Foreward

In these two short papers Professor Tobler demonstrates both the originality and analytical ability so typical of his work. Some of the ideas he expresses have been developed from conversations held by the members of the Michigan Community of Mathematical Geographers over the past year. The advantages of extending analytical techniques of one dimension to other dimensions is one example.

His analysis of map generalization by numerical means has produced surprises. I did not expect that some smoothing operators would prove to have inverses. A class of such operators is demonstrated here. Some of our readers may be surprised at the usefulness of computer produced maps. Professor Tobler has assembled programs which produce such maps with ease and speed and at costs under the costs of manually producing the same.

The two ideas I find most exciting in these papers are the emergence of an explicit operational definition of site or neighborhood conditions by use of local operators and the suggestion that by use of frequency filters and similar techniques periodicities in space may be revealed.

The Editor

January 1966

Numerical Map Generalization

W. R. Tobler
Department of Geography
University of Michigan
Ann Arbor

ABSTRACT

Mathematical procedures for the generalization of isarithmic maps are derived. It is shown that the level of generalization for different purposes may be specified in advance, and that under certain conditions the original map can be recreated from the generalized map.

ACKNOWLEDGEMENTS

The author is indebted to the computing center of the University of Michigan for support of this study. J. Harbaugh made available the two-dimensional Fourier analysis computer program. Colleagues and students in the Michigan Inter-University Community of Mathematical Geographers contributed numerous comments. The data employed for the examples came from a digitalization by R. Yuill of a contour map given on page 92 of Garnier (1963).

The process of map generalization bears a certain resemblance to the process of abstracting textual materials, such as books and articles. As a condensation one expects that only the most relevant and important items will be retained in the simplified version. It is of some theoretical interest to speculate on the extent to which this process can be formalized. As a form of "picture abstracting" the process of map generalization is probably a simple case of a more general problem of pattern analysis, which in turn may be considered to be the basis for the important inductive approach to any knowledge. In particular, it might be anticipated that a study of map generalization might also illuminate the process of scientific generalization, especially as practiced in fields which traditionally make frequent use of maps.

Mapped data may be considered to consist of point (or zero-dimensional) phenomena, and line, area, intensity, and flow symbols. Generalization consists of the application of a transformation which modifies the map data. Pillewizer and Topfer (1964) have considered generalization of point symbols, and Perkal (1958) has contributed to the problem of generalizing lines and areas. The present discussion examines the generalization of "three-dimensional" phenomena, which are assumed to be continuous. Information of this type is most often shown on maps by isarithms of various types; contours, isopleths, isobars, and so on.

As a priori criteria one would expect that generalization of isarithmic maps should somehow eliminate "small scale" features. If generalization is considered as a transformation it is natural to inquire whether the process can be reversed. If topographical maps are generalized the result should conform to appropriately modified accuracy standards. Finally, certain statistical parameters (mean, standard deviation, and so on) should be preserved by the transformation.

Trend analysis provides one form of map generalization. The geographical trend of population density (persons per square mile) in Michigan is (Tobler, 1964a)

$$D = 6314 - 68.7 * \text{latitude N} - 36.9 * \text{longitude W},$$

or, in words, the population declines towards the northwest. The "plane" defined by this equation is in some sense a geographical generalization of the distribution of population in the state of Michigan. The average population density is preserved and the generalization represents about 8% of the observations. A quadratic generalization of the same data,

$$D = 207529 - 4547 * \text{lat. N} - 2389 * \text{long. W} + 52 * \text{lat. N} * \text{long. W},$$

accounts for some 15% of the data. In both of these instances the generalization is extreme, though cases can be cited (Krumbein and Graybill, 1965) for which this technique of trend analysis is most useful.

Another possible method of contour generalization is to smooth the values by a moving average, as in the treatment of time series data. For expository convenience consider first only a profile. This can be represented by a single valued function $Z = f(x)$. From this profile select a finite sample of values Z_i at a set of discrete locations $x_1, x_2, x_3, \ldots, x_i, \ldots x_n$. For notational convenience let $\Delta x = x_i - x_{i-1}$ = a constant for all i. Smoothed values Z^* can be obtained from the original values from the following linear equation

$$Z_i^* = \frac{\sum_{m=-k}^{m=k} W_m Z_{m+i}}{\sum_{m=-k}^{m=k} W_m}$$

except for obvious modifications at the ends of the interval. The W's are weights, so that Z^* is a weighted moving average. This can be written, for $k = 1$ and with the sum of the weights equal to unity, as the matrix equation

$$Z^* = Z \cdot S$$

where Z is the (1 by n) vector of Z_i and S is an (n by n) smoothing matrix of the form

$$S = \begin{bmatrix} 3/4 & 1/4 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 1/4 & 1/2 & 1/4 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 1/2 & 1/4 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 1/2 & 1/4 & & 0 & 0 & 0 \\ \vdots & & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & & 1/2 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1/4 & 3/4 \end{bmatrix}$$

which represents a three point binomially weighted moving average. For $k > 1$ the matrix is simply less sparce. Note that a second smoothing can be obtained as

$$Z^{**} = Z^* \cdot S,$$

or, more generally

$$Z^M = Z \cdot S^M .$$

Repeated application of the smoothing operation of course results in increased generalization.

Under certain conditions the original values can be restored from the smoothed values by solving the above matrix equation to obtain

$$Z = Z^* \cdot S^{-1} .$$

In other words, a generalized profile can be ungeneralized ! The condition is the existence of the inverse matrix S^{-1}. The necessary theorems demonstrating the existence of this inverse are in the appendix. In general,

the exact inverse requires an inverse weighting using all n of the smoothed values. An approximate, three weight, inverse for the matrix shown can be constructed from the weights -0.5, +2.0, -0.5, however (Holloway, 1958). This is less exact but is more practical for large n.

If the sum of the weights in the smoothing operation is equal to unity, then the mean of the series is invariant. The extremes are usually reduced, however. In some cartographical situations it seems important that the extrema be retained. Several simple operators can be devised which restore the extremes. For example, let

$$Z_i^r = A + BZ_i^*$$

where

$$B = \frac{Zmax - Zmin}{Z^*max - BZ^*min.}$$

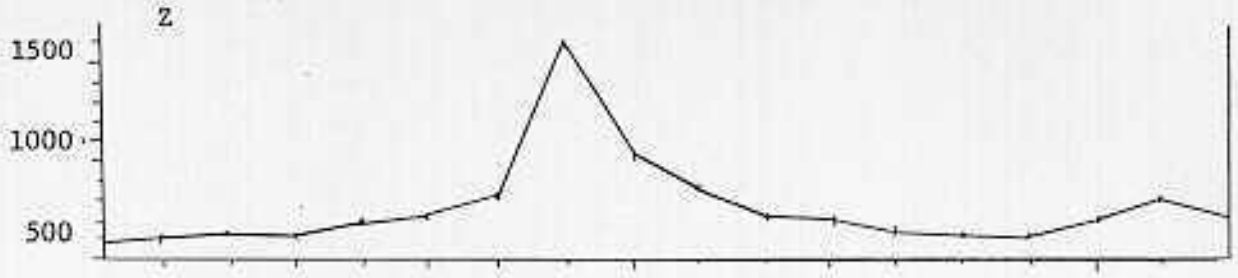
This normalization restores the maximum and minimum of the series.

