Numerical Annealing of Low-Redundancy Linear Arrays

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Abstract—**An algorithm is developed that estimates the optimal distribution of antenna elements in a minimum redundancy linear array. These distributions are used in thinned array interferometric imagers to synthesize effective antenna apertures much larger than the physical aperture. The optimal selection of antenna locations is extremely time consuming when large numbers of antennas are involved. This algorithm uses a numerical implementation of the annealing process to guide a random search for the optimal array configuration. Highly thinned low-redundancy arrays are computed for up to 30 array elements. These arrays are equivalent to the optimal solutions that are known for up to 11 elements. The arrays computed for 12–30 elements have the least redundancies reported to date.**

I. Introduction

INTERFEROMETRIC aperture synthesis is a technique widely used in the radio astronomy community to improve the angular resolution of radio telescopes [1],[2]. Pairs of antennas are cross correlated, acting as spatial filters. Different spatial frequencies are sampled by element pairs with different separations. The set of all sampled spatial frequencies can then be inverted to estimate the original scene radiance.

The question of where to place the antenna elements in order to optimally sample the spatial frequency spectrum of the scene has received considerable attention. In most radio astronomy applications, earth rotation is used to obtain additional relative spacings between antenna pairs [3]–[5]. For some applications, however, an adequate sampling of the spatial frequency spectrum is required in a relatively short time. Such a sampling scheme is often referred to as snapshot imaging. Snapshot array performance is judged by the instantaneous set of relative spacings between pairs of elements [6–8].

The interferometric imaging technique is also an attractive means of improving angular resolution for applications in microwave radiometer Earth remote sensing [9]–[11]. An interferometric imager in low Earth orbit would probably use a snapshot sampling scheme because of the rapid relative motion between the antenna and the scene. One practical implementation of a snapshot imager is the thinned linear array [6]. The individual element patterns are fan beams oriented cross track to the direction of motion of the imager, and interferometric pattern synthesis is performed in that direction only. This configuration is illustrated in Fig. 1. An aircraft prototype has been built and flown, which operates at 1.4 GHz and synthesizes 16 antenna beams from a five-element thinned linear array [12],[13].

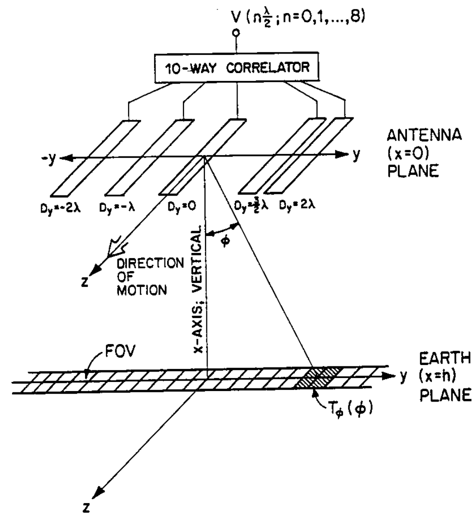

Fig. 1. Antenna array configuration of a thinned linear array of antennas used for interferometric imaging. The fan beam antenna patterns of all array elements overlap to form the instantaneous field of view of the imager. Synthesized resolution within the field of view is achieved by cross correlating all possible pairs of antenna elements. Each cross-correlation samples a spatial harmonic of the scene radiance in the field of view. These samples can be combined by Fourier inversion to estimate the original scene (from [10]).

II. Thinned Linear Array Sampling

Elements of a thinned linear array are arranged in discrete steps along a line. If each element of the array is assigned an integer representing its position, then the set of all differences between pairs of integers constitutes the sample space of the interferometer. A sample space with no missing integers insures optimal spatial sampling. The definition of an optimal (minimum redundancy) array then reduces to: What set of n integers will generate continuous differences from 1 up to the largest possible number. This problem has been investigated as a purely number theoretic issue [14],[15]. It was shown that the following relationship exists between the maximum difference between integers, N, and the number of integers, n, in an optimal (minimum redundancy) set

$$\lim_{n\to\infty} \frac{n^2}{N} \simeq 2.6 \qquad (1)$$

This limit provides an upper bound on the efficiency with which large thinned linear arrays can sample spatial harmonics.

