Given X a scheme, $\mathcal{F}$ a coherent sheaf.
WANT to define $H^i(X,\mathcal{F})$ w/ $H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$.

MOTIVATIONS.

1. $X \overset{i}{\subseteq} \mathbb{P}^n_k$

$$P_X(m) = \text{Hilb. poly}$$
$$= \chi(X, \mathcal{O}_X(m))$$
$$= \sum (-1)^i \dim_k H^i(\mathcal{O}(m)).$$

2. Riemann-Roch: C a curve

$$\chi(C,L) = \deg C + \chi(C, \mathcal{O}_C).$$

...

MODEL #1. Čech cohomology

X a top. space
$\mathcal{U} = \{U_\alpha\}_{\alpha \in S}$ an open covering of X
$\mathcal{F}$ a presheaf of abelian groups.

Chains

$$C^i(\mathcal{U},\mathcal{F}) = \text{"group of } i\text{-chains w/ values in } \mathcal{F}\text{"}$$

$$= \prod_{\alpha_0,\dots,\alpha_i \in S} \mathcal{F}(\underbrace{\mathcal{U}_{\alpha_0}\cap\dots\cap\mathcal{U}_{\alpha_i}}_{\mathcal{U}_{\alpha_0\dots\alpha_i}})$$

$$\mathcal{U}_{\alpha_0}\cap\dots\cap\mathcal{U}_{\alpha_i} = \mathcal{U}_{\alpha_0\dots\alpha_i}$$

I write elements

$$C^i(\mathcal{U},\mathcal{F}) \ni s = \{s(\alpha_0,\dots,\alpha_i)\}$$

Differentials

$$C^i(\mathcal{U},\mathcal{F}) \xrightarrow{\delta} C^{i+1}(\mathcal{U},\mathcal{F})$$

$$(\delta s)(\alpha_0,\dots,\alpha_{i+1}) = \sum_{j=0}^{i+1}(-1)^j \operatorname{res}\left[s(\alpha_0,\dots,\hat{\alpha}_j,\dots,\alpha_{i+1})\right].$$

$$\cap$$

$$\mathcal{F}(\mathcal{U}_{\alpha_0,\dots,\alpha_{i+1}})$$

Exs. $0 \to 1$: $(\delta s)(\alpha_0,\alpha_1) = \operatorname{res} s(\alpha_1) - \operatorname{res} s(\alpha_0)$
$\in \mathcal{F}(\mathcal{U}_{\alpha_0,\alpha_1})$

$1 \to 2$: $(\delta s)(\alpha_0,\alpha_1,\alpha_2)$

$$= \operatorname{res}(s(\alpha_1,\alpha_2)) - \operatorname{res}(s(\alpha_0,\alpha_2)) + \operatorname{res}(s(\alpha_0,\alpha_1)).$$

Chain Complex.

Need to show: $\partial \circ \partial = 0$.

Let $t = \partial(s) \in C^{i+1}(X, F)$

$$(\partial t)(\alpha_0, \dots, \alpha_{i+2}) = \sum (-1)^j \operatorname{res} t(\alpha_0, \dots, \hat{\alpha}_j, \dots, \alpha_{i+2})$$

$$= \sum (-1)^j \operatorname{res} \left[\sum_{k<j} (-1)^k \operatorname{res} t(\alpha_0, \dots, \hat{\alpha}_k, \dots, \hat{\alpha}_j, \dots, \alpha_{i+2}) + \sum_{j<k}^{i+2} (-1)^{k-1} \operatorname{res} t(\alpha_0, \dots, \hat{\alpha}_j, \dots, \hat{\alpha}_k, \dots, \alpha_{i+2}) \right]$$

$$\begin{tikzcd}
t(\alpha_0, \dots, \alpha_j, \dots, \alpha_k, \dots, \alpha_{i+2}) \arrow[r, "k \text{ first}"] \arrow[d, "j \text{ first}"] & (-1)^k \operatorname{res} t(\alpha_0, \dots, \alpha_j, \dots, \alpha_k, \dots, \alpha_{i+2}) \arrow[d] \\
(-1)^j t(\alpha_0, \dots, \alpha_j, \dots, \alpha_k, \dots, \alpha_{i+2}) \arrow[r] & \begin{array}{c} (-1)^k \cdot (-1)^j \operatorname{res} t(\alpha_0, \dots, \alpha_j, \dots, \alpha_k, \dots, \alpha_{i+2}) \\ + \\ (-1)^j (-1)^{k-1} \operatorname{res} t(\alpha_0, \dots, \alpha_j, \dots, \alpha_k, \dots, \alpha_{i+2}) \\ \shortparallel \\ 0 \end{array}
\end{tikzcd}$$