A numerical example is given in Table I and is illustrated in the accompaning figures.

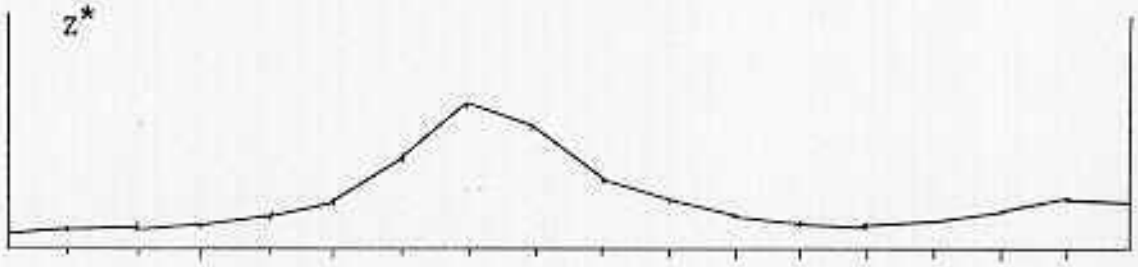
TABLE I
PROFILE GENERALIZATION

i	Z	Z^*	Z^r	Z^{**}	Z'	Z"	$\hat{Z}$	$\hat{\hat{Z}}$
1	490	493	490	495	644	645	490	492
2	500	500	501	500	647	647	501	499
3	510	505	508	508	652	644	499	512
4	500	523	536	529	624	641	511	498
5	580	565	600	568	662	644	559	585
6	600	620	684	670	627	597	520	593
7	700	875	1073	881	472	553	863	709
8	1500	1155	1500	1051	992	751	1365	1489
9	920	1020	1294	986	547	681	1088	934
10	740	750	882	788	637	609	674	724
11	600	633	704	648	614	632	604	619
12	590	575	615	579	662	643	567	571
13	520	533	551	537	634	643	526	539
14	500	505	508	517	642	635	482	483
15	500	523	536	537	624	633	496	516
16	590	595	646	590	642	652	605	578
17	700	648	726	629	699	666	686	708
18	600	625	691	631	622	641	614	597

