorphic to $\mathbb{A}^n_k$ given by considering the standard affine cover of $\mathbb{P}^{n-1}_k$.

(a) Write down these *standard charts*, i.e. write down the n maps

$$\mu_i \colon \mathbb{A}^n_k \to \mathbb{A}^n.$$

(b) What is the polynomial g_i that defines the exceptional divisor

$$E = \mu^{-1}((0, \cdots, 0))$$

in each of these charts?

(c) Let $f \in k[x_1, \cdots, x_n]$ be a polynomial. Show that

$$g_i^m | \mu_i^* f \iff f \in (x_1, \cdots, x_n)^m.$$

(the maximum such m is the *multiplicity* of m at the point $(0, \cdots, 0) \in \mathbb{A}^n_k$).

10. Now working over $\mathbb{Z}$.

(a) Write down affine charts for the blow-up of the ideal $(2, x)$ in $\mathbb{A}^1_{\mathbb{Z}}$ (or (p, x) for a prime p).

(b) Write down the equation for the exceptional divisor in these charts.