On the Length of Empirical Curves

by

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by

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In this paper, I shall point out difficulties encountered in the measurement of the length of empirical curves, define a new method for approximating lengths of order ϵ, and describe a longimeter used for the measurement of such lengths.

1. We will consider arcs of curved lines on a plane (henceforth simply arcs) as pictures of homeomorphical sections. By the length of an arc we understand the limit of the sum of segments of length inscribed in this arc, by letting the segments go to zero. Such a limit (finite or infinite) always exists. Arcs of finite length we will call rectified. By the distance between arcs A and B we take a small number r, such than an arbitrary point on either arc A or B is not further than r from some (nearest) point on the other arc (See Fig. 1). The distance between two arcs is zero if and only if these arcs coincide.

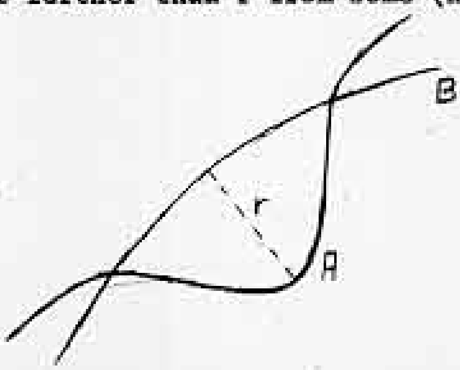

Fig. 1

If we know the manner in which the arc was constructed (for example if we know the equation of the arc) we can attempt to state whether the arc is rectifiable or not. If it is rectifiable we can ask for the length of the arc. If however, we do not know the manner in which the arc was constructed then we cannot assume that the arc is rectifiable.

This is the situation with respect to empirical curves. The circumference of leaves, the length of the seacoast, or of the edges of sharp razors are examples of arcs which may not be rectifiable. H. Steinhaus points out (Ref 6), that sharp razors observed with the naked eye have the shape of a straight line, but the same razor observed through a magnifying glass has an entirely different shape because of small notches; the same razor observed under a microscope again has an entirely different shape, depending on the structure of the steel. Figure 2 shows the edge of a leaf (Peucedanum sp.) observed (a) with the naked eye, (b) magnified ten times, (c) 100 times, and (d) 1000 times (these illustrations were kindly provided at my request by Z. Hejnowicz from the Academy of Anatomy and Cytological Plants at the University of Wroclaw). The true appearance of the edges of leaves and razors is not known, and it is not known whether the edges are arcs which are rectifiable.

Can these small deviations, perceptible only through a microscope, actually give rise to a real difference in the length of the arcs? The answer is yes. The length of an arc is not a continuous functional. Arbitrarily near a given arc A we can draw another arc B (saw-toothed, as in Fig. 3) whose length is considerably greater than that of arc A. Thus a nearly rectilinear sharp razor observed with the naked eye appears as a great deal longer and more complicated curve when observed under the microscope. And if we observe the razor under progressively stronger microscopes we will see a progressively more complicated curve, and consequently one of greater length. In fact we cannot even talk of the true and final shape of the curve. This is why we cannot talk of the true length of a razor. The area of a flat surface bounded by a curve behaves differently. The area of a plane is a continuous functional. If two curves lie near each other, than there is little difference in the areas bounded by the curves.

There exists a naive method of measuring the length of empirical curves. Lay a thread along the length of the arc, and then measure the length of the straightened thread. In practice however, it is impossible to complete the first of these instructions. No matter how we lay the thread, when we look through a microscope we will find that there are places where the thread deviated from the arc. A similar result holds for all other methods of measuring the length of empirical curves, whether it be leading a tracing wheel

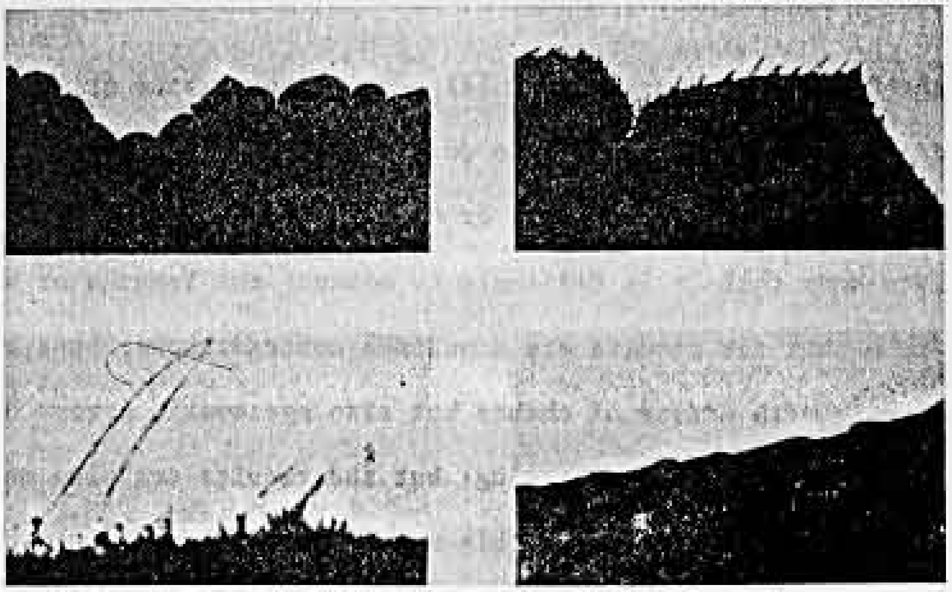

Figure 2
a b
c d

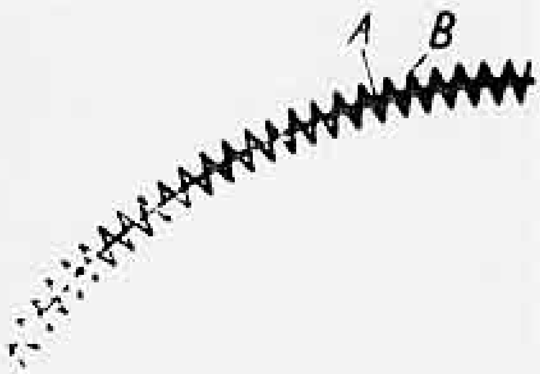

Figure 3

along the length of the arc, or incremental measurements using dividers. They must fail, for as I mentioned earlier we do not know how to ascertain if the measured empirical curve is rectifiable or not.

Naturalists know that it is difficult to measure the lengths of empirical curves; they know that the results are sometimes contradictory; that they are encumbered not only with errors of chance but also systematic errors depending on the method and on the person measuring; but the results are also encumbered with important general errors. This was first realized by geographers; the Viennese geographer A. Penck in 1894 presented the length of the seacoast between two points on the peninsula of Istrii (Ref 1) measured on various differently generalized maps. These are the results:

Map No.	scale	length in km
1	1:11:1600000000	105
2	1: 3,700,000	132
3	1: 1,500,000	157.6
4	1: 750,000	199.5
5	1: 75,000	233.8

Note that the scale and the degree of generalization both effect the divergence of the results. But the degree of generalization should not be used. If map 4 is photographically enlarged ten times, we would have a map of the same scale as map 5, but the generalization thus obtained would be the same as generalized map 4, for this reason it would be expected that on this map the length of the measured section of the coast would be 199.5 and not 233.8 km. This would be so because the photographically enlarged map 4 would not bring out any new featuresthat are apparent on the more suitable map 5.

In the presence of such difficulties and divergencies the question arose: Is there, in general, any sense in talking about the length of empirical curves? It appears that a new notion of length should be developed, which should also approximate the classical, but should be accepted in lieu of the classical method. Such a new length should serve for all empirical curves and should be simple for measurement.

H. Steinhaus has contributed in this direction (Ref 5 and other works cited there) and gives methods for measuring and comparing lengths of order n. These notions are based on results due to Crofton. Crofton's theorem relates the length of a curve lying in the original plane with a set contained in Crofton's plane. Take a curve lying on the first plane and a series of straight lines cutting this curve. Each one of these lines becomes a point of Crofton's plane. Now a measure of this set of points is equal to the length of the curve. However, points describing straight lines intersecting the measured curve twice must be counted twice, and in general, points describing straight lines intersecting our curve n-times must be counted n-times. Crofton's theorem should be expressed thus: the series K_i is a set of points of Crofton's plane, which describe the straight lines of the first plane intersecting the measured curve at least i-times. The length of the arc is the sum of the infinite series

$$(1) \qquad \sum_{i=1}^{\infty} K_i$$

2. Steinhaus' known longimeter is based on this equation. This is a sheet of transparent paper with parallel lines. This sheet is placed on the curve to be measured and the number of points that intersect the curve with the straight longimeter are counted. The number of these points of intersection are approximately proportional to the length of the curve. Many-fold repetition of this measurement increases the accuracy in that the error of chance becomes smaller.