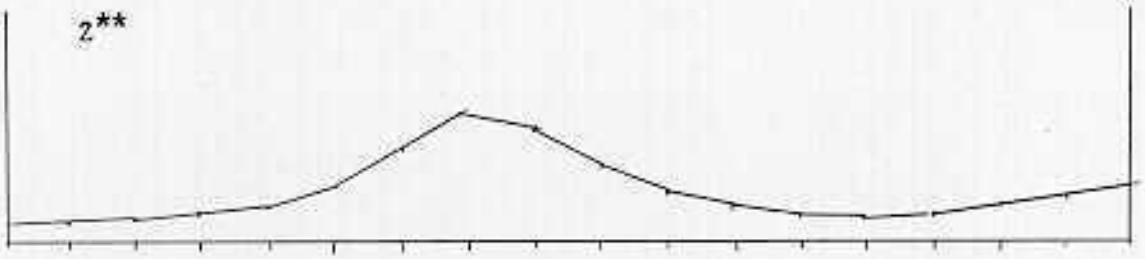
PROFILE GENERALIZATION



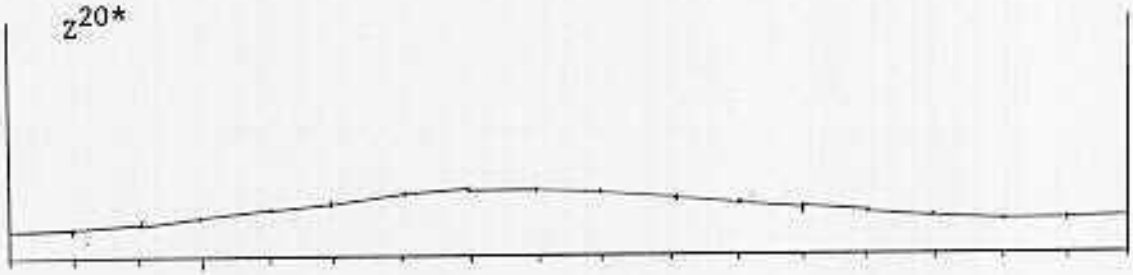
Original Profile



Smoothed Profile



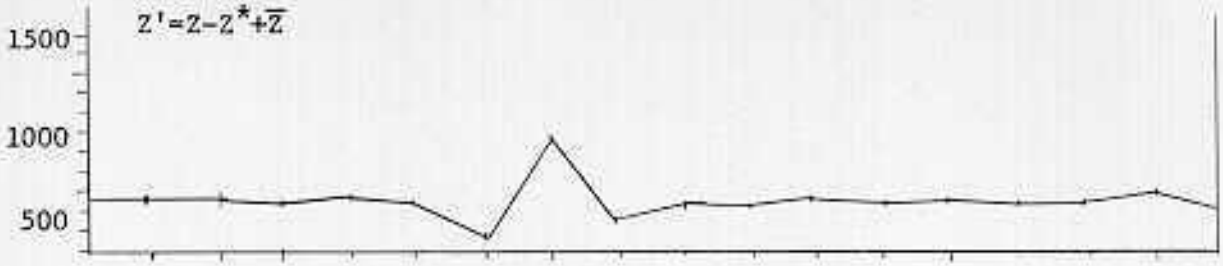
Twice Smoothed Profile



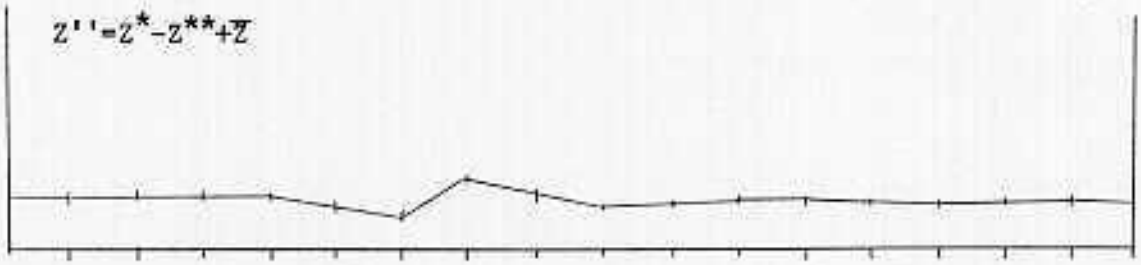
Profile Smoothed Twenty Times

Figure 1a

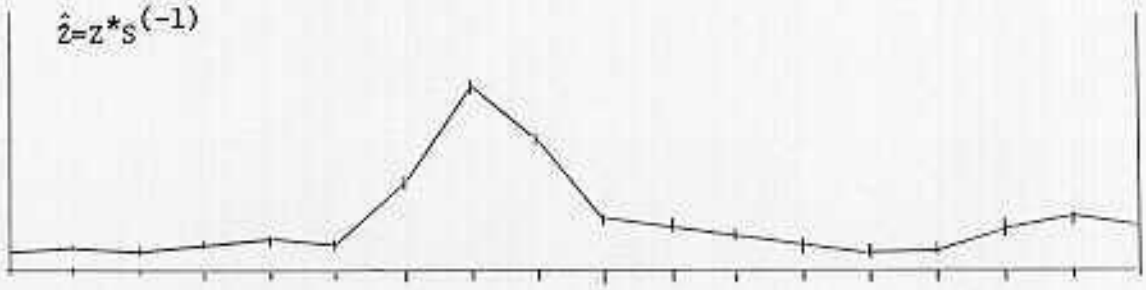
PROFILE GENERALIZATION



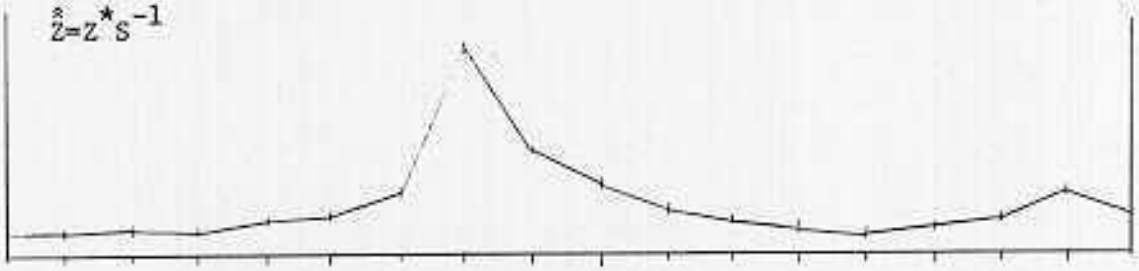
The Small Scale Features



The Medium Scale Features



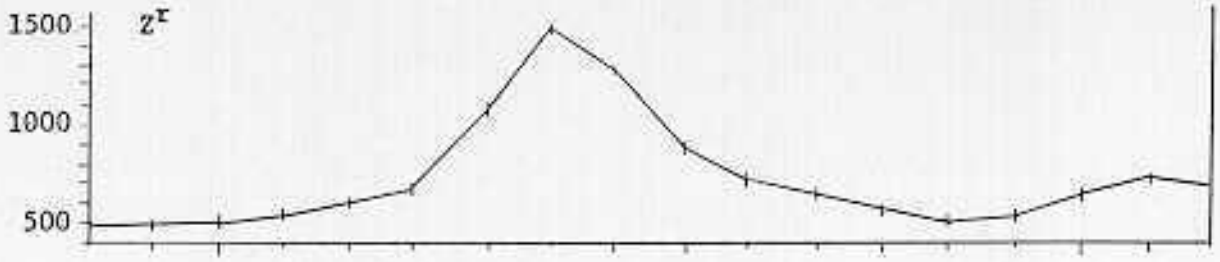
Restored Values obtained from Z^{*} by use of the 3 weight approximate inverse $S^{(-1)}$



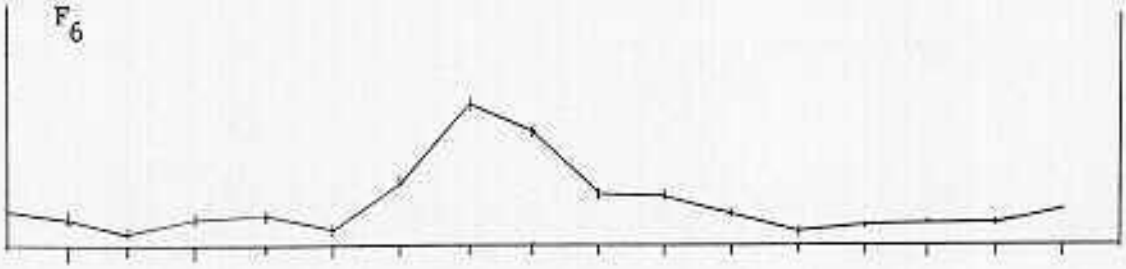
Restored values obtained from Z^{*} by use of the exact inverse S^{-1}

Figure 1b

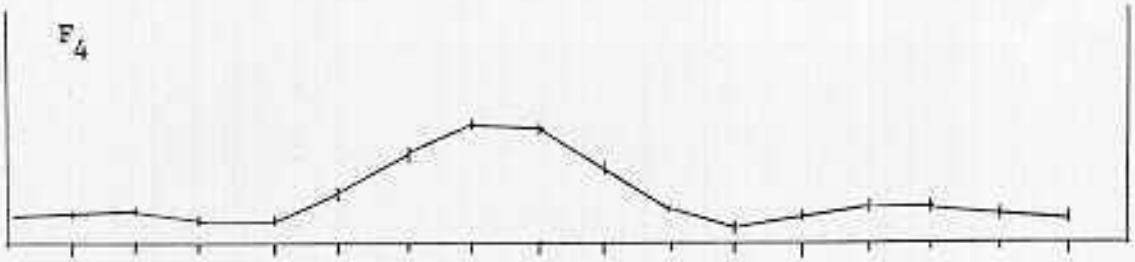
PROFILE GENERALIZATION



Smoothed, then Normalized



Fourier expansion of 12 terms



Fourier expansion of 8 terms

Fourier expansion of 4 terms

Figure 1c

The smoothing operation can be applied to generalize contour maps, not only profiles. Let the value of a surface at any point ij be defined by $Z_{ij} = f(x_{ij}, y_{ij})$. In most cartographical situations this is a single valued function. For convenience take $\Delta x = \Delta y =$ unity, and a kth two-dimensional weighted moving average can then be defined by

$$Z^*_{ij} = \frac{\sum_{p=-k}^{p=k} \sum_{q=-k}^{q=k} W_{pq} Z_{i+p,\, j+q}}{\sum_{p=-k}^{p=k} \sum_{q=-k}^{q=k} W_{pq}}$$

This is easily implemented on a digital computer. In matrix form, the process can be written as

$$Z^* = S \cdot Z \cdot S,$$