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Optimal configurations have been described for up to 11 elements [6]. With more elements, the number of possible array configurations rapidly becomes too large to sort through completely. Several earlier attempts have been made to construct low-redundancy linear arrays. One algorithm selectively sorts through the possible configurations by adding locations one by one in efficient locations [14]. This approach was implemented numerically to produce the lowest redundancy 14 element thinned array at that time [10]. A different algorithm was developed that constructs large, low-redundancy thinned arrays from subarrays of small minimum redundancy arrays [7]. This approach has provided a large class of the lowest redundancy array configurations known to date.

III. Numerical Annealing

Determining the optimal thinned linear array for a given number of elements is a many-variable maximization problem in which the function to be maximized is the largest difference between element positions. Alternatively, for a given maximum difference, N, and a given number of elements, n, the optimization problem becomes determining whether the other $n-2$ elements can be placed between the two outermost ones so that all the required difference spacings are present. If such a configuration is found, N can be incremented and the procedure repeated until N grows too large. This approach is more amenable to a numerical implementation. A random search for the proper positions of the $n-2$ elements begins with a particular array configuration. The element positions are then randomly perturbed and the sampling performance of the new array is tested. A natural measure of sampling performance is the number of difference spacings that have not been sampled. This measure is to be minimized. If a better array is always accepted and a worse array is always discarded, the search will rapidly locate an array configuration that has a local (in the space of array configurations) minimum number of missing samples and then stay there. This is a result of the very large number of local minima present in this many-variable optimization problem.

One effective method for locating a global minimum in many-variable systems is numerical, or simulated, annealing [16]. In this procedure, element positions are randomly perturbed as above. Again, if the new array performs better, it is accepted. However, in this case, a worse array is also occasionally accepted. The probability of accepting a worse array is dependent on a variable called the temperature of the annealing process. A higher temperature corresponds to a higher probability of accepting a worse array. At the beginning of the random search, the temperature variable is set fairly high. High temperatures provide the agitation needed to free the array configuration from local minimum wells and drop it into the global minimum. As the performance of the array improves, the temperature is gradually lowered and the selection of worse arrays becomes less likely.

Numerical annealing has been used to locate efficient distributions of antenna elements lying in a ring for two-dimensional imaging with applications in radio astronomy [8]. The relative spacings between elements in a two-dimensional array do not generally lie on a discrete grid of sampling points analogous to the discrete line present in the one-dimensional case. For this reason, there are different measures of optimum performance possible in the two-dimensional case. Cornwell attempts to minimize a function of the mean distance between elements, which emphasizes the closer spaced elements [8]. This tends to spread out the set of relative spacings between elements and provide a more uniform coverage of the sampling space. Another example of numerical annealing of a ring of antenna elements attempts to put at least one relative spacing sample in each cell of the discrete grid of sample points [17]. Additional samples not available in the snapshot mode are assembled by rotating the array.

IV. Description of Algorithm

The algorithm begins with a set of n integers representing the positions of the array elements at discrete intervals along a line. The maximum difference between integers, N, is also specified. This represents the maximum spacing between array elements and is proportional to the inverse of the angular resolution of the synthesized antenna pattern [10]. The set of initial integers includes 0, 1, and N. These members must be included in any array solution that contains all differences between 1 and N. This can be seen by noting that the elements 0 and N are needed to generate the difference N, and the element 1 is needed to generate the difference $(N-1)$. (The third element can be placed in either position 1 or $(N-1)$, but by symmetry the choice is arbitrary. The fourth element, needed to generate the difference $(N-2)$, could be placed in any of positions 2, $(N-2)$, or $(N-1)$. This choice is left up to the annealing process.) The other $n-3$ integers are initially selected at random from between 2 and $N-1$ without repetition.