Defn: $Z^i(\mathcal{U}_\bullet, F) = \mathrm{Ker}\left[\delta : C^i \longrightarrow C^{i+1}\right]$ = grp of i-cocycles

$B^i(\mathcal{U}_\bullet, F) = \mathrm{Im}\left[\partial : C^{i-1} \longrightarrow C^i\right]$ = grp of i-coboundaries

$$\frac{Z^i(\mathcal{U}_\bullet, F)}{B^i} = H^i(\mathcal{U}_\bullet, F)$$

EXAMPLES

A. $H^0(\mathcal{U}_\bullet, F) = Z^0(\mathcal{U}_\bullet, F)$

$= \{s(\alpha_0)\} \in C^0(\mathcal{U}_\bullet, F)$ such that $s(\alpha_1) - s(\alpha_0) = 0$ (i.e. $s(\alpha_1) = s(\alpha_0)$)

If F a sheaf then $H^0(\mathcal{U}_\bullet, F) = \Gamma(X, F)$.

B. $\mathbb{P}^1 : \mathcal{U}_\bullet = \{\mathbb{A}^1_z, \mathbb{A}^1_{z^{-1}}\}$ $F = \Omega_{\mathbb{P}^1/k}$.

$$k[z]dz \times k[\tfrac{1}{z}]d(\tfrac{1}{z}) \longrightarrow k[z^{\pm 1}]dz$$

$$(f(z)dz, 0) \longmapsto f(z)dz$$

$$(0, g(\tfrac{1}{z})d(1/z)) \longmapsto g(\tfrac{1}{z}) \cdot \tfrac{-1}{z^2}dz.$$

$$B^1(\mathcal{U}_\bullet, F) = k[\tfrac{1}{z}] \cdot \tfrac{1}{z^2}dz \oplus 0 \oplus k[z]dz$$

$$\boxed{H^1(\mathcal{U}_\bullet, F) = k \cdot \frac{dz}{z}.}$$

Issues: Want a theory that doesn't depend on the covering.

Refinement

Suppose $\mathcal{V}_\bullet = \{V_\beta\}_{\beta \in J}$ is a refinement of $\mathcal{U}_\bullet$

Means:

$\forall \beta \in J \ \exists \ \alpha \in I$ such that $V_\beta \subset \mathcal{U}_\alpha$.

Fix: $\sigma : J \to I$ s.t. $V_\beta \subset \mathcal{U}_{\sigma(\beta)}$.

$$\underset{\mathcal{U}_\bullet \to \mathcal{V}_\bullet}{\mathrm{ref}} : C^i(\mathcal{U}_\bullet, F) \longrightarrow C^i(\mathcal{V}_\bullet, F).$$

$$\underset{\mathcal{U}_\bullet \to \mathcal{V}_\bullet}{\mathrm{ref}} (s)(\beta_0, \dots, \beta_i) = \mathrm{res}(s(\sigma(\beta_0), \dots, \sigma(\beta_i))).$$

Claim: (1) This gives a map on chain complexes

AND (2) The induced map on cohomology is independent of σ.

Proof of (2) Given 2 maps $\sigma, \tau : J \to I$

$$\operatorname{ref}_\sigma(s) - \operatorname{ref}_\tau(s) = \delta t$$

for $t(\beta_0, \dots, \beta_{i-1}) = \sum_{j=0}^{i-1} (-1)^j s(\sigma(\beta_0), \dots, \sigma(\beta_j), \tau(\beta_j), \dots, \tau(\beta_{i-1}))$

Defn: For X, F the Čech cohomology of F is defined to be:

$$\check{H}^i(X, F) := \operatorname{colim}_{\mathcal{U}_\bullet} \check{H}^i(\mathcal{U}_\bullet, F).$$