where Z is now a matrix of "elevational" data, and S is a conformable smoothing matrix as in the previous case. Repeated smoothings can be obtained by repeated matrix multiplications. The inverse operation ("ungeneralizing, or roughening") is given by

$$Z = S^{-1} \cdot Z^* \cdot S^{-1}.$$

The data matrix Z, of course, need not be square, but the smoothing matrices are then of different orders.

A somewhat more practical procedure than the matrix formulation, and perhaps easier to visualize, is to consider the contour generalization to be the resultant of two operations taken in sequence, one applied to all the east-west profiles, then the other applied to all of the north-south profiles. The amount of smoothing in these two directions is usually the same, but could also differ. Construction of a nine-point two dimensional weighting function from the one dimensional case then follows by cross multiplication from a table such as given here:

	1/4	1/2	1/4
1/4	1/16	1/8	1/16
1/2	1/8	1/4	1/16
1/4	1/16	1/8	1/16

Construction of an approximate inverse weighting function is similar, e.g.:

	-1/2	2	-1/2
-1/2	1/4	-1	1/4
2	-1	4	-1
-1/2	1/4	-1	1/4

The common cartographic rational for map generalization is retention of legibility under reductions of scale. It is suggested here, however, that it may also be appropriate to generalize maps, not only because of scale changes, but also simply for different map purposes. In other words, the map information is to be filtered in accordance with the use of the map. A brief discussion of spatial frequency filters is therefore appropriate.