But this longimeter, like all other equipment used for measuring classical lengths, is not suitable for measuring the length of empirical curves since it is not known whether the pertinent series is convergent.

The method presented permitted H. Steinhaus to introduce a new concept of length, length of order n. H. Steinhaus calls the length of order n the sum of the first n expressions of the series (1). A measurement of this length of order n occurs with the aid of the Steinhaus longimeter just as in the classical measurement of length, with the one difference that for every straight longimeter no more than n points of the intersection of the line with the measured curve are counted. If the straight lines intersect the curve less than n-times, or n times, then all of the points of intersection of the line and curve are counted. If however any line intersects the curve more than n times, it is accepted that this line intersects the measured curve only n times. In agreement with the accepted program, the length of n^{th} order is approximately the classical length in the sense that it tends towards the classical length when n increases to infinity. The length of order n serves for all empirical curves and is easy to measure. It would appear therefore that the problem is solved and that the length of order n instead of classical length can be introduced conventionally. The concept of the length of order n solves many questions pertaining to the measurements of empirical curves. This approximated length is useful in general for the comparison of the length of curved lines on geographical maps or on several maps of various generalizations. It can be admitted that in applying the length of the indicated order, for example the 5th, Penck would not have obtained such variations in the measurement of the length of the seacoast of Istrii. There are however natural problems for which the aforementioned length of order n is insufficient.

The arguments are as follows:

(1st) A difficulty with the method is that there is a slightly weak connection between the length of order n and the classical length. Naturalists want to measure the true length, the so called classical length of the empirical curve, and want to believe that such a length exists. It seems that it is easier for the naturalist to digest generalization of a curve rather than to introduce a new abstract measurement, such as the length of order n. It is known that the length of order 1 gives the length of rectilinear classical curves of first degree and the lengths of order 2 give the lengths of convex classical curves. It is not known however for what class of curves of length of order n is the classical length when n>2. It is further known that by increasing n the length of order n increases and approaches the classical length. It is not known what is the connection between n and a suitable length of order n, the difference between classical length and the length of order n. If the naturalists would obtain an approximated length of measurement he would want to visualize a curve approximating the measured (generalized) one whose classical length would be equal to the length of order n of the measured arc.

(2^{nd}) Every method of approximating measured length must embrace a convention. In H. Steinhaus' method the number n is conventional in that it is the order of length. This integer is not dependent on customary units of length (centimers or inches). But a difficulty with this convention is the lack of correspondence between the arbitrary n and the suitable measurements of classes of curves whose length, classical and of order n, are equal. In certain cases it would appear to be more convenient to use a convention connected with suitably approximated measurements and which would define classes of curves for which the approximated length would be equivalent to the classical.

(3^{rd}) The length of order n, as in classical length, is a discontinuous functional which could lead to the aforementioned paradoxes of length. In measuring the length of order 10, for example, for intricate empirical curves it may occur that points of intersection of the arc with the longimeter line would appear under the microscope not as one point but as greater number of points of intersection. The length of order 10 is therefore dependent on whether the curve is observed with the naked eye or with a microscope or on the generalization of the curve. It could occur, that every point of intersection of the empirical curve with the lines of the longimeter is actually such a collection of points (in practice we could not prove this otherwise). In the meanwhile misgivings would arise that the length of order n would be simply n times length of order 1.

3. I will now describe a concept of approximate length of order ϵ, where ϵ is a real number. The length of order ϵ will be a continuous function of the curve and will depend on the number ϵ in a continuous manner. The length of order ϵ does not have the fault I previously indicated for the length of order n. A more comprehensive description of this concept can be found in my paper (Ref 3).

In the paper (Ref 2) I defined the collection A (X); that is, the ϵ-halo of the arc X, as the collection of all points on the plane not more than ϵ distant from the arc X.

$$(2) \qquad A_\epsilon(X) = \underset{x}{E}\left[(x,X)\leq\epsilon\right]$$

where E is Lebesgue's symbol, and (x,X) indicates the distance of the point x from the arc X; that is, the distance of point x from the nearest point on arc X.

Consider the area $a_\epsilon(X)$ of $A_\epsilon(X)$. Figure 4 shows the ϵ-halo of the arc X. As is evident it consists of a belt whose width is 2ϵ enclosing the arc X plus two semicircles of radius ϵ.

The length of order ϵ of the arc X I indicate with the symbol $L_\epsilon(X)$ and define by the following formula:

$$(3) \qquad L_\epsilon(X) = \frac{a_\epsilon(X) - \pi\epsilon^2}{2\epsilon}$$

For the arc shown in Figure 4 the length of order ϵ is therefore the area enclosing the arc X (the area of the ϵ-halo) minus the area of the two semicircles, divided by the width of this belt. If the arc X

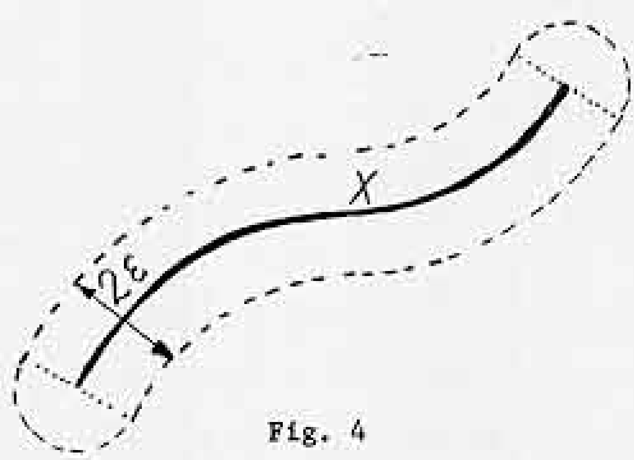

Fig. 4

were a straight line, this belt would be a rectangle of width 2ε and of a length equal to the length of arc X. The area of the belt will not change if the arc X is subject to small variations. For straight arcs the definition (3) is intuitive and the length of order ε is equal to the normal length.

It is evident (see Ref 3) that the length of order ε is a continuous function of the arc; that is, if arc X lies close to arc Y, then the length of order ε of these arcs differs only slightly. This length is a diminishing and continuous function with respect to a changing ε, for example, if ε diminishes the length of order ε increases and the reverse. By this, if the change in ε is insignificant, the length of order ε either will not be subject to change, or the change will likewise be insignificant.

In an earlier investigation (Ref 4) I wrote on ε-convex sets. An arc is ε-convex, if a circle of diameter ε could fit on both sides of this arc. In other words, an arc is ε-convex, if every point on it has a radius of curvature of not less than $\frac{1}{2}\varepsilon$. For curves of 2ε convexity (with the ends separated by at least 2ε) the length of order ε is equal to the classical length. For other curves the length of order ε can be equal to or smaller than classical.

Figure 5 represents an ε-convex set (ε = 5mm) and an arc X which is not 2ε-convex. The thin continuous line traces the ε-halo of the arc X. As

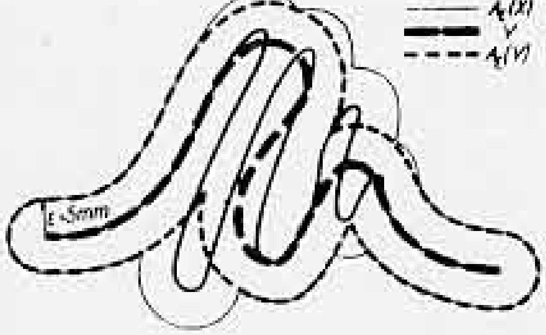

Figure 5

formula (3) shows, every arc, whose ε-halo is equal to $A_\varepsilon(X)$, has the same length of order ε as arc X. In general, if it would be possible to draw the arc Y as 2ε-convex and with ε-halo equal to $A_\varepsilon(X)$, the classical length of arc Y would be equal to its length of order ε and equal to the length of order ε of the arc X. Furthermore, we would be able to show that arc Y approximates the given arc X (lying in $A_\varepsilon(X)$), whose classical length would then be the length of order ε of arc X. Unfortunately, such an arc Y in general does not exist. We can only inscribe the 2ε-convex arc Y in such a manner that one has $A_\varepsilon(Y) \subset A_\varepsilon(X)$. Such an arc Y with its ε-halo is drawn in Figure 5 as a broken line. In the meanwhile it is known that $L_\varepsilon(X) \geqslant L_\varepsilon(Y) = L(Y)$, where the last symbol represents the classical length. We therefore know how to draw arc Y approximately the same as arc X, with a classical length less than $L_\varepsilon(X)$ with a difference depending on ε. It is easy to estimate this difference; that portion of the collection $A_\varepsilon(X)$ that is not covered by the ε-halo of arc Y is divided by 2ε, or $\left[a_\varepsilon(X) - a_\varepsilon(Y)\right] / 2\varepsilon$.