If $0 \to F_1 \to F_2 \to F_3 \to 0$

is a s.e.s. of presheaves:

(means $0 \to F_1(U) \to F_2(U) \to F_3(U) \to 0$ exact $\forall U$)

then get:

$$\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C^{i-1}(\mathcal{U}_\bullet, F) \arrow[r] \arrow[d] & C^{i-1}(\mathcal{U}_\bullet, F) \arrow[r] \arrow[d] & C^{i-1}(\mathcal{U}_\bullet, F) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^{i}(\mathcal{U}_\bullet, F) \arrow[r] \arrow[d] & C^{i}(\mathcal{U}_\bullet, F) \arrow[r] \arrow[d] & C^{i}(\mathcal{U}_\bullet, F) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^{i+1}(\mathcal{U}_\bullet, F) \arrow[r] \arrow[d] & C^{i+1}(\mathcal{U}_\bullet, F) \arrow[r] \arrow[d] & C^{i+1}(\mathcal{U}_\bullet, F) \arrow[r] \arrow[d] & 0 \\
& \vdots & \vdots & \vdots &
\end{tikzcd}$$

Given a long exact sequence of the $H^i(\mathcal{U}_\bullet, -)$.

Taking colimits gives l.e.s.

$$0 \to H^0(X, F_1) \to H^0(X, F_2) \to H^0(X, F_3)$$
$$\to H^1(X, F_1) \to \cdots$$

Issue: s.e.s. of sheaves are not s.e.s. of presheaves. (usually) $\ddot{\frown}$

SOLUTION Assume X is separated.

Then if

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

is exact seq. of q.coh. sheaves, for any affine open:

$$0 \to F_1(U) \to F_2(U) \to F_3(U) \to 0$$

is exact.

Note: * X separated $\Rightarrow$ $\cap$ of affine is affine.

* Affine covers are cofinal. ($\Rightarrow$ can take colimits over them).

So: s.e.s. of q.coherent sheaves on X (separated)

$\rightsquigarrow$

l.e.s. of cohomology.

Acyclic Resolutions

Suppose $\exists$ a long exact sequence:

$$0 \to F \to G_0 \to G_1 \to G_2 \to \cdots$$

such that:

① $H^i(X, G_k) = 0 \quad \forall\ i>0,\ k \geq 0.$

② If $K_k = \ker(G_k \to G_{k+1})$

$C_k = \operatorname{cok}(G_{k-1} \to G_k)$

presheaf.

Assume $H^i(X, C_k) \cong H^i(X, K_k)$.

Then $H^i(X,F) \cong$ ith cohomology of the complex.

Ex. $H^1(X,F) \cong \operatorname{coker}(H^0(G_0) \to H^0(C_0))$

$\vdots$

SERRE'S THEOREM

Let $\mathcal{U}_\bullet, \mathcal{V}_\bullet$ be two affine open coverings of a separated scheme X such that $\mathcal{V}_\bullet$ refines $\mathcal{U}_\bullet$. Then:

$$\mathrm{res}: H^i(\mathcal{U}_\bullet, F) \to H^i(\mathcal{V}_\bullet, F)$$

is an isomorphism.

Prop. Let $\operatorname{Spec} R = X$ be an affine scheme.

- $\mathcal{U}_\bullet = \{\operatorname{Spec}(R_{f_i})\}_{i \in I}$ a finite dist. open cover.
- $\tilde{M}$ a q.coherent module on X ass'd to M.

Then $H^i(\mathcal{U}_\bullet, \tilde{M}) = 0 \quad \forall i > 0$.

PROOF

Call $\mathcal{U}_f = \operatorname{Spec} R_f$.
Know: $M(\mathcal{U}_f) = M_f$.
Have the Čech complex:

$$C^\bullet(\mathcal{U}_0, \tilde{M}) = \prod M_{f_{i_0}} \longrightarrow \prod M_{f_{i_0} f_{i_1}} \longrightarrow \cdots$$

There are errors in this argument. Refer to Mumford Oda.