If the current set of integers does not contain all the differences needed, the annealing process is implemented by randomly switching one of the $n-3$ intermediate integers with another integer between 2 and $N-1$ that has not yet been selected. The new set of integers is accepted over the previous one if the following condition is satisfied:

$$e^{(k_0-k)/T} > x \tag{2}$$

where k_0 is the number of differences missing between 1 and N for some reference set of integers, k is the number of differences missing for the current set of integers, T is the temperature variable, and x is a random number chosen from a uniform distribution between 0 and 1. The new integers will always be selected if they have missing differences equal in number to or fewer than the reference (*i.e.*, if $k < k_0$). If the new integers have more missing differences, then the probability of accepting them is given by

$$Pr(k) = Pr\left[e^{(k_0-k)/T} > x\right] \tag{3}$$

If k is viewed as a random variable with statistics determined by its probability of acceptance, then its normalized probability distribution function is

$$\text{pdf}\,(k) = \frac{1}{T}e^{(k_0-k)/T} \qquad \text{for } k_0 < k < \infty \tag{4}$$

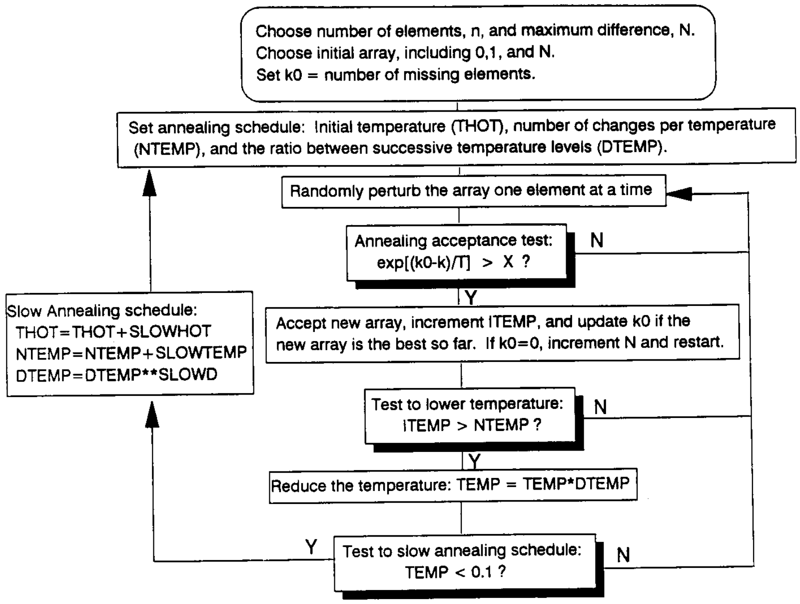

Fig. 2. Flow chart of the numerical annealing algorithm.

The mean value of k then follows as

$$\langle k\rangle = \int_{-\infty}^{\infty} k\, pdf(k)\, dk = k_0 + T \qquad (5)$$

Equation (5) illustrates the significance of the temperature variable T. Numerical annealing at a temperature T will maintain the random search at an average level of T missing samples higher than the reference.

An annealing schedule is followed which slowly drops the temperature variable toward zero. The reference level k_0 used in (2) is the lowest number of missing samples k found up to that point. In this way, the reference level drops as the array performance improves and the mean level of the perturbed arrays above the reference level drops as the temperature is reduced. Once T has dropped below approximately 0.1, the probability of improving the performance of the perturbed arrays over that of the reference array becomes very small, and the annealing schedule is reset. A higher initial temperature and slower rate of temperature decrease are selected and the annealing process is repeated. If, on the other hand, the algorithm locates an array with no missing samples, the maximum difference N is increased and the annealing process is restarted. This process is repeated until several consecutive reductions in the annealing rate produce no acceptable array configuration. An algorithm flow chart is shown in Fig. 2.

V. Run Time Example

Following is a simplified example that illustrates the logic by which this annealing procedure searches for an optimal array configuration. Given a five-element array, a solution is attempted with a maximum of nine-unit spacings between elements. The positions of three of the elements are predetermined as 0, 1, and 9. The locations of the remaining two elements are initially chosen at random. Assume the fourth element is randomly located at position 5. Possible positions for the fifth element are 2, 3, 4, 6, 7, and 8. The number of missing difference spacings for each of these positions is 1, 1, 3, 2, 1, and 2, respectively. (For example, if the fifth element is in position 8, then differences of 2 and 6 are missing.) The random search will accept or discard possible locations for the fifth element according to the criteria specified by (2). Assume for the moment that this search does not include an annealing component. This is equivalent to only accepting new array configurations which are improvements over the previous ones. Such a search would locate the fifth element in either position 2, 3 or 7, since these all result in a minimum number of missing difference spacings. Assume that position 3 was selected. Now the fourth array element could be moved to attempt to further improve the array. Possible new locations for the fourth element include 2, 4, 5, 6, 7, and 8. The number of missing difference spacings for each of these positions is 2, 1, 1, 2, 1, and 1, respectively. None of these positions improve the array, so the algorithm is stuck in a local minimum number of missing difference spacings.