The length of order ε includes the conventional parameter ε (of known length) and indicates a difference between the classical length of rectifiable arcs and the length of order ε. I wrote above that arcs with small curvature (radius of curvature not less than ε) have as the length of order ε the classical length. It can be seen that the length of order ε measures accurately an arc of very small curvature and other arcs in an approximate matter; consequently the error of approximation is larger the greater the curvature of the measured arc. For unrectifiable arcs this error is infinitely large, since the classical length of such arcs is infinitely large, but the length of order ε is finite. From this it is evident that the actual error, or the difference between classical length and length of order ε should not be a standard of accuracy of measurement, for the error depends not so much on ε as on the arc. For unrectifiable arcs the error is infinite (independent of ε) and therefore for empirical arcs is indeterminate. It can be agreed that the standard of precision will be the ratio ∕ = ‒/L (X) (H. Steinhaus' suggestion). When ‒ decreases, the length of order ε (the denominator) will increase, and the ratio will also decrease. It can be conventionally required that in measuring the length of the arc X, a length of such order be used so that the ratio is not greater than 10% or 5%. The number ‒ can be chosen in another, more natural manner, by considering which small curves of the arc can be discarded (generalized) without harm to the problem under consideration. If we decide to disregard curves of radius less than ‒, then the length of order ‒ will really be the length of such curves. We will return to this problem again after discussing generalization.

The length of order ‒ has properties which should be required of empirical length. It is a continuous function, and completely removes the paradox of length on the consideration that a very long arc may still be found near a short arc. For 2 -convex arcs the length of order is equal to the classical. For other arcs it is smaller than the classical length but increases in a continuous manner when we decrease . If an arc is rectifiable, then by decreasing to zero the length of order can in a sense be treated as an approximation of classical length, and, for a given measured arc, another arc can be constructed, whose classical length is similar (the error can be estimated) to the length order of the measured arc. The parameter of length , can be fitted to the arc, or to the problem, in a natural manner. This is related to the generalization of the arc. In the following portion of the paper I will describe a simple tool for measurement of length of order .

4. Figure 4 presents an arc X and its -halo. In formula (3), in order to designate the order length of this arc, the area a (X) of the -halo of the arc X must be measured. This can be accomplished with a so-called point planimeter (see Ref 7); this is a plane with a network of points arranged

regularly, as for example on a square lattice (see Figure 6) with points with coordinates (am,an) where m and n are whole numbers and a is an arbitrarily assumed real number (constant for the entire problem); or a triangular lattice with sides b (see Figure 7). Every such regular arrangement of points corresponds to a plane figure whose centers are points of the figure. The figures are sketched in Figure 6 and 7. In the case of rectangular points these are squares with areas a^2, and in the triangular coordinates they are hexagonal with area $\frac{1}{2}\sqrt{3b^2}$. The point planimeter is a network of points regularly distributed on a sheet of transparent paper. This sheet is placed on the set to be measured (Fig. 8) and the number of points falling within the set are counted. The area of the set is equal to the expected number of these points increased by the area of the pertinent figure. The area can be estimated with the aid of arithmetic means from a number of random applications of the planimeter to the figure. From the Jarniaka-Steinhaus' theorem the error with respect to the measurement carried out with the point planimeter is directly proportional to the length of the arc bordering the measured feature and inversely proportional to the area of this feature (for which the unity is a). If we decrease a q-times, then the length of the arc increases q-fold and the area of the feature q^2-times. In view of this the error with respect to the measurement decreases q-times. The denser is the network of rectangular points (for triangular network likewise), the more efficient is the planimeter, therefore the more accurate is the measurement carried out with the planimeter.

The measurement of area is carried out in the following manner: the point planimeter (rectangular with side a, triangular with side b) is placed k-times on the feature and the number of points falling in the area are counted. Denote the first application as n_1, the second n_2, ... the k^{th} as n_k. The area of the set is

$$\frac{a^2}{k}\sum_{i=1}^{k} n_i \qquad \text{(rectangular network)}$$

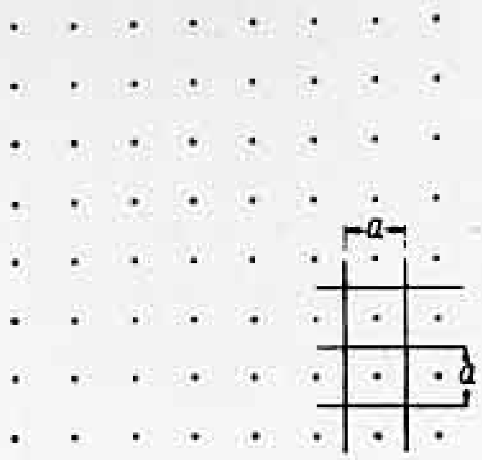

Figure 6

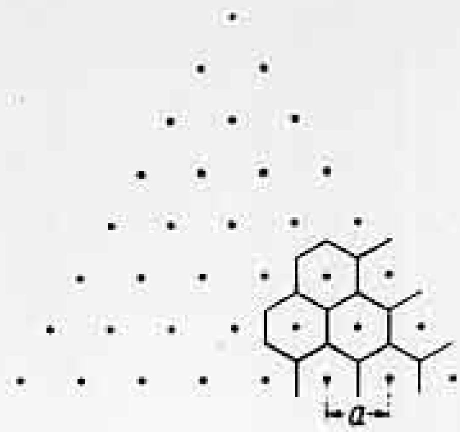

Figure 7

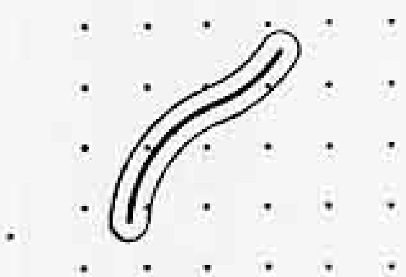

Figure 8

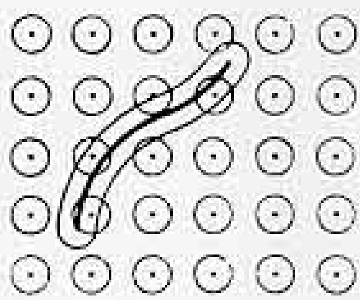

Figure 9

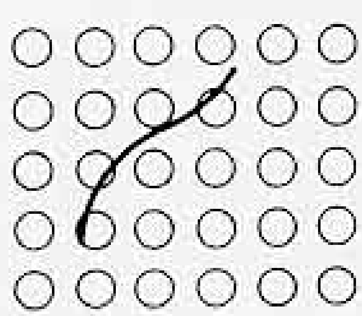

Figure 10

$$\text{or} \quad \frac{b^2\sqrt{3}}{2k}\sum_{i=1}^{k} n_i \qquad \text{(triangular network)}$$

From formula (2) we conclude (Fig. 9) that the point P falls in $A_\varepsilon(X)$ if and only if a circle of radius ε centered at point P encounters the arc X. This permits us to measure the area $a_\varepsilon(X)$ from a drawing on which only the arc X is traced (Fig. 10). It will be satisfactory to replace points with circles of radius ε on the planimeter. The drawing at the end of the paper gives such an arrangement based on a rectangular point planimeter with sides of the lattice $a = 10$ mm, that is, for $\varepsilon = 5$ mm. The second drawing gives a planimeter based on a triangular network with sides $b = 8$ mm, or for $\varepsilon = 4$ mm. With these drawings we use a light table and can count the number of circles encountered by the arc X. This incidence number increased by a^2 for the rectangular arrangement of circles and by $\frac{1}{2}\sqrt{3b^2}$ for the triangular, will be equal to $a_\varepsilon(X)$, the area of ε-halo of the arc X. For better estimation of the number of circles cutting the arc X, the measurement should be repeated k times. Summing the number of circles encountering the arc X for all k measurements and using the earlier mentioned formula we obtain from equation (3) the following:

$$(4) \qquad L_\varepsilon(X) = -\frac{\pi}{2}\varepsilon - \frac{a^2}{2k\varepsilon}\sum_{i=1}^{k} n_i \qquad \text{(rectangular lattice)}$$

$$\text{and} \qquad L_\varepsilon(X) = -\frac{\pi}{2}\varepsilon + \frac{b^2\sqrt{3}}{4k\varepsilon}\sum_{i=1}^{k} n_i \qquad \text{(triangular lattice)}$$

We add to equation (4) constants a, b, and k for the various ε's so that the tool will be most convenient and effective. We will call our tool for short an ε-longimeter (either rectangular or triangular). Because the planimeter with a denser number of points in the network is more effective it follows that a denser layout of circles in the longimeter will also be more effective. The longimeter in Figure 10 is less effective then in the larger drawings, but the circles would overlap and the use of such a longimeter would be inconvenient. The most convenient longimeter is one with circles touching each other, with $a = 2\mathcal{E}$ in the case of a square lattice, and with $b = 2\mathcal{E}$ in the case of a triangular one.