Finiteness $\Rightarrow$ Any cochain $\exists N$ fixed s.t.

$$m(i_0, \dots, i_k) = \frac{m_{i_0 \cdots i_k}}{(f_{i_0} \cdots f_{i_k})^N}$$

$$\delta m(i_0, \dots, i_{k+1}) = \frac{m_{i_1 \cdots i_{k+1}}}{(f_{i_0} \cdots f_{i_{k+1}})^N} - \frac{m_{i_0 i_2 \cdots i_{k+1}}}{(f_{i_0} f_{i_2} \cdots f_{i_{k+1}})^N} + \cdots + (-1)^{k+1} \frac{m_{i_0 \cdots i_k}}{(f_{i_0} \cdots f_{i_k})^N}.$$

$$= \frac{f_{i_0}^N m_{i_1 \cdots i_{k+1}} - f_{i_1}^N m_{i_0 i_2 \cdots i_{k+1}} + \cdots + (-1)^{k+1} f_{i_{k+1}}^N m_{i_0 \cdots i_k}}{(f_{i_0} \cdots f_{i_{k+1}})^N}$$

$\partial m = 0$

$$\Rightarrow (f_{i_0} \cdots f_{i_{k+1}})^{N'} \left[f_{i_0}^N m_{i_1 \cdots i_{k+1}} - f_{i_1}^N m_{i_0 i_2 \cdots i_{k+1}} + \cdots + (-1)^{k+1} f_{i_{k+1}}^N m_{i_0 \cdots i_k} \right]$$

$$= 0 \quad (\text{in } M \text{ for } N' \gg 0)$$

WLOG:

$$f_{i_0}^N m_{i_1 \cdots i_{k+1}} - f_{i_1}^N m_{j_0 \cdots i_{k+1}} + \cdots + (-1)^{k+1} f_{i_{k+1}}^N m_{i_0 \cdots i_k}.$$

Know: $1 = \sum_{i \in I} g_i f_i^N$. (some $g_i \in R$)

Define: $n \in C^{k-1}$ by

$$n(i_0 \cdots i_{k-1}) = \frac{n_{i_0 \cdots i_{k-1}}}{(f_{i_0} \cdots f_{i_{k-1}})^N}$$

$$n_{i_0 \cdots i_{k-1}} = \sum_{\ell \in I} f_\ell m_{\ell, i_0, \dots, i_{k-1}}$$

$$\partial n(i_0 \cdots i_k) = \sum_{j=0}^{k} (-1)^j \frac{n_{i_0 \cdots \hat{i}_j \cdots i_{k+1}}}{(f_{i_0} \cdots \hat{f}_{i_j} \cdots f_{i_k})^N}$$

$$= \frac{1}{(f_{i_0} \cdots f_{i_k})^N} \sum_{j=0}^{k} (-1)^j f_{i_j}^N \sum_{\ell \in I} g_\ell m_{\ell i_0 \cdots \hat{i}_j \cdots i_k}$$

$$= \frac{1}{(f_{i_0} \cdots f_{i_k})^N} \sum_{j=0}^{k} (-1)^j m_{i_0 \cdots \hat{i}_j \cdots i_k} = m(i_0 \cdots i_k).$$

∎

Theorem.

Let X be separated + Noetherian.
$\mathcal{U}_i = \operatorname{Spec} R_i \underset{\varphi_i}{\subset} X$ a finite affine open cover.
F q.coherent. Then:

$$H^i(\mathcal{U}_\bullet, F) = H^i(X, F).$$

Proof.

1. $H^i(X, \varphi_{i*}(\widetilde{F(\mathcal{U}_i)})) = 0 \quad \forall i > 0.$

(use the previous Propn.)

2. Consider the complex of sheaves:

$$0 \to F \to \bigoplus_{i_0} \underbrace{\varphi_{i_0 *}(\widetilde{F(\mathcal{U}_i)})}_{G_0} \to \bigoplus_{i_0, i_1} \underbrace{\varphi_{i_0 i_1 *}(\widetilde{F(\mathcal{U}_{i_0 i_1})})}_{G_1} \to \cdots$$

Then **A.** This is exact at the level of sheaves.

B. $H^i(G_k) = 0 \quad i > 0 \quad k \geq 0.$

C. $H^0 G_\bullet \cong C^\bullet(\mathcal{U}_\bullet, F)$

as complexes.

$\blacksquare$

COHOM. of $\mathcal{O}_{\mathbb{P}^1}(d)$

$\mathbb{P}^1$, $\mathcal{O}_{\mathbb{P}^1}(d)$. H^0 ✓ H^1 ?