If an annealing component is included in the search decision, then the above outcome can be avoided. First, the fourth element is randomly located at position 5 and the fifth element is moved to position 3 to minimize the number of missing differences. This results in $k_0 = 1$ missing differences. The possible state of future iterations of the array is determined by the temperature of the system. A temperature of $T = 1$ will, according to (5), set the mean number of missing differences for arrays that are accepted in future iterations at 1 higher than the current minimum number, k_0. Thus the fifth element can be located at position 6, with two missing samples. The next iteration of the 4th element, with the fifth set at position

TABLE I
NUMERICAL ANNEALING LOW REDUNDANCY ARRAY CONFIGURATIONS

n	Location of Array Elements																
4	0	1	4	6													
5	0	1	4	7	9												
6	0	1	2	6	10	13											
7	0	1	2	6	10	14	17										
8	0	1	4	10	16	18	21	23									
9	0	1	3	6	13	20	24	28	29								
10	0	1	3	6	13	20	27	31	35	36							
11	0	1	3	6	13	20	27	34	38	42	43						
12	0	1	5	9	16	23	30	37	44	47	49	50					
13	0	1	3	6	17	20	27	35	45	49	53	57	58				
14	0	1	2	8	14	20	31	42	53	58	63	66	67	68			
15	0	1	2	5	10	15	26	37	48	59	65	71	77	78	79		
16	0	1	2	5	10	15	26	37	48	59	70	76	82	88	89	90	
17	0	1	2	8	14	20	31	42	53	64	75	86	91	96	99	100	101
18	0	1	2	8	14	20	31	42	53	64	75	86	97	102	107	110	111
	112																
19	0	1	2	3	4	5	50	58	65	71	77	83	89	95	101	107	112
	117	121															
20	0	1	2	4	9	16	23	24	37	50	63	76	89	102	115	121	127
	131	132	133														
21	0	1	2	3	4	5	6	59	68	74	79	87	94	101	108	115	122
	129	133	139	145													
22	0	1	3	9	15	23	24	37	50	58	63	76	89	102	115	128	141
	146	148	153	157	160												
23	0	1	3	4	5	6	11	16	21	35	49	63	77	91	105	119	133
	147	156	160	169	172	173											
24	0	1	3	6	7	8	13	15	20	27	43	59	75	91	107	123	139
	155	164	173	177	184	186	188										
25	0	1	2	3	4	12	13	22	31	40	57	74	91	108	125	142	159
	176	184	192	200	205	206	207	208									
26	0	1	2	3	5	8	16	24	32	49	66	83	100	117	134	151	168
	185	194	203	204	213	222	223	224	225								
27	0	1	2	4	10	13	19	25	41	57	73	89	105	121	137	153	169
	185	201	211	214	218	221	228	231	235	236							
28	0	1	2	3	8	16	23	24	31	48	65	82	99	116	133	150	167
	184	201	218	227	236	245	251	254	255	256	257						
29	0	1	2	5	7	9	29	31	32	51	70	89	108	127	146	165	184
	203	222	238	239	243	246	255	256	259	267	269	270					
30	0	1	2	14	16	22	28	36	42	61	80	99	118	137	156	175	194
	213	232	251	262	264	269	275	278	280	282	285	286	287				

6, could locate it at position 2. This results in a five-element array configuration of 0, 1, 2, 6, and 9, which has no missing samples and is a minimum redundancy linear array.