In the case of the square longimeter, the constant coefficients in equation (4) are simplified if they take the form $a^2/2k\mathcal{E} = 2\mathcal{E}/k$. This occurs if $k = 2\mathcal{E}$, where $\mathcal{E}$ is expressed in suitable units of length (for instance in millimeters) and the results will be in the same units. Formula (4) then becomes:

$$(5) \qquad L_{\mathcal{E}}(X) = -\tfrac{1}{2}\pi\mathcal{E} + \sum_{i=1}^{k} n_i$$

Thus, for example, on the large drawing the longimeter for $\mathcal{E} = 5$ mm is shown. The side of the lattice comes out as $a = 2\mathcal{E} = 10$ mm, and the number of repetitions of the measurement is $k = 2\mathcal{E} = 10$. The subtrahend $\tfrac{1}{2}\pi\mathcal{E} = 7.854 \approx 8$. The method for the measurement of the length of order $\mathcal{E} = 5$ mm of the arc X is very simple:

We lay the longimeter at random 10 times on the arc X and count the (grand total) number of circles falling on the arc X. Eight is subtracted from this sum to obtain the length of 5 mm order of the arc X in millimeters. The numbers a and k can be obtained similarly for arbitrary $\mathcal{E}$'s. Such numbers together with the subtrahend $\tfrac{1}{2}\pi\mathcal{E}$ for various $\mathcal{E}$'s are presented in Table 1:

TABLE I

$\mathcal{E}$	1	1.5	2	2.5	3	4	5	6	8	10	15	20	30	50
a	2	3	4	5	6	8	10	12	16	20	30	40	60	100
k	2	3	4	5	6	8	10	12	16	20	30	40	60	100
$\tfrac{1}{2}\pi\mathcal{E}$	2	2	3	4	5	6	8	9	13	16	24	31	47	79

Each column of this table with ε expressed in arbitrary units of length, permits the construction of suitable ε-longimeters. Thus, for instance, the longimeter in the large drawing is made with the aid of the column whose initial number is 5.

In the case of the triangular lattice the coefficient in equation (4) is $b^2\sqrt{3}/4k\varepsilon$; after substitution for $b = 2\varepsilon$ it will be equal to $\frac{\varepsilon\sqrt{3}}{k}$. It will be equal to unity for $k = \varepsilon\sqrt{3}$, which in the view that k (the number of times the measurements are made) must be a natural number, makes it possible to construct an equally suitable longimeter only for certain values of ε. For example, the ε-longimeter in the case of the triangular coordinates shown on the drawing has circles of radius $\varepsilon = 4.05$ mm. This is a number chosen so that $4.05\sqrt{3} \approx 7 = k$. The subtrahend $\frac{1}{2}\pi\varepsilon$ here is equal to $6.37 \approx 6$. The rule to use this longimeter is: we obtain the length of order $\varepsilon = 4.05$ mm for the arc X by sevenfold (random) application of the longimeter to the arc X, and subtract 6 from the total number of circles hit by the arc. Similarly, we can construct other sets of numbers for ε-longimeters using the triangular lattice. They are shown in Table 2.

TABLE 2

ε	1.15	1.73	2.31	2.89	4.05	5.78	8.67	11.5	14.4	28.9	57.8
b	2.30	3.46	4.62	5.78	8.10	11.6	17.3	23.0	28.8	57.8	115.6
k	2	3	4	5	7	10	15	20	25	50	100
$\frac{1}{2}\pi\varepsilon$	2	3	4	5	6	9	14	18	23	45	91

In the equations used for longimeters I recommend k-fold random applications to the arcs. In some cases it is advisable to use systematic applications of the longimeter instead of random. With this aim, a k-pointed star can be drawn (on the rectangular longimeter 10 points, on triangular longimeter 7 points) in the center of the longimeter. On the arc a section should be traced and one of its ends marked. Then the longimeter should be applied to the arc to be measured so that the center of the star falls on the marked point, and the sections of the arc traced on the points of the star. I-measurements of I-pointed stars should be applied to the section. In this case, it is not necessary to write down the several quantities of circles of the longimeter falling on the arc in the Ith application of the longimeter. It is sufficient to obtain the number at one counting, and after finishing k measurements (after going around the whole star) the proper subtrahend should be deducted. The length of order ε of the measured arc is then available immediately.

5. The above described measurement with the ε-longimeter gives the length of order ε of the arc X. If the arc X is 2ε-convex and has ends at least 2ε apart, then the length of order ε is equal to the classical length. If however the arc X is not 2ε convex, that is, has a radius of curvature at some place of less than ε, the length of order ε is a new quantity which we conventionally accept as the length. Because for empirical curves we cannot confirm, as mentioned earlier, whether they are rectifiable or not, the more so it cannot be confirmed whether they are 2ε-convex. It therefore should be noted that for all empirical curves the length of order ε is a new conventional measurement.

The shape of empirical curves can be very capricious. In an earlier paper (Ref 4) I brought attention to the need of simplifying empirical sets and proposed the use of ε-convexity. We will apply this to the simplification of the geometrical shape of an arc. Usually empirical curves are of interest as boundaries between two areas, as for example, the seashore, leaves and their background, and razors and their background. If we were interested in a curve which is not such a boundary we could artifically introduce, for example, a circle connecting the ends of the arc and two areas would be created which our arc divides.

Let A and B be two areas divided by the arc X (Fig. 11). In paper (4) I designated the 2ε-convexity of an area A by $C_{2\varepsilon}(A)$, or the smallest set of 2ε-convexity enclosed in area A. This is called an ε-generalization of area A. The edge of this set, or rather that portion of it which is enclosed in the closed area B will be designated by $G_\varepsilon(A/B)$ and will be called the ε-generalization of the edge of area A in area B:

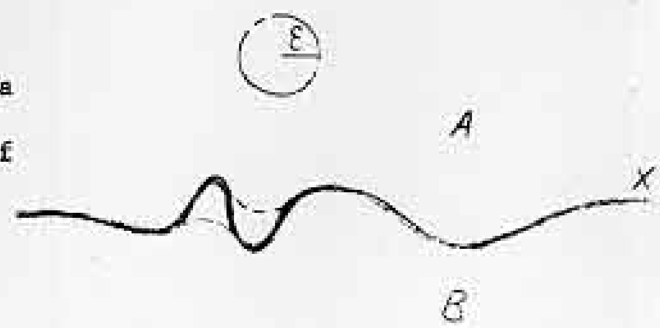

Fig. 11

$$(6) \qquad G_\varepsilon(A/B) = BF_r\left[C_{2\varepsilon}(A)\right].$$

The arc $G_\varepsilon(A/B)$ is shown in Figure 11 by a broken line. It can be formed in the following manner. Along the arc X we will roll (on the B side) a circle with a radius of ε. The envelope of all positions of the circle consists of two branches; that branch of the ε-generalized edge of area A in area B will be that which lies nearest the arc X. In the case where area A is 2ε-convex, $G_\varepsilon(A/B)$ coincides with arc X.