It is known that a single valued profile can adequately be represented as the Fourier sum of a large number of sine and cosine curves of varying frequency, amplitude, and phase. Similar results hold for the two dimensional case. The moving average operates as a transformation to modify the amplitudes and phases of the individual trigonometric terms (which are usually referred to as frequencies), and this modification is not the same for all terms. The amount (which can be calculated) by which the amplitudes and phases of individual frequencies are modified depends on the weights employed in the moving average. The binomial weighting employed in the previous examples has certain advantages (e.g., large scale features are retained and small scale features are filtered out, and phases are unchanged) but is only one of many possible weighting schemes. The simple (or equally weighted)

moving average might, at first glance, appear more advantageous, but it has certain undesirable properties; for example, some peaks may become troughs. Since the frequency response of particular weighting schemes may be calculated, it might be expected that one could apply the procedure in reverse. To some extent filters (weights) can be devised to eliminate, or emphasize, features of any size (frequency), and approximate inverse filters can also be obtained. This suggests that it is possible to specify the level of generalization in an a priori fashion for different map scales. For example, at a map scale of 1:500,000 features having a wavelength of less than one half kilometer can probably not be shown. Holloway (1958) demonstrates explicitly the amount of generalization (that is, the weighting function) required (in his notation set $\sigma = 0.167$ kilometers). This example is elaborated in the appendix.

Also implicit in the construction of frequency filters is the notion that, if one knows the geographical scale to which some process is sensitive, one can mathematically generalize a map specifically for this purpose, and this map will differ from that for some other process. For example, topographic features of wavelengths greater than 25 kilometers and less than 0.10 meter are probably not important for the construction of airports. An appropriate filter would eliminate these large and small scale features but would leave medium sized features. Unfortunately we do not have much information concerning the wavelengths to which geographical phenomena are sensitive. It is also suggested that Perkal's (1958) ε - generalization procedure consists of a form of frequency cut-off (truncation of the Fourier series).

An example of contour map generalization by the procedures described is given in Table II and the accompaning illustrations.

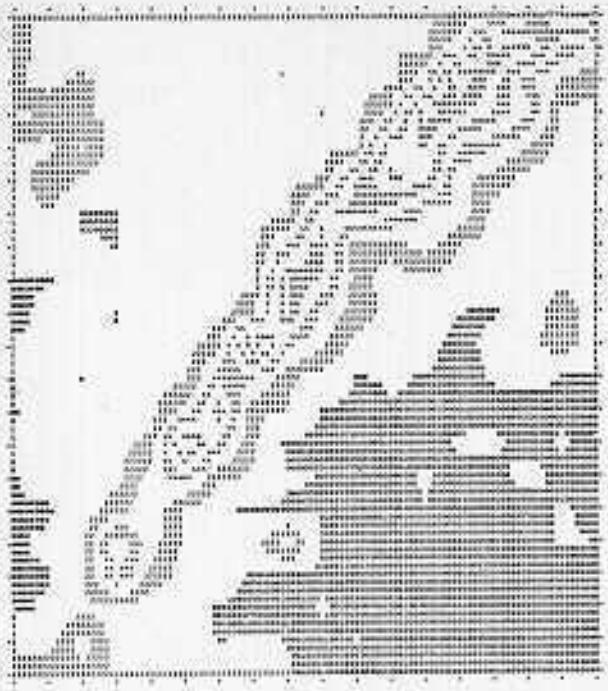
TABLE II

ELEVATIONS EMPLOYED IN CONTOUR GENERALIZATION[1]

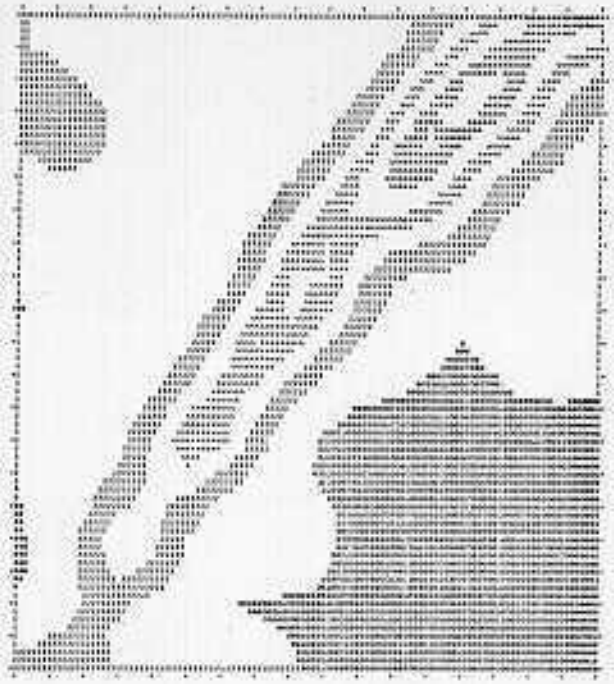
630	620	605	595	590	605	590	605	605	590	580	600	610	640	710	1050	1400	1340
640	605	580	560	610	595	598	590	600	600	575	610	620	860	1500	1380	980	800
640	600	650	570	570	600	580	605	570	580	590	585	850	1600	1300	910	720	660
800	690	640	600	570	600	585	605	570	630	580	800	1530	1400	1210	1030	595	598
580	590	750	570	570	560	550	560	570	600	595	1300	1370	995	900	750	580	590
540	730	590	575	560	550	560	570	595	630	1100	1380	1300	790	690	580	570	570
550	610	490	500	530	600	570	560	605	1050	1200	1050	850	850	590	570	580	610
540	520	510	500	530	570	560	595	900	1200	645	670	805	630	570	550	580	605
500	495	540	540	550	565	580	600	1250	1100	690	570	605	570	570	570	580	585
475	525	520	500	520	560	580	700	1250	850	700	580	550	505	490	550	700	550
520	505	510	505	550	570	700	1500	910	750	585	550	525	480	520	585	700	550
500	530	500	520	605	590	950	900	650	585	500	505	500	470	510	520	500	480
500	510	530	560	590	790	1200	650	620	495	475	500	500	475	475	480	490	500
505	510	550	595	650	1350	950	640	475	480	475	500	480	510	520	460	510	460
495	540	575	580	800	930	650	610	510	500	460	475	510	480	480	530	480	455
500	480	560	700	850	595	510	495	485	500	500	470	500	460	460	470	510	480
480	530	590	1100	800	600	520	575	800	470	460	450	500	480	460	470	510	500
485	485	605	1000	700	570	540	495	500	490	475	470	440	420	420	445	465	460
470	525	600	585	520	530	495	475	470	510	480	450	430	380	400	430	440	440
580	580	800	570	505	515	500	510	500	480	505	420	400	400	430	440	450	460
700	700	705	605	580	580	615	605	505	470	440	400	380	420	410	450	490	485

1) Elevations in feet, grid interval of 0.5 mile.