While this example necessarily oversimplifies the search process, it illustrates the procedure by which an annealed array overcomes local minima in the space of array configurations. A sufficiently high temperature T can remove the search from any local minimum. The temperature is initially set quite high to allow the search to include many different regions of the configuration space. As the temperature is lowered, the search is restricted to those regions with the deepest local minima. The search is terminated when either a satisfactory array is found or the temperature has dropped too low to escape the current configuration.

VI. RESULTS AND COMPARISON WITH EARLIER ARRAYS

The numerical annealing algorithm was applied to up to 30 array elements. The resulting configurations are listed in Table I. In each case, the algorithm produces many intermediate sized arrays that also have no missing samples. Only the arrays with the largest maximum spacing between elements, and hence the highest spatial resolution, are listed. Equation (1) provides one point for comparison with the performance of the annealing algorithm. Because (1) is only valid in the limit of large arrays, it does not necessarily predict the true optimum performance level for arrays of finite size. In other words, the arrays located by annealing may in fact be optimum. This claim can only be proved by an exhaustive search through

TABLE II
ARRAY CONSTRUCTION ALGORITHM PERFORMANCE COMPARISON

	Maximum Separation Between Array Elements[1]			
# of Elements	Annealed	Theory[2]	Ishiguro[3]	Leech/UMass[4]
4	6	6		
5	9	9		
6	13	13		
7	17	17		
8	23	23		
9	29	29		
10	36	36		
11	43	43		
12	50	55.4	45	
13	58	65.0		
14	68	75.4	52	64
15	79	86.5	66	
16	90	98.5	84	
17	101	111.2		
18	112	124.6	94	
19	121	138.8		
20	133	153.8	123	
21	145	169.6	122	
22	160	186.2	130	
23	173	203.5		
24	188	221.5	175	
25	208	240.4	180	
26	225	260.0		
27	236	280.4	206	
28	257	301.5	227	
29	270	323.5		
30	287	346.2	256	

Notes:
[1]All arrays have no missing relative spacings.
[2]For 4 to 11 elements, the true optimum array is given. For 12 to 30 elements, the number quoted is the theoretical optimum array in the limit of very large arrays (see Equation (1)). These limits overestimate the degree of thinning possible with arrays of less than 12 elements and may do so for larger arrays.
[3]Large Ishiguro arrays are predicted for numbers of elements which have two prime factors which are both between 2 and 11 [7].
[4]Leech algorithm implemented by UMass for 14 elements only [10].

all possible alternatives; it can be disproved by identifying a better array.

As a second point of comparison, the true minimum redundancy arrays are known for up to 11 elements. They were all successfully located by the annealing algorithm. For larger numbers of elements, the arrays computed using the Ishiguro algorithm [7] provide another point of comparison. In all cases, the annealing algorithm located arrays with larger maximum spacings. One other comparison is available using the University of Massachusetts implementation of the Leech algorithm for the case of 14 array elements [10]. While this technique out performed the Ishiguro 14–element array, it, too, is surpassed by the annealing algorithm. The maximum difference available from each of these algorithms is summarized in Table II.

It should be emphasized that the new array configurations located by this annealing procedure cannot be guaranteed to have a minimum number of redundancies. This would require a comparison search through all possible array configurations with the same number of elements. It was the intractability of that search that originally motivated the use of an annealing approach. Likewise, the new solutions presented here cannot be guaranteed to be unique. Different array configurations may exist that perform as well with the same number of elements. The significant and useful result of this approach is a set of array configurations that perform better than previous ones. However, these configurations are best regarded as low-redundancy, not minimum-redundancy, arrays for the present.

VII. CONCLUSIONS

Minimum-redundancy linear arrays permit the maximum number of antenna elements to be removed from a uniform array without degrading the antenna pattern that results from interferometric aperture synthesis. This approach has found applications in both radio astronomy and passive Earth remote sensing. Exact solutions are known for the locations of the elements in small minimum-redundancy linear arrays. A number of algorithms exist that construct larger low-redundancy arrays. Whether or not these arrays minimize their redundant samples can usually only be disproved by counterexample. A new algorithm is presented that constructs arrays with fewer redundant samples than other procedures. At present, these arrays provide the best angular resolution possible for a given number of elements, and they provide a given angular resolution with the fewest elements possible.

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