Analogously by $G_\varepsilon(B/A)$ is meant the ε-generalized edge of area B in area A. This is obtained by running a circle of radius ε along arc X on the A side. The arc X is traced with a heavy line in Figure 12, and arcs $G_\varepsilon(A/B)$ and $G_\varepsilon(B/A)$ by thin lines. Both area A and B are 2ε-convex at the same time, if and only if the arc X is 2ε-convex; then arcs $G_\varepsilon(A/B)$ and $G_\varepsilon(B/A)$ coincide with arc X. If only one of the areas A or B is 2ε-convex, then one of the arcs $G_\varepsilon(A/B)$ or $G_\varepsilon(B/A)$ coincides with the arc X, and the other is different. If finally neither of the areas A or B is 2ε-convex then all three arcs differ. In the last two cases, that is if the arc X is not 2ε-convex, then between arcs $G_\varepsilon(A/B)$ and $G_\varepsilon(B/A)$ there exists a two dimensional set, the area $C_{2\varepsilon}(X)$. This area is called the ε-generalized edge of areas A and B:

$G_\varepsilon(A,B) = C_{2\varepsilon}(X)$.

This is the natural interpretation of these definitions. Imagine that area A is the ocean and area B the land and let us consider the boundary between the land and the sea (Figure 12). We will limit ourselves to an instantaneous moment, that is, we will not consider the changes caused by tides and waves. A series of sufficiently accurate photographs are taken at the chosen moment. But then the boundary should distinguish the individual grains of sand from the water around these grains, and, with greater accuracy the curve will become even more complicated. Thus the following arc will be the ε-generalized

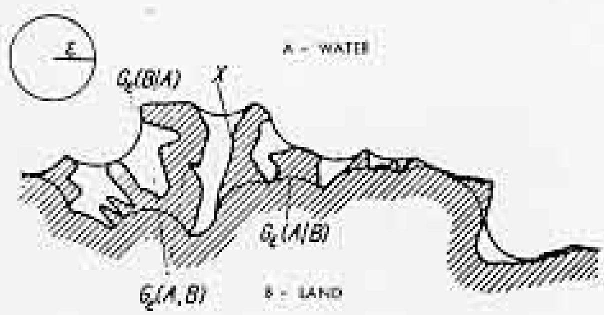

Figure 12

edge of the land in the sea: we will place a plane circle or radius on the sea, floating on top of the water. We will try to swim as close as possible to the land with the circle, so that at every moment at least one point on the circle is in contact with land and so that the circle is at all times on the surface of the water. Bays narrower than 2 (that is, in those in which our circlé will not fit) will be counted as part of the 2 -convex land the line along which our circle will run will be the envelope of this circle under all conditions, that is, it will be the -generalized edge of land in the sea.

Similarly, if we permit the circle to move as near as possible to the sea but in such a way that it is entirely on land and does not enter any peninsulas narrower than 2ε, they will be counted as 2ε-convexity of the sea. The line formed by the movement of this circle will be the ε-generalized edge of the ocean in the land. Note that these two generalized curves do not coincide. The curve $G_{\varepsilon}(B/A)$, the -generalized edge of the land in the ocean, will leave some narrow bays on the side of the land and water on the side of the sea. On the other hand, arc $G_{\varepsilon}(A/B)$, leaves narrow peninsulas on the side of the sea and dry land on the landward side. An ε-generalized edge of land and sea will be contained between these two curves; G (A,B) will not be a line but a domain of certain added area, an area consisting of narrow bays of the oceans and peninsulas of the land. Similarly small lakes with sea water and islands of continental land can appear there. However, in this area, we will not find a single piece of land or sea which could contain a circle with a radius of .

The length of order ε of two curves can be measured with the aid of the ε-longimeter described in paragraph (4): the ε-generalized edge of area A in area B and ε-generalized edge of area B in area A can be measured without tracing these edges. This means that the measurement of curves G (A/B) and G (B/A), or only one of these curves, can be carried out having only one drawing of the curve X dividing areas A and B (or these features in nature) and an -longimeter. For this we need in addition a separate circle with a radius of (the same as one of the circles in the -longimeter) cut out of pasteboard.

In measuring the length of order of the arc X, the circles of the -longimeter falling on arc X are counted. We will differentiate between circles attainable from area A, attainable from area B, and not attainable. We will call circles of the -longimeter attainable from area A if we can fit in area A additional circles cut from pasteboard so that they have points in common with the circles of the longimeter under consideration. Circles of the longimeter not attainable either in area A or B we call unattainable. In Figure 13 is shown a curve X dividing area A and B and the -longimeter placed on curve X. Circle a of this longimeter is attainable equally from area A as from B. Circle b is

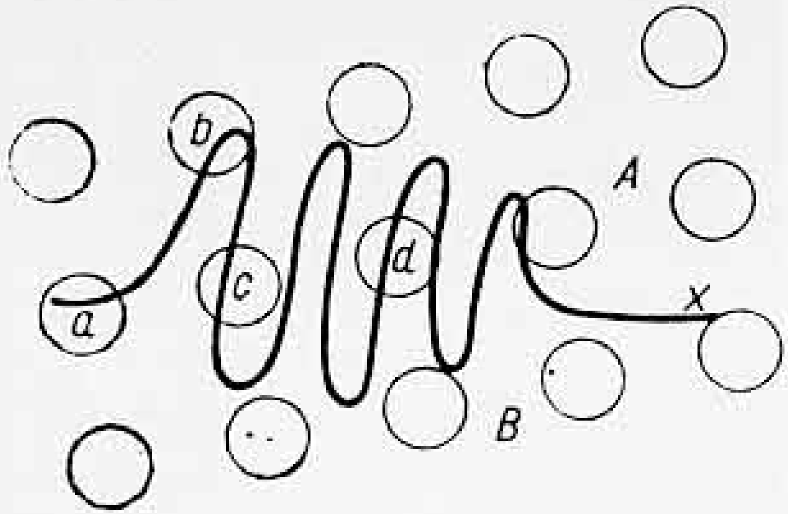

Figure 13

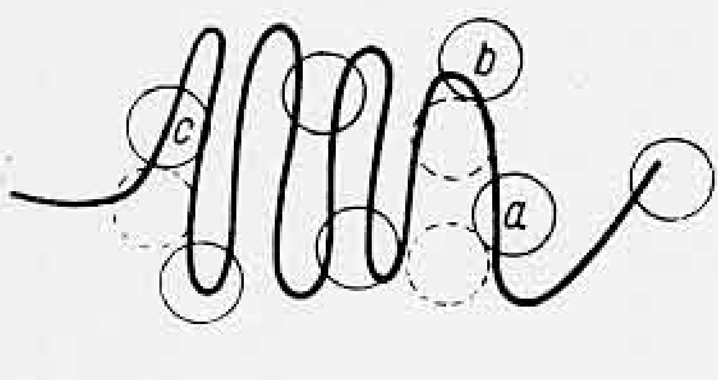

Figure 14

attainable only from A, circle c only in B, and circle d is unattainable. Occasionally we can ascertain without additional circles if the circles are or are not attainable in some area. In certain cases we have to use additional circles (from pasteboard) thus making the measurement more difficult. Examples of simple and difficult determinations are presented in Figure 14 (in the latter case, the additional circles are shown as dotted circles). For the circles of the ε-longimeter to fall on the arc $G_\varepsilon(A/B)$, it is necessary and sufficient for the circle to fall on the arc X and to be attainable from area B. Actually, if the circle of the longimeter falls on the arc $G_\varepsilon(A/B)$ then it cannot lie completely either in area A, or in area B. Therefore it is concluded that it must fall on the arc X dividing areas A and B. If furthermore the circle falls on the curve $G_\varepsilon(A/B)$, then the points of the set $C_{2\varepsilon}(B)$ are enclosed; the points of this set have the property that a circle of radius can be added to every such point that lies within area B. Therefore, the circles of the longimeter falling on curve $G_\varepsilon(A/B)$ are attainable from area B. Conversely, if the circle of the longimeter falls on the arc X and is attainable from area B, this includes points in series $C_{2\varepsilon}(B)$ and points of area A, therefore it falls on the edge of area $C_{2\varepsilon}(B)$ or curve $G_\varepsilon(A/B)$.

Analogously, for a circle of the ε-longimeter to fall on the curve $G_\varepsilon(B/A)$ it is necessary and sufficient that the circle fall on curve X and that it be attainable from area A.

From this results the following method of measuring the length order ε of the ε-generalized edge of area A in area B; we apply the ε-longimeter to the curve X and count the number of circles of the longimeter falling on curve X and at the same time attainable from area B. The measurement is repeated k times (the number k from Table 1 or 2 should be recorded on the longimeter) and from the combined quantity of counted circles subtract the pertinent coefficient $\frac{1}{2}\pi\varepsilon$ (also given on the longimeter). In an analogous manner the length of order ε of the curve $G_\varepsilon(B/A)$ can be measured. It should be noted that curves $G_\varepsilon(A/B)$ and $G_\varepsilon(B/A)$ do not have to be 2ε-convex and in view of this their length of order ε does not have to be a classical length.