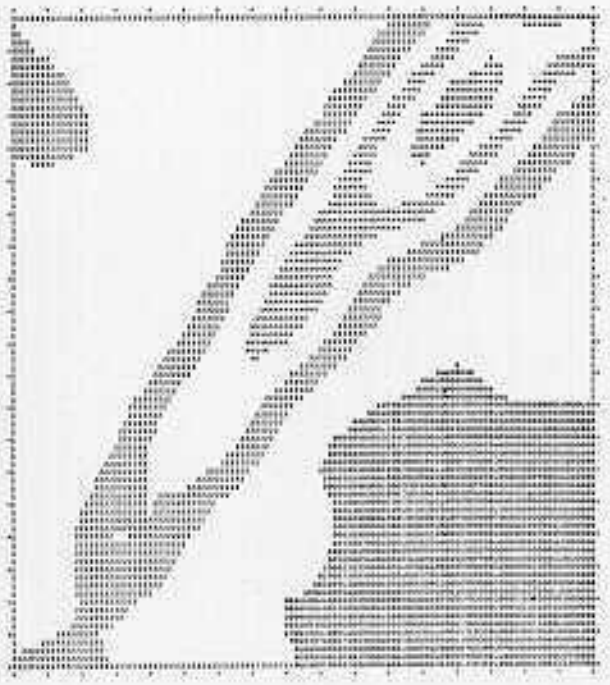
MAP GENERALIZATION



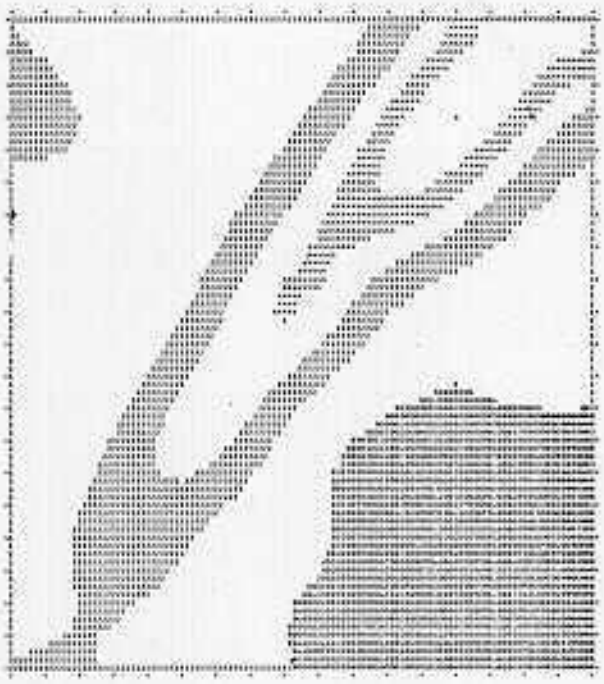
Original Map Z



Smoothed Map Z^{*}

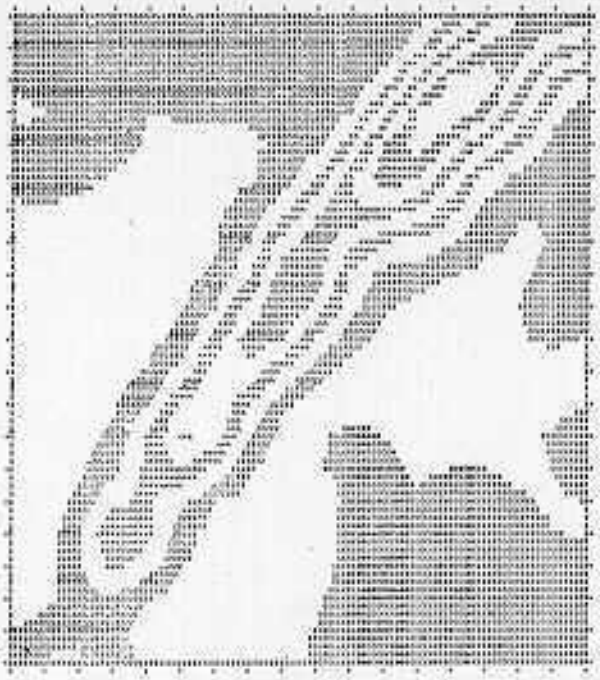


Twice Smoothed Map Z^{**}

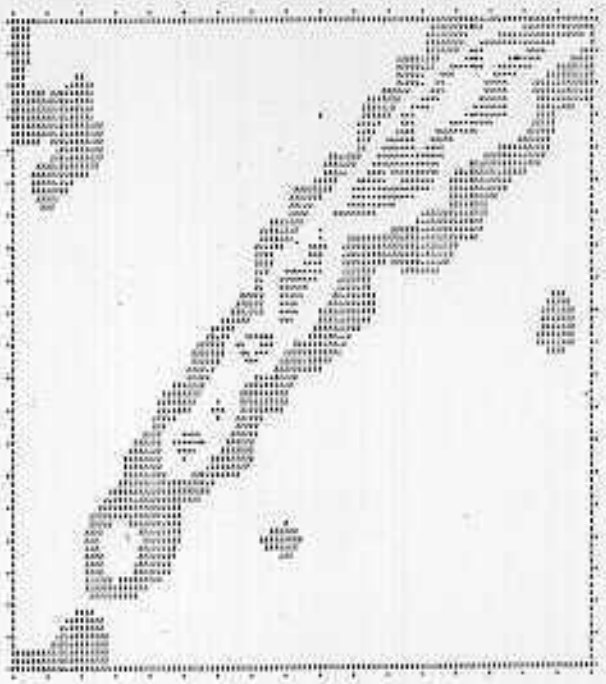


Thrice Smoothed Map Z^{***}

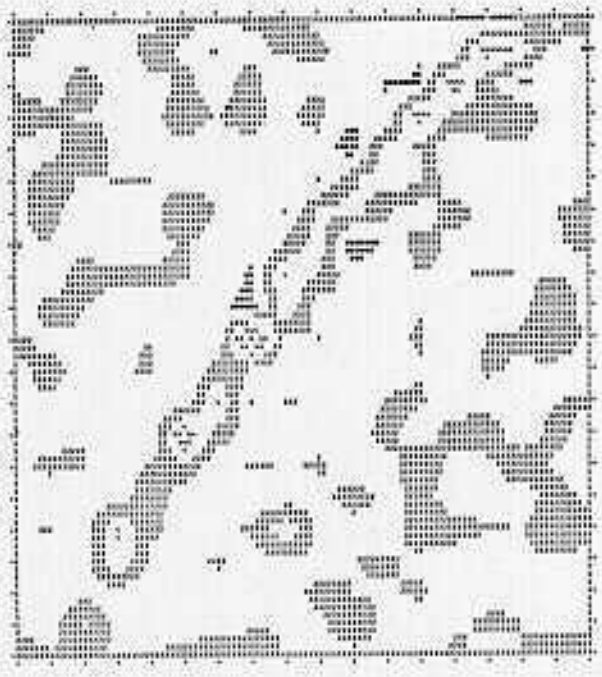
MAP GENERALIZATION



Smoothed - Then Normalized



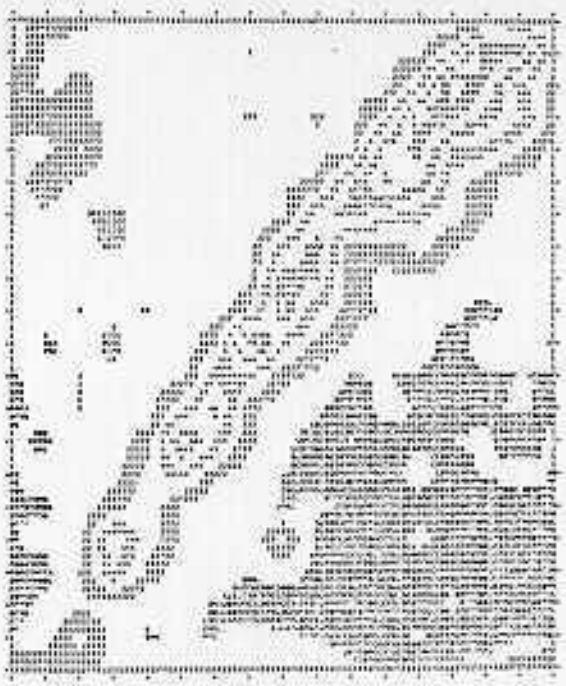
Original Map - Doubled Contour Interval



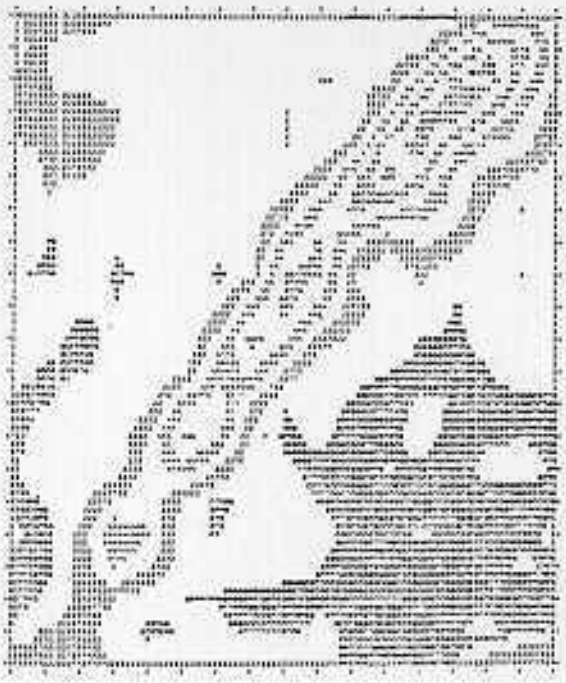
Small Scale Features
Removed During 1st Smoothing