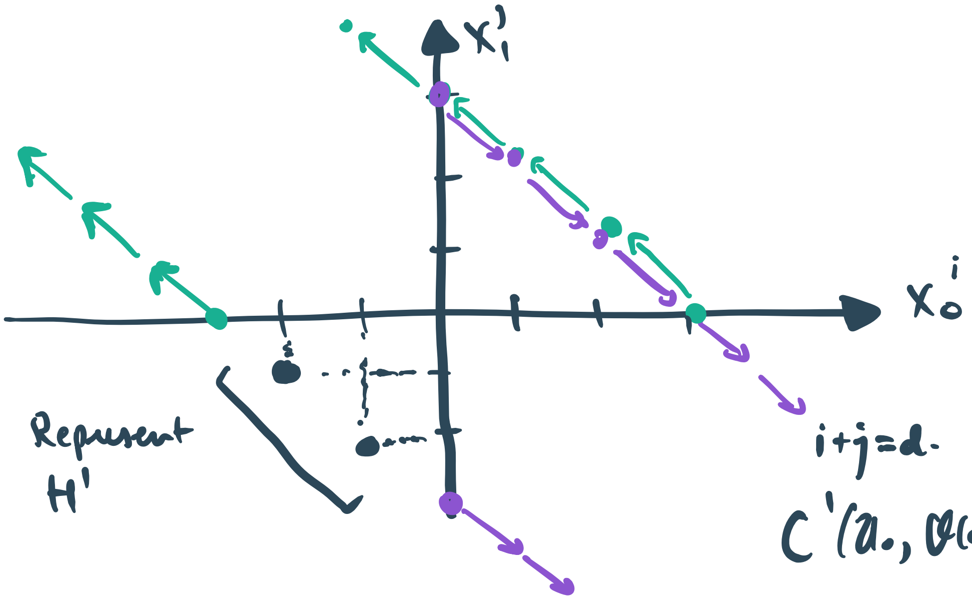
2 standard affines.

$$\mathbb{A}^1_{\frac{x_1}{x_0}} : \quad \mathcal{O}_{\mathbb{P}^1}(d)(\mathbb{A}^1_{\frac{x_1}{x_0}}) = x_0^d \cdot A[\tfrac{x_1}{x_0}]$$

$$\mathbb{A}^1_{\frac{x_0}{x_1}} : \quad \mathcal{O}_{\mathbb{P}^1}(d)(\mathbb{A}^1_{\frac{x_0}{x_1}}) = x_1^d \cdot A[\tfrac{x_0}{x_1}].$$

$$\downarrow$$

$$\mathcal{O}_{\mathbb{P}^1}(d)(\mathbb{A}^1_{\frac{x_0}{x_1}} \cap \mathbb{A}^1_{\frac{x_1}{x_0}}) = x_0^d\, A[\tfrac{x_1}{x_0}^{\pm 1}]$$



Represent H^1

$i+j=d$.

$C^1(\mathfrak{U}, \mathcal{O}(d))$.

$$\Longrightarrow H^1(\mathbb{P}^1_A, \mathcal{O}(d)) = \bigoplus_{\substack{i+j=d \\ (i,j) \text{ strictly} \\ \text{in quadrant III}}} A \cdot x_0^i x_1^j .$$

$$\left\{ (i,j) \;\middle|\; \begin{array}{l} i+j=d \\ (i,j) \text{ strictly} \\ \text{in quadrant III} \end{array} \right\} \xleftrightarrow{1:1} \left\{ (i,j) \;\middle|\; \begin{array}{l} i+j=d+2 \\ (i,j) \text{ in quad III} \end{array} \right\}.$$

[shift by (1,1)]

$$\underset{(-1)}{\xleftrightarrow{1:1}} \left\{ (i,j) \;\middle|\; \begin{array}{l} i+j=-2-d \\ (i,j) \text{ in quad I} \end{array} \right\}.$$

$$\underset{1:1}{\longleftrightarrow} \text{Homog. polys of degree } -2-d .$$

Cohomology of Projective Space

Theorem:

A If $d \geq 0$ then $H^0(\mathbb{P}^n_A, \mathcal{O}(d))$ is a free A-module with basis degree d monomials.