The number of circles of the ε-longimeter falling on the arc X, but attainable neither from area A or B, in a certain manner indicates the size of the ε-generalized edge of areas A and B. A circle is unattainable if it lies entirely within the set $G_\varepsilon(A,B)$, or if the center of this circle lies within the ε-core (see Ref 2) of the set $G_\varepsilon(A,B)$. The expected number of unattainable circles falling on the arc X is an indication of the size of the ε-generalized edge of areas A and B (the boundary between the land and sea, consisting of narrow bays and peninsulas). Just because this number is zero does not necessarily mean that the ε-generalized edge of areas A and B has an area equal to zero; this means only that this edge is narrow, and that the ε-core of this edge has an area equal to zero.

6. The circles of the ε-longimeter are numbered 1, 2, ... , r and the event Z_i is considered as being dependent on placing the i-th circle on the arc X, when using the longimeter. To x_i we assign a random variable (which can have the value one) when the event Z_i is approached, and a value of 0 when the opposite event approaches, that is, when the i-th circle does not fall on the arc X. By $x = x_1 + x_2 + \cdots + x_r$, we mean the number of circles of the longimeter which fall on the arc X in one application (in paragraph 4 the variable x is indicated by $\sum n_i$).

If the occurrences of Z_i were independent of each other and if the probability P of the occurrence of Z_i were independent of i, the random variable x would have a binomial distribution with an average rp, equal to the anticipated number of circles of the ε-longimeter falling on the arc X, with Bernoulli's variation rpq (where $q = 1 - p$). But the events Z_i are not independent but are precisely correlated. For that matter, if one circle of the longimeter falls on the arc X, then the probability that neighboring circles of the longimeter also fall on this arc is greater than the probability that a random circle of the longimeter will fall on the arc X. The correlation between the events Z_i depends on the shape of the arc X; for some it is greater and for others less. This positive correlation increases the variance of the random variable x, making it greater than normal.

On the other hand, the probability of the occurrence of Z_i is dependent on i. By the application of the longimeter to the arc X the center of the longimeter is utilized and for that reason the circles lying in the center of the tool have a greater possibility of falling on the arc X than the circles lying on the edges of the tool. This brings about a decrease in the variance of the random variable x (to less than normal) and the size of this change depends on the person making the measurement; that is, on whether he applies the center or the edge of the longimeter to the arc X.

These two influences to a certain extent cancel, and one can with a certain amount of approximation maintain that the random variable x has a normal variance rpq. The variance of the random variable N which is the summation over k of the independent random variable x (the combined number of circles falling on the arc in k independent applications of the longimeter) is therefore approximately equal to krpq, and with a large longimeter, where r is great in comparison to the anticipated value of variable x, q is close to unity, and the variance in the random variable N is equal to krp/N. From this we can draw the conclusion that one should expect that the variance in the length of order ε of the arc X is of the order of the measured length, and the average quadratic deviation is on the order of the root of this length. Therefore the greater the length (of order ε) of the arc, the greater the precision (percentwise) in measuring with the longimeter. If, for example, the length of order ε of some curve is equal to 20, then it is expected that the probable average error of this length is of the order of 4 to 5, that is, 20 to 25%. If the length is equal to about 100, then the probable average error is on the order of 10, that is, about 10%. If, as I mentioned at the end of paragraph 3, ε is taken so that the ratio $\eta = \varepsilon/L_{\varepsilon}(X)$ is not greater than 10%, then in view of the fact that $\varepsilon \geqslant 1$ and $L_{\varepsilon}(X) \geqslant 10$, the probable average error would become not greater than 30% of the length of order ε.

In practice measurement with the longimeter proved to be far more precise than would appear from the above considerations. Evidently the influence of the different rolls played by the central and the extreme circles of the longimeter to diminish the variance is far stronger than the influence among the events Z_i. The result of this is that in practice the variance is remarkably less than normal.

Together with laboratory assistants of the Mathematical Institute, J. Dobrowolske, A. Huskowski, and M. Kusiatkow, we measured several curves with the longimeter (the measurements were repeated a number of times, in one case 200 times). The aim of these measurements was the examination of the systematic error in practice, the random measurement of different curves with different longimeters, and the determination of the time required to accomplish the measurements.

For the investigation of the systematic errors three persons; A, B, and C applied at random the ε-longimeter ($\varepsilon = 3$ mm) 60 times on arc 2 (Figure 15) and counted the number of circles of the longimeter that hit the arc. In Table 3 are presented in a series of columns the number of circles falling on arc, the average, the variation, and the average quadratic deviation of these numbers. As can be seen, there are no systematic differences between the average results obtained by

TABLE 3

Number of circles hitting arc 2	Number of applications per person		
	A	B	C
12	-	1	2
13	2	5	8
14	9	13	24
15	29	12	15
16	15	23	10
17	4	4	1
18	-	1	-
19	1	1	-
Average of the count of circles	15.23	15.20	14.43
Variation	1.02	1.73	1.15
Average deviation of the quadratic mean	0.13	0.17	0.14

persons A and B. However, there is a real difference between A and C (Student's criterion gives the likelihood of a hypothesis saying that there is no difference as being equal to 0.0001). The difference in the results obtained between B and C is equally real. In the variance, the results obtained by B differ greatly (at a reliability level of 1%) from those of A; the variances obtained by A and C do not really differ.

The result of this is that one might worry about the systematic errors encountered in the results of measuring with the longimeter. In our case the extreme difference was 0.80, or 5 to 6%. It should be realized from the systematic differences in errors, that these differences could be 20% of the errors.