MAP GENERALIZATION

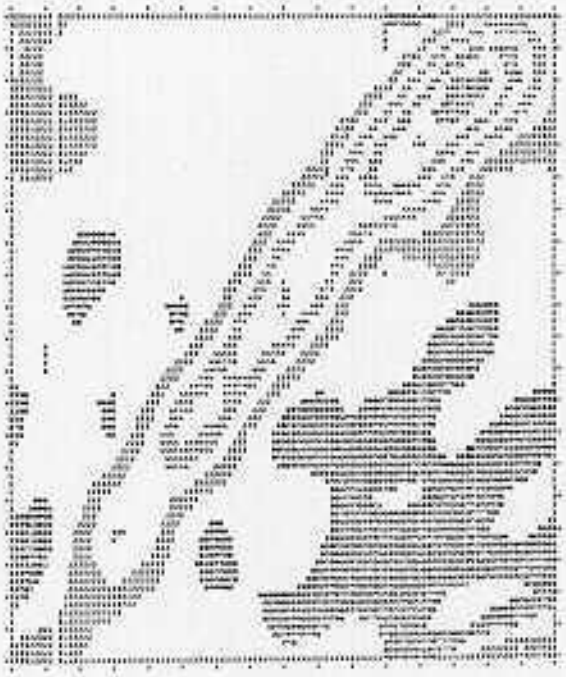
Using Trigonometric Polynomials



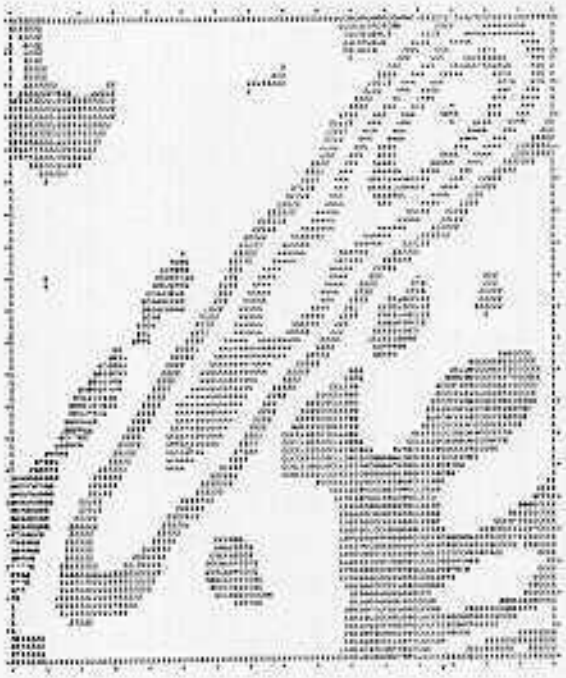
285 Terms



143 Terms

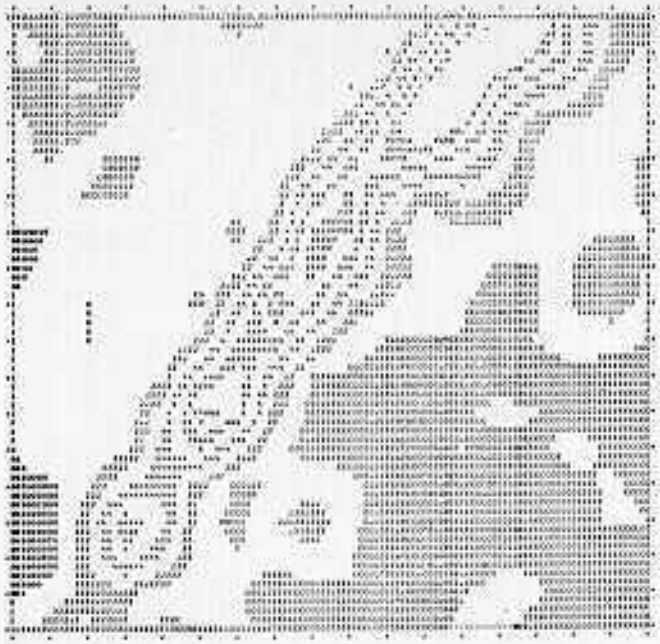


81 Terms

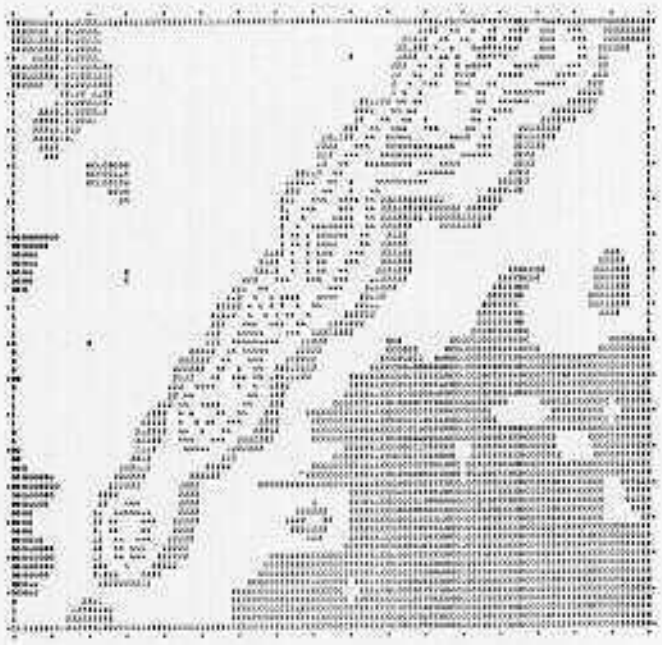


49 Terms

MAP GENERALIZATION



Map Restored Using Approximate Inverse



Map Restored Using Exact Inverse

It is also possible to consider any individual contour as a line, and attempt line-generalization, as of say, a coastline. As the map scale becomes smaller and smaller, less and less of the coastline detail can be shown. This situation is discussed at some length in the available literature. Alternately, as Perkal suggests, different generalizations of the coastline may be appropriate for different purposes. One computer technique for coastline generalization consists of thinning. Given a string of positional coordinates for sequential points along the coast, points j are eliminated if

$$D_{ij} < \varepsilon$$

where $j = i + k$, $k = 1,2,\ldots,n$,

and in the cartesian case,

$$\left[D_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2\right]^{1/2}$$

and ε is some minimal distance, for example, twice the minimal plotting size of the automatic drawing instrument. This approach has already been employed with considerable success and is more fully discussed elsewhere (Hershey, 1963; Tobler, 1964b). Thinned data cannot be restored, however.

If the coastline were a single valued function it could be generalized as a topographic profile. The coastline, however, is not single valued. It might be decomposed into single valued intervals but a parametric representation in terms of arc length s is more convenient. Let the coastline be represented by the complex function

$$Z = f(x + iy), \quad i^2 = -1,$$

where $s = s(s)$ and $y = y(s)$. For equal intervals Δs of s the jth point of the coastline is

$$Z_j = x(s_j) + iy(s_j).$$

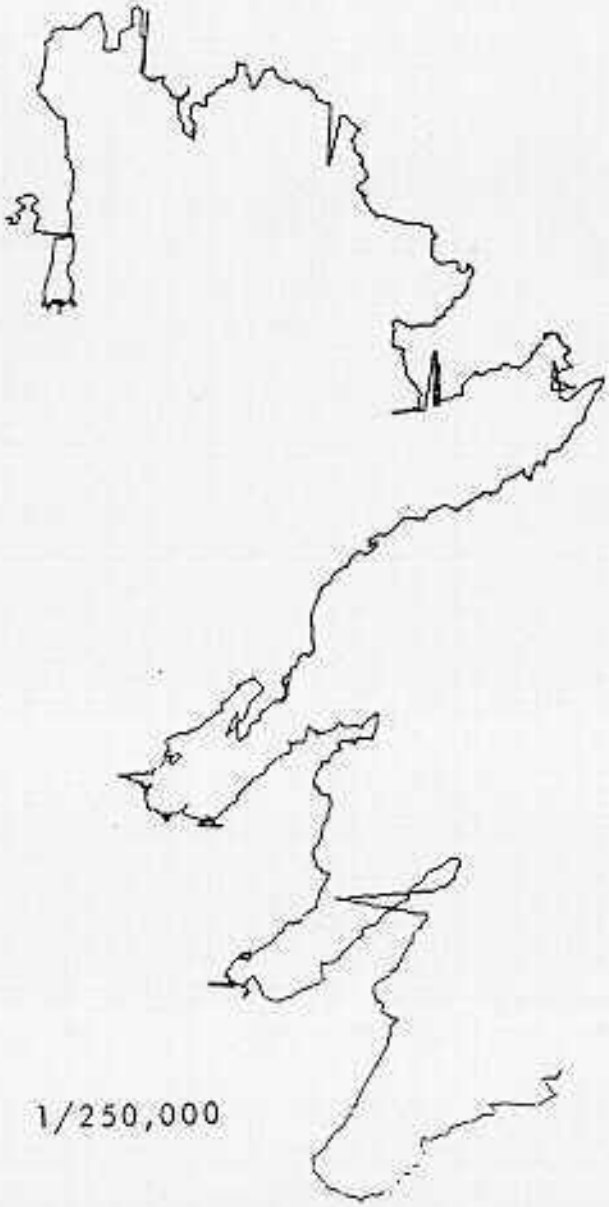
A smoothing can then be accomplished by taking

$$Z^* = Z \cdot S$$

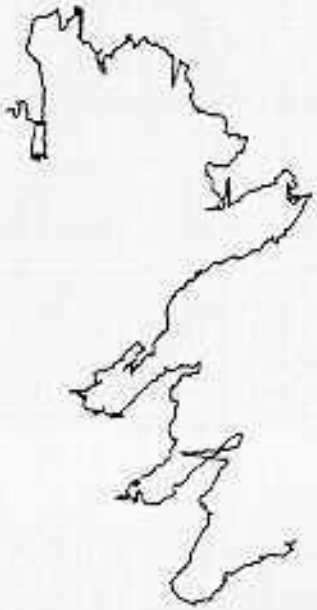
where S is a smoothing matrix as before. Since the coastline closes, k values

Generalization of Coastline by Thinning. After Tobler (1964b).

Machine drawn from data compiled at 1/50,000



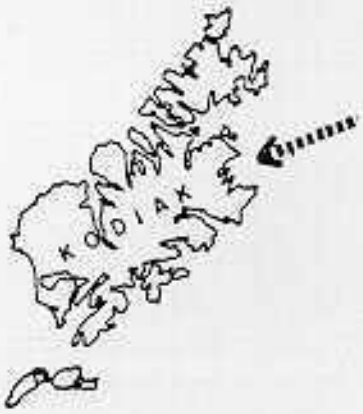
1/250,000



1/500,000

1/1,000,000

1/5,000,000



Tracing from 1/5,000,000
Jet Navigation Chart
showing area involved

of Z_j could be added to the beginning and end of the complex vector Z, and S becomes an n + 2k by n + 2k matrix. Different smoothings could in fact be applied to the real and imaginary parts of Z, though it is difficult to imagine why this might be done.