B If $d \leq -n-1$, $H^n(\mathbb{P}^n_A, \mathcal{O}(d))$ is a free A-module with basis degree d Laurent monomials:

$$\frac{1}{x_0^{a_0} \cdots x_n^{a_n}} \quad (a_i > 0).$$

C These are the only non-zero cohomology groups $H^i(\mathbb{P}^n_A, \mathcal{O}(d))$.

Corollary:

A Let X be a projective A-scheme w/ A Noetherian. If F is coherent on X then $H^i(X,F)$ is a f.g.'d A-mod.

B (Serre Vanishing) $\exists M$ s.t. $\forall d > M$

$$H^i(X, F(d)) = 0. \quad (\forall i > 0).$$

Q. Give an example of X proj., A not Noetherian s.t. $H^i(X,F)$ is not f.g.'d.

Pf of Cor. Downwards induction for all coherent sheaves simultaneously.

Can compute cohom. on $\mathbb{P}^n_A$ (see HW).

1. $\exists\, m$ s.t. $F(m)$ is globally gen'd.

$$0 \to K \to \mathcal{O}(-m)^{\oplus N} \to F \to 0$$

2. $H^i(X,F)=0 \quad i>n.$

LES gives...

$$H^n(X,\mathcal{O}(-m)^{\oplus N}) \to H^n(F) \to 0$$

$\Rightarrow H^n(F)$ is a finitely gen'd A-mod. $\forall F$ coherent.

$$\Rightarrow\ 0 \to \underset{\substack{\uparrow \\ \text{(sub. }A\text{-module)}}}{H^{n-1}(F)} \to \underset{\substack{\uparrow \\ \text{f.g'd.}}}{H^n(K)}$$

$\Rightarrow H^{n-1}(F)$ f.g'd continue...
(proves **A**).

For **B**. Know $H^n(\mathbb{P}^n, \mathcal{O}(d+m))=0 \quad \forall\, d \geq -n-m.$

$\Rightarrow H^n(\mathbb{P}^n, F(d))=0$ (same d).
So we know it $\forall$ coherent F.

Continuing the induction:

$$0 \to H^{n-1}(\mathbb{P}^n, F(d)) \to H^n(\mathbb{P}^n, K(d))$$

($\overset{\shortparallel}{0}$ for $d > M$).

AND continue ... $\blacksquare$

Proof of Theorem

Pt A ✓

Pt B By Čech cohomology, we have:

$$H^n(\mathbb{P}^n_A, \mathcal{O}(d))$$
$$\|$$
$$\mathrm{COKER}\left[\begin{array}{l} A\left[\left(\frac{x_1}{x_0}\right)^{\pm 1}, \dots, \left(\frac{x_{n-1}}{x_0}\right)^{\pm 1}, \frac{x_n}{x_0}\right] x_0^d \\ \oplus \\ \vdots \\ \oplus \\ A\left[\frac{x_0}{x_1}, \left(\frac{x_2}{x_1}\right)^{\pm 1}, \dots, \left(\frac{x_n}{x_1}\right)^{\pm 1}\right] x_1^d \end{array} \longrightarrow A\left[\left(\frac{x_1}{x_0}\right)^{\pm 1}, \dots, \left(\frac{x_n}{x_0}\right)^{\pm 1}\right] x_0^d\right]$$

all degree d monomials w x_n coeff ≥ 0.

all degree d monomials w/ x_0 coeff ≥ 0.

all degree d monomials.

$$\| $$

$$\bigoplus_{\substack{a_i \geq 0 \\ \sum a_i = -d}} A \cdot \frac{1}{x_0^{a_0} \cdots x_n^{a_n}}$$ ✓

P+C

Consider $F = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}(d)$.

The Čech complex for F is easy to write down: If $S = A[x_0, \ldots, x_n]$ then:

$$C^\bullet(\{A_i\}, F)$$

$$\|$$

$$\bigoplus S_{x_i} \longrightarrow \bigoplus_{i<j} S_{x_i x_j} \longrightarrow \cdots \longrightarrow \bigoplus_{i=1}^{n} S_{x_0 x_1 \cdots \hat{x}_i \cdots x_n} \longrightarrow S_{x_0 \cdots x_n}$$

(taking degree d part gives $H^\bullet(\mathcal{O}(d))$).