I next took up random errors. From what I have written previously the

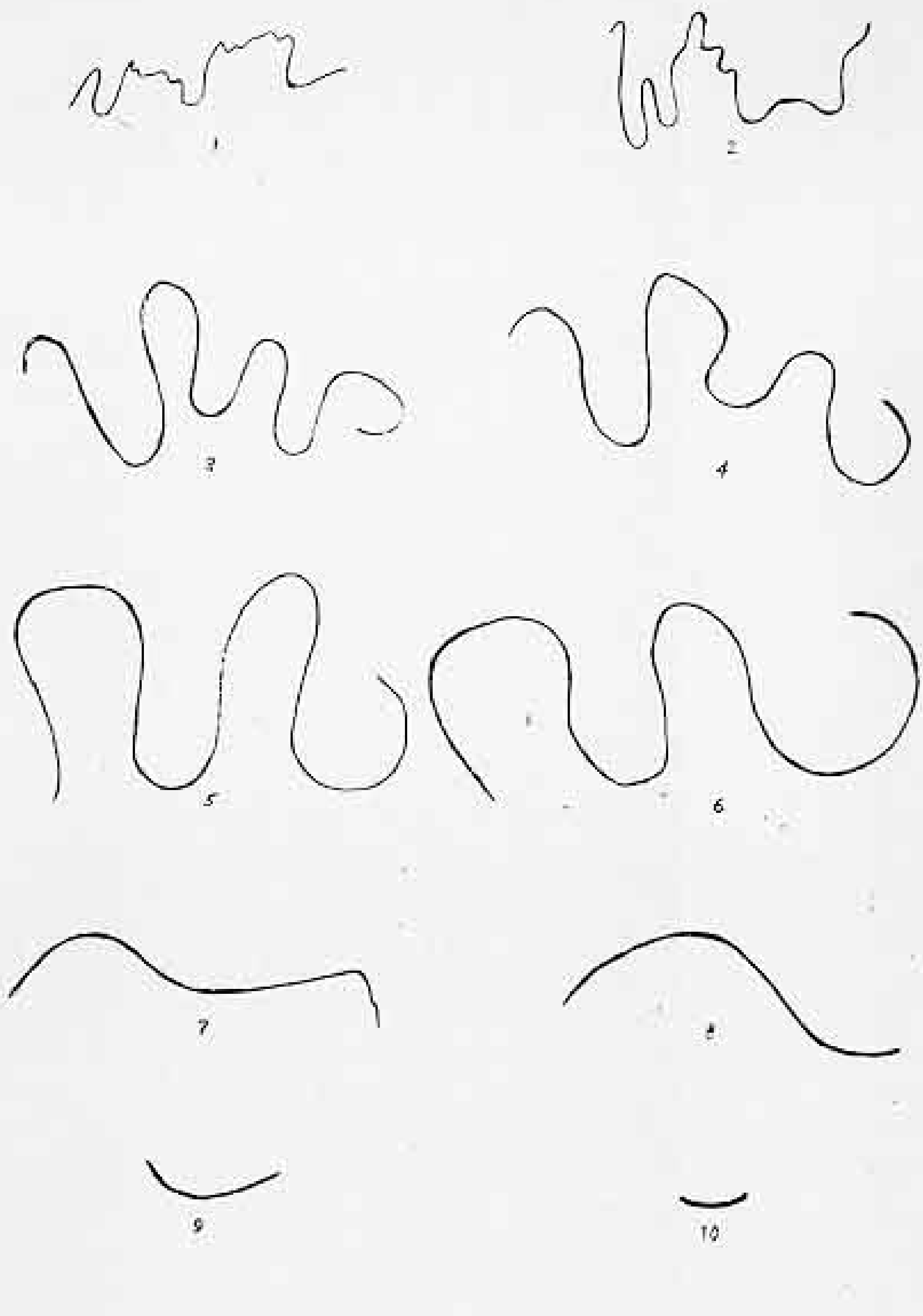

Fig. 15

results show that the random error depends to a large extent on the person making the measurements; for person A the error was 20% less than that for person B. The variation in the results of one application of the longimeter we indicate by S^2. The length of order 3 mm we obtain using the sum of six measurements; consequently the variation in the length of order 3 mm will be the sum of the variation of six random independent variables of variation S^2, or $6S^2$ (a constant coefficient does not affect the variation).

The average length of order 3 mm of curve 2 obtained by persons A, B, and C are presented in Table 4. The average quadratic deviation of these lengths calculated from 6 applications and the same deviations expressed in percent of the length of the arc are also given. In the beginning of the paragraph, we calculated that in the case of normal variation the average quadratic deviation should be in the order of $\sqrt{90} \approx 9.5$, therefore the empirical variation of length is actually subnormal.

TABLE 4

	A	B	C
Average length of the order of 3 mm of arc 2	86.7	86.5	81.9
Average quadratic deviation of the length obtained from 6 applications	2.47	3.22	2.62
The same deviations in %	2.9	3.7	3.2

The random error of measurement (average quadratic deviation) can be estimated on the basis of one measurement made up from 6 applications of a longimeter with $\epsilon = 3$ mm. The error will actually be estimated with less precision, but for comparative purposes it will be sufficient.

Ten curves are presented in Figure 15. Double measurements of length were made on each of these, of order 3 mm, 5 mm, and 8 mm with the appropriate longimeter (square lattics). The results are presented in Table 5. From the table we can investigate how the lengths get smaller as the order of the arc increases (ε = 3, 5, 8 mm), for various curvatures and for large (curves 1 and 2) and small (No. 8) curves. Curves 9 and 10 were particularly short. In the last

TABLE 5

LENGTH \ ARC		1	2	3	4	5	6	7	8	9	10
$L_3(X)$	A	69	87	145	135	165	158	64	58	26	10
	B	67	86	145	138	175	166	65	63	20	10
$L_5(X)$	A	61	79	132	132	168	163	65	56	20	11
	B	54	75	121	128	174	171	61	60	20	13
$L_8(X)$	A	56	63	111	107	156	159	66	57	21	11
	B	51	69	106	111	158	162	66	58	20	9
$L(X)$		76	105	151	138	176	171	65	58	22	10

line are given the lengths L(X) of these curves, measured by an incremental method. This method is based on counting small steps made along the length of the curve with point dividers (the span of the points is adjusted with a screw). The length of the steps are determined by dividing 100 steps along a straight line. As can be seen, arcs of sufficiently small curvature have the length of order ε very close to the length obtained by the incremental method. Letters A and B represent the persons performing the measurements. The results from both persons are given in order to see the difference between two measurements (each consisting of 6 applications) performed by the different persons.

As can be seen from the table (Table 5), the first three arcs have lengths of order 3 mm which are smaller than those lengths measured incrementally. The lengths of order 5 mm of these curves are smaller than the lengths of order 3 mm, and still smaller are the lengths of 8 mm order. The following curve (4) has a length of 3 mm order equal to the length established by the stepwise method. On the other hand, its length of 5 mm order is somewhat smaller and the length of 8 mm order is considerably smaller. The next two curves (5 and 6) have lengths of 3 mm and 5 mm order equal to the lengths measured stepwise, but the length of 8 mm order is smaller. Finally the last four curves 7, 8, 9, and 10 have the lengths of 3 mm, 5 mm and 8 mm order equal to the stepwise measured lengths.

Curves 9 and 10 are especially short. The ratio $\eta = \varepsilon/L(X)$ for these curves approaches 75% (when $\varepsilon = 8$ mm). The remaining curves have lengths exceeding 60 mm and for these the ratio η is decidedly smaller, although they occasionally exceed 10%, which earlier we had accepted as the upper limit. Despite this the random errors inherent in the results of measurements are not large. Table 6 consists of ratios η_ε and the indices of variation W_ε, that is, the percent error in the length of order ε. The number of applications of the longimeter necessary for one measurement of length of the order ε are given in the column titled k . Actually if the entire measurement were repeated n times (that is the longimeter were applied nk times) the average error of length of order ε obtained from these n measurements would become $\sqrt{n}$ times smaller.

TABLE 6

50

K	ARC	1	2	3	4	5	6	7	8	9	10
6	3	4.4	3.5	2.1	2.2	1.8	1.9	4.7	5.0	13.0	30.0
	W_3	3.5	2.8	1.7	1.7	3.3	1.7	3.7	3.9	6.3	12.2
10	5	8.2	6.3	3.8	3.8	3.0	3.1	7.7	8.9	25.0	45.0
	W_5	3.6	3.6	3.6	2.1	3.1	1.8	2.2	5.8	9.5	15.5
16	8	14.3	13.	7.1	7.5	5.1	5.0	12.1	14.0	38.1	72.8
	W_8	7.9	5.7	3.0	4.2	3.6	2.1	5.5	6.0	13.2	18.2

As can be seen, in cases where the ratio η does not exceed 10%, the error of measurement does not exceed 6% and in only two cases does it exceed 4%. It is further seen that when ε increases, in general, the error becomes greater, and η increases simultaneously.

In Table 7 are presented the lengths x of the ε-generalized edges of areas which are separated by the plane curves of Figure 15. We will call P the area lying above the arc X, and Q below this arc. The length of the arc $G_\varepsilon(Q/P)$ we will call $\overline{L}_\varepsilon$ and the length of arc $G_\varepsilon(P/Q) = \underline{L}_\varepsilon$. Each one of these lengths were measured by two persons, A and B. Each measurement consisted of k applications of the longimeter. These measurements contain greater errors, both random and systematic. Person A had results systematically greater than those of person B

TABLE 7

ARC \ ORDER OF LENGTH		1	2	7	8	9	10
$\overline{L}_3$	A	50	73	59	56	24	9
	B	37	62	59	56	22	11
$\underline{L}_3$	A	54	81	62	57	23	11
	B	34	73	60	57	22	11
$\overline{L}_5$	A	39	63	67	58	24	13
	B	42	54	57	54	22	12
$\underline{L}_5$	A	52	65	65	57	22	12
	B	58	56	59	52	18	14
$\overline{L}_8$	A	42	49	66	61	19	10
	B	41	46	58	56	23	10
$\underline{L}_8$	A	46	67	65	61	20	10
	B	42	52	57	56	19	9

by a factor of 4. After eliminating these systematic errors it is found that the random error is established at about 5% of the measured length.

The lengths of curves 1, 2, and 6 measured by Steinhaus' line-longimeter are included in Table 8 for comparison. This table presents the averages obtained for fifty applications of the longimeter for each result. These averages contain random errors more or less of the order resulting from single measurements with the $\mathcal{E}$-longimeter. For example, the average error (from 50 applications of Steinhaus' longimeter) of the length of order 1 of arc 1 is established at 4.5% and for the order 8 it is established at 3.3% of the length of the measured arc.