Examination of a number of published contour maps at different scales but of the same area suggests that the practical cartographer generalizes by thinning after enlarging the contour interval in approximately the inverse ratio of the map scales. This process is easily formalized, e.g., on a digital computer, and appears suitable for the navigational usage of maps. The filtering procedure, on the other hand, seems to have advantages for more advanced theoretical investigations. For example, property values for individual parcels of real estate within a city (as might be obtained from the assessor's office) typically show considerable variation from one parcel to the next. The space smoothing technique can be employed to investigate the existence of geographical patterns of property values. The generalization facilitates the recognition of patterns because it appears to be true, as Holloway (1958, p.386) suggests, that we do "high-pass filtering in our 'mind's eye'."

Several additional smoothing techniques can be found in works on numerical methods, and these can generally be extended to the two-dimensional case fairly easily. Virtually all of these techniques make the assumption that "there is a strong antecedent probability that if the observations had been more accurate the curve would have been smooth" (Whittaker and Robinson, 1944), where smooth means that higher order differences vanish. In the topographic situation, at least for purposes of generalization, it can be assumed that the errors are negligible and the concept of a filter seems more appropriate. In terms of geographical theory one might argue that the pattern-generating processes are subject to random disturbances, which we wish to

eliminate by the generalization procedure.

APPENDIX I

The Inverse of the Smoothing Matrix

Definitions

(1) A matrix $A = [a_{ij}]$ is diagonally dominant if

$$\left|a_{ii}\right| \geq \sum_{\substack{J=1 \\ J\neq i}}^{n} \left|a_{ij}\right| \quad , 1 \leq i \leq n$$

(2) A matrix A is reducible, $n \geq 2$, if there exists an n by n permutation matrix P such that

$$P \cdot A \cdot P^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} is an r by r submatrix, A_{22} is an (n-r) by (n-r) submatrix, $1 \leq r \leq n$. If such a P does not exist then A is irreducible.

(3) A directed graph is strongly connected if for any ordered pair of nodes N_j and N_k there exists a directed path $N_j\ N_{m1},\ N_{m1}\ N_{m2}, \ldots, N_{mr}\ N_k$ connecting N_j and N_k.

(4) A matrix A is irreducibly diagonally dominant if A is irreducible and diagonally dominant with the strict inequality valid for at least one i.

Theorems

(I) A matrix is irreducible if, and only if, its directed graph is strongly conected.

(II) A matrix which is irreducibly diagonally dominant is non-singular.

Remarks

The 18 by 18 smoothing matrix S used in the generalization is irreducibly diagonally dominant. The inverse matrix S^{-1} can in fact be exibited explicitly, but is omitted here (interested parties can obtain copies from the

author). An "inverse" $S^{(-1)}$ constructed from the approximate inverse weighting function (-.5, +2, -.5) has the property that

$$S^{(-1)} \cdot S \neq I$$

as was expected.

APPENDIX II

Deterimination of a Binomial Weighting Function

Procedure

(1) Given (in appropriate units, e.g. kilometers):

(a) The observation interval Δx

(b) The "cut-off" wavelength

(2) Set the cut-off wavelength = 3σ

(3) The interval between observations then corresponds to

$$\frac{\Delta x}{\sigma}$$

on the abscissa of the normal curve.

(4) Record (e.g., CRC Standard Math Tables, 12th ed., pp. 245-249) the ordinates of the normal curve from zero to at least $\pm 3\sigma$ in increments of $\frac{\Delta x}{\sigma}$.

(5) Adjust the values obtained to yield a weighting function which sums to unity by dividing by $\frac{\sigma}{\Delta x}$.

Example

(1) a) Suppose data are available at every 0.1 km. (e.g. $\Delta x = 0.1$).

b) On a 1/500,000 scale map features having a wavelength of less than one half km. cannot be shown. The cut-off frequency is 0.5 km.

(2) $3\sigma = 0.5 \rightarrow \sigma = 0.167$ km.

(3) $\frac{\Delta x}{\sigma} = \frac{0.1}{0.167} \rightarrow \sim \frac{0.1}{0.2} = 0.5$

(4) From the table of the normal curve:

Abscissa	Ordinate	Normalized	Weights
0.0	0.3989	0.1994	Wo
0.5	0.3521	0.1760	W±1
1.0	0.2420	0.1210	W±2

Continued on next page

Abscissa	Ordinate	Normalized	Weights
1.5	0.1295	0.0648	$W_{\pm 3}$
2.0	0.0540	0.0270	$W_{\pm 4}$
2.5	0.0175	0.0088	$W_{\pm 5}$
3.0	0.0044	0.0022	$W_{\pm 6}$

The 13 weights obtained can be used to smooth the data. Note that by taking $\frac{\Delta x}{\sigma} = 0.5$, the map is over generalized, since $3\sigma = 0.6$ km not 0.5 km as specified in the initial conditions. This is hardly of consequence since perfect cut-off filters cannot be realized in practice (cf. Holloway).

References

B. J. Garnier, 1963, Practical Work in Geography, London, E. Arnold Co.

A. V. Hershey, 1963, The plotting of Maps on a CRT Printer, NWL Report No. 1844

J. Harbaugh and F. Preston, 1965, "Fourier Analysis in Geology", Computers and Computer Applications in Mining and Exploration, J. Dotson and W. Peters, eds., Tuscon, University of Arizona, Vol. I, pp. R1 - R46.

J. L. Holloway, 1958, "Smoothing and Filtering of Time Series and Space Fields", Advances in Geophysics, 4, New York, Academic Press, pp. 351-389.

E. Imhof, 1957, "Generalizierung der Hohenkurven", Kartographische Studien, Erganzungsheft Nr. 264, Pet. Geogr. Mitt., pp. 89-99.

W. Krumbein, and Graybill, 1965, An Introduction to Statistical Models in Geology, New York, McGraw-Hill.

D. H. Maling, 1963, "Some Quantitative Ideas about Cartographic Generalization" ICA Bulletin, No. 4.

W. D. Montgomery, and P. W. Brome, 1962, "Spatial Filtering" Journal of the Optical Society of America, 52, 11, pp. 1259-1276.

A. J. Pannehoek, 1962, "Generalization of Coastlines and Contours," International Yearbook of Cartography, II, pp. 55-74.

J. Perkal, 1958, "Proba Obiektywnej Generalizacji", Geodezja i Kartografia, VII, 2, pp. 131-142.

W. Pillewizer, and F. Topfer, 1964, "Das Auswahlgesetz: Ein Mittel zur Kartographischen Generalisierung", Kartographische Nachrichten, 14, 4, pp. 117-121.

W. Tobler, 1964a, "A Polynomial Representation of Michigan Population", Papers, Michigan Academy of Science, Arts, and Letters, XLIX, pp. 445-452.

W. Tobler, 1964b, An Experiment in the Computer Generalization of Maps, Technical Report #1, Office of Naval Research task No. 389-137.

A. Wiin-Nielsen, 1965, "Smoothing and Filtering of Two-Dimensional Fields", in A. Dingle and C. Young, Computer Applications in the Atmospheric Sciences, Ann Arbor, University of Michigan (NSF G-25204), pp. 214-221.

E. Whittaker, and G. Robinson, 1944, The Calculus of Observations, 4th ed. London, Blackie and Son, pp. 285-316.