This is a complex of graded S-modules.

Localizing w.r.t. x_0 gives the Čech complex for S_{x_0} on $\mathbb{A}^{n+1} \setminus (x_0 = 0)$,

$\mathbb{A}^{n+1} \setminus (x_0 x_1 = 0)$,

$\cdots$

$\mathbb{A}^{n+1} \setminus (x_0 x_n = 0)$.

S_{x_0} is a module on an affine variety $\Rightarrow H^i = 0$ for $i > 0$.

So localizing the original complex @ x_0 kills the S-modules H^i $(i>0)$

$\Rightarrow$ For any cycle $\alpha \in H^i(\mathcal{O}(d))$ $\exists$ a power l s.t. $x_0^l \alpha = 0$.

Now we induct! Consider:

$$0 \to \mathcal{O}(d) \xrightarrow{\cdot x_0} \mathcal{O}(d+1) \to \mathcal{O}_H(d+1) \to 0$$

$(H = (x_0 = 0))$

Given:

$$H^0(\mathbb{P}^n_A, \mathcal{O}(d+1)) \to H^0(\mathbb{P}^n_A, \mathcal{O}_H(d+1)) \to$$

$$H^1(\mathbb{P}^n_A, \mathcal{O}(d)) \xrightarrow{\cdot x_0} H^1(\mathbb{P}^n_A, \mathcal{O}(d+1)) \to H^1(\mathbb{P}^n_A, \mathcal{O}_H(d+1)) \to$$

$$\cdots$$

$$\to H^i(\mathbb{P}^n_A, \mathcal{O}(d)) \xrightarrow{\cdot x_0} H^i(\mathbb{P}^n_A, \mathcal{O}(d+1)) \to H^i(\mathbb{P}^n_A, \mathcal{O}_H(d+1))$$

Induction $\Rightarrow$ $x_0 : H^i(\mathbb{P}^n_A, \mathcal{O}(d)) \hookrightarrow H^i(\mathbb{P}^n_A, \mathcal{O}(d+1))$.

So the groups must vanish! $\blacksquare$

Remark:

The groups: $$H^0(\mathcal{O}_{\mathbb{P}^n}(d)) = \bigoplus_{\substack{a_0 \dots a_n \geq 0 \\ \sum a_i = d}} A \cdot x_0^{a_0} \cdots x_n^{a_n}.$$

& $$H^n(\mathcal{O}_{\mathbb{P}^n}(-n-1-d)) = \bigoplus_{\substack{a_i > 0 \\ -\sum a_i = -n-1-d}} \frac{1}{x_0^{a_0} \cdots x_n^{a_n}}.$$

Have a natural perfect pairing that lands in $H^n(\mathcal{O}_{\mathbb{P}^n}(-n-1)) \cong A \dfrac{1}{x_0 \cdots x_n}$.

(This is the first appearance of SERRE DUALITY)

EXAMPLE

Let $C \subseteq \mathbb{P}^2_k$ be a smooth degree d plane curve.

$$0 \to \underset{\mathcal{O}_{\mathbb{P}^2}(-d)}{\overset{\mathcal{I}_C}{\wr\wr}} \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0$$

So for all e:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(e-d) \to \mathcal{O}_{\mathbb{P}^2}(e) \to \mathcal{O}_C(e) \to 0$$

So these sequences are exact:

$$0 \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(e-d)) \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(e)) \to H^0(C, \mathcal{O}_C(e)) \to 0.$$

AND

$$0 \to H^1(C, \mathcal{O}_C(e)) \to H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(e-d)) \to H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(e)) \to 0.$$

Allows us to compute $H^i(C, \mathcal{O}_C(e))$ for any e.

($\Rightarrow C$ is connected)

$$H^1(C,\omega_C) \cong H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) = k$$

$$H^1(C,\mathcal{O}_C) \cong H^2(\mathbb{P}^2, \mathcal{O}(-d)) \xleftrightarrow[\text{dual.}]{} H^0(C,\omega_C) \cong H^0(C, \mathcal{O}_C(d-3))$$

$$\cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-3)).$$

$$H^0(C,\mathcal{O}_C) \cong H^0(\mathbb{P}^2, \mathcal{O}) = k.$$