TABLE 8

ORDER OF LENGTH / ARC	1	2	3	4	5	6	7	8
1	44	62	74	78	79	80	80	81
2	47	71	90	100	108	110	111	111
6	77	126	153	163	168	168	168	168

In conclusion, I will give the time required to accomplish the measurement of the various lengths. For the measurement of length of order ($\mathcal{E}$ = 3, 5, or 8 mm) of any arc in Figure 15, two to four minutes are required. Such a measurement (consisting of 6, 10 or 16 applications of the longimeter) results in an error of about 4%. To obtain the length of order n of such an arc with the same precision by Steinhaus' method would require 50 applications which would take 15 minutes or four times longer than in required by the $\mathcal{E}$-longimeter. The incremental method of measurement (a double measurement and double marking of the length with steps) requires more than 10 minutes. The precision of this method depends on the shape of the curve. For arc of small curvature (as in arc 6) the stepwise method gives very precise results (the error is established at about 0.3% of the length), but for arcs of greater curvature the precision decreases decidedly (for example, in arc 1 the error is established at about 4%).

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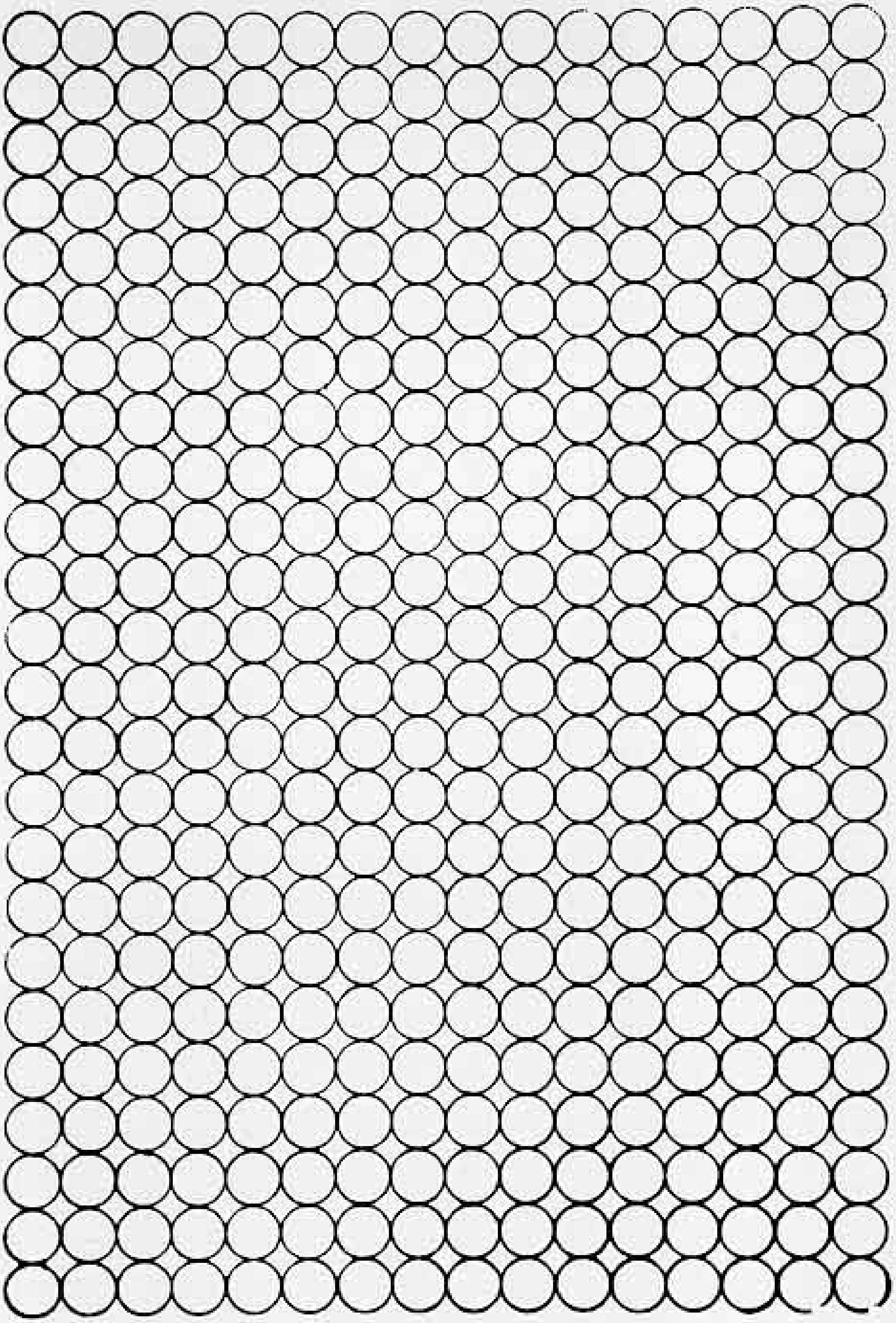

Figure 16

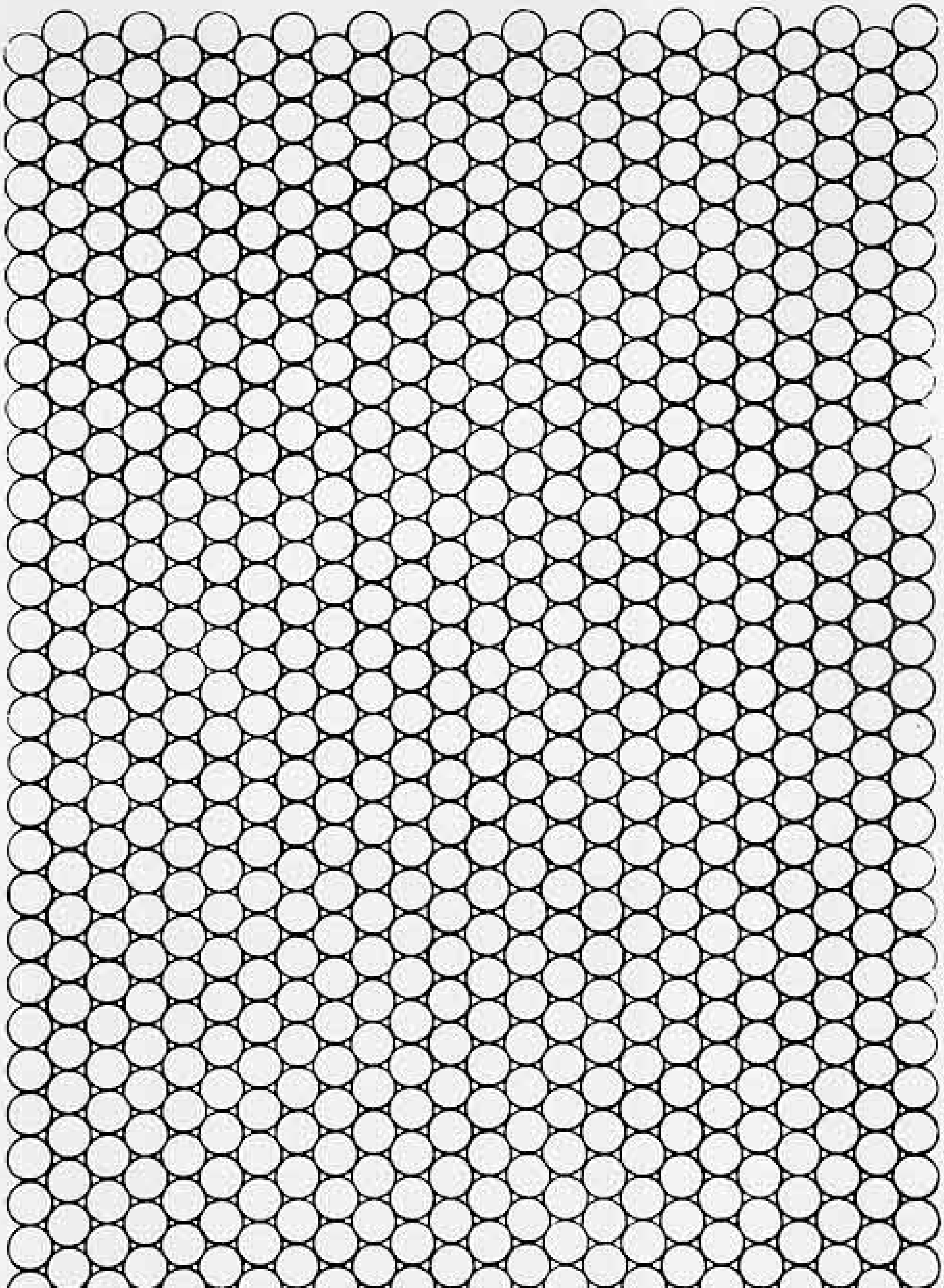

Figure 17