FOREWORD

Does the coastline have a length? Like Lewis Richardson's "Does the wind possess a velocity?" "This question, at first sight foolish, improves on acquaintance" (1926, Proc. Roy. Soc., A, 110, p. 709). As Nystuen points out in the accompanying paper, these questions should not be thought of as theoretical curiosities. Close examination instead leads to very useful results. Richardson, in his paper, abandons the notion of limits in favor of nearest neighbor relations. Steinhaus and Perkal involke epsilon neighborhoods to overcome the paradox of length. Nystuen adopts this latter strategy to examine geographical boundaries.

Ratzel would have appreciated Nystuen's paper, with its clear definition of the area of an edge, for he once wrote "Der Grenzraum ist das Wirkliche, die Grenzlinie die Abstraktion davon." The well known literature of political goegraphy contains numerous boundary studies, to which Nystuen's paper now adds concepts and operational procedures which should have great impact. Even the cartographer should take note because it is now clear that generalizing the inside of a boundary differs from generalizing the outside of the same boundary.

One of the justifications for an organization such as the Michigan Inter-University Community of Mathematical Geographers is the exchange of ideas which it fosters. Professor Bunge brought the works of Steinhaus and Perkal to the attention of Professor Nystuen, who characteristically developed numerous and insightful geographical interpretations of the materials. Nystuen's first presentations on this topic were to the Community at its meeting place in Brighton during the spring of 1965. He subsequently gave a short paper (reproduced here) on the subject at a Regional Science Association meeting. A more recent paper, which includes further valuable geographical generalizations and extensions to problems of international concern, will be published in the *Papers* of the Peace Research Society (Volume VII).

To assist the reader, we have included translations from the Polish of two of Perkal's original papers. These also refer to the publications in French and English which Professor Nystuen had available. We were distressed to learn of Perkal's demise during the summer of 1966 and regret that he was unable to become familiar with these applications of his work.

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EFFECTS OF BOUNDARY SHAPE

and the

CONCEPT OF LOCAL CONVEXITY

by

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Many spatial processes depend upon the shape of the partitions created by their boundary patterns. If the boundary shape is changed the process itself is changed, in fact, the very existence of the process may depend on the boundary shape.

In this paper I report on a concept which I believe is useful in analyzing the interplay between boundaries and spatial processes. This notion may be called _local convexity_ or perhaps, _convex-in-the-small_.

It seems to me three classes of subject matter are affected by boundary shapes. In one case the boundary affects the processes in the domains on one or both sides of the boundary. In the second case the boundary affects processes which are crossing it. And finally, I recognize a class of processes or spatial elements which exist only at boundaries. The main purpose of this paper is to suggest some measurement concepts which will be of aid in analysis of all three of these types of spatial processes.

Boundary Effects in Spatial Processes.

Some spatial domains have clearly defined boundaries. The boundary between land and sea is often well identified. At least it is well identified at certain scales. As an aside, I note that one is hard pressed to find the land-sea boundary to centimeter accuracy even in an instant in time such as would be available in analyzing an air photograph. Leaving that question for the moment, we can agree that at accuracies of one-tenth a kilometer, we can usually know where the land ends and the sea begins.

Sea processes are confined by this land-sea boundary and are affected by it. Consider the tides in the Bay of Fundy. The funnel shape of the bay causes a steep face on the tidal bore and creates at this place some of the highest tides in the world. Clearly here is a boundary shape which affects the spatial process it bounds.

The dimension of a spatial boundary is one less than the domain it bounds. For example, the tidal forces in the Bay of Fundy operate on the volume of water contained in the bay. The boundary shape which is important in determining the height of the tide, is the shape of the two-dimensional surface of the bottom and sides of the bay. Other processes can be considered to operate in essentially two-dimensional domains. They are bounded by one-dimensional edges. An example is the wave pattern on the surface of the bay. The wave pattern is sensitive to the shape of the one-dimensional shore line.

The first step in any analysis of the effect of shape is to determine the relevant dimension of the domain in which the spatial process operates. Many geographically significant processes can be considered two-dimensional. Line boundaries are the critical constraints to these processes. For simplicity, most of my remarks in this paper are concerned with two-dimensional domains and line boundaries.

A second class of boundary effects involves transfer processes. In processes or activities crossing a boundary there is often observed a transfer impedance which acts to reduce the effectiveness of communication across the boundary. Consider the customer contact of a supermarket across a river in a city. Figure I shows a sample of home places of customers to a supermarket in Ann Arbor. The river impedes contact across it. In part, this is geometric. There are only two bridges close enough to be of effect in the supermarket's trading area. Actually, people on the opposite side of the river must take a dog-leg route to the bridges in traveling to the store. In addition because the barrier is impermeable to automobile and pedestrian traffic except at the bridges, congestion occurs at the bridges which further reduces drawing power beyond the river. In analyzing this type of situation we need to be concerned with several general problems. The function of the barriers in terms of such questions as whether it is permeable all along its length or only at certain points is one problem. The second problem is geometric in the sense of the need to analyze the effects of the configuration and relative position of the barrier. Furthermore there is a trade-off implied between costs of additional crossings versus costs of congestion or queuing at a few points. Boundary function and boundary shape clearly affect transfer problems. Notice in passing that in order to analyze the function of a boundary, the activity involved in the crossing must be identified. The river would have a completely different role if the process under consideration was the spread of polluted air from nearby factories. Boundaries must be functionally defined.

Boundary shapes create another situation of great interest. Consider the coastline again. This boundary is no doubt the most important on earth. It is actually the line dividing three volumetric domains: the ocean mass, the ocean of air, and the land mass. Three volumes can meet at most in a line. Scientists postulate that life itself started at this interface.

There are many phenomena which exist only at the edge of a domain or in a region which we may call the boundary zone. Coast lines are in some places smooth and possessing very small average curvatures while at other places they may be

FIGURE 1

A Portion of Ann Arbor, Michigan

Sample of Supermarket Customers' Homeplaces, 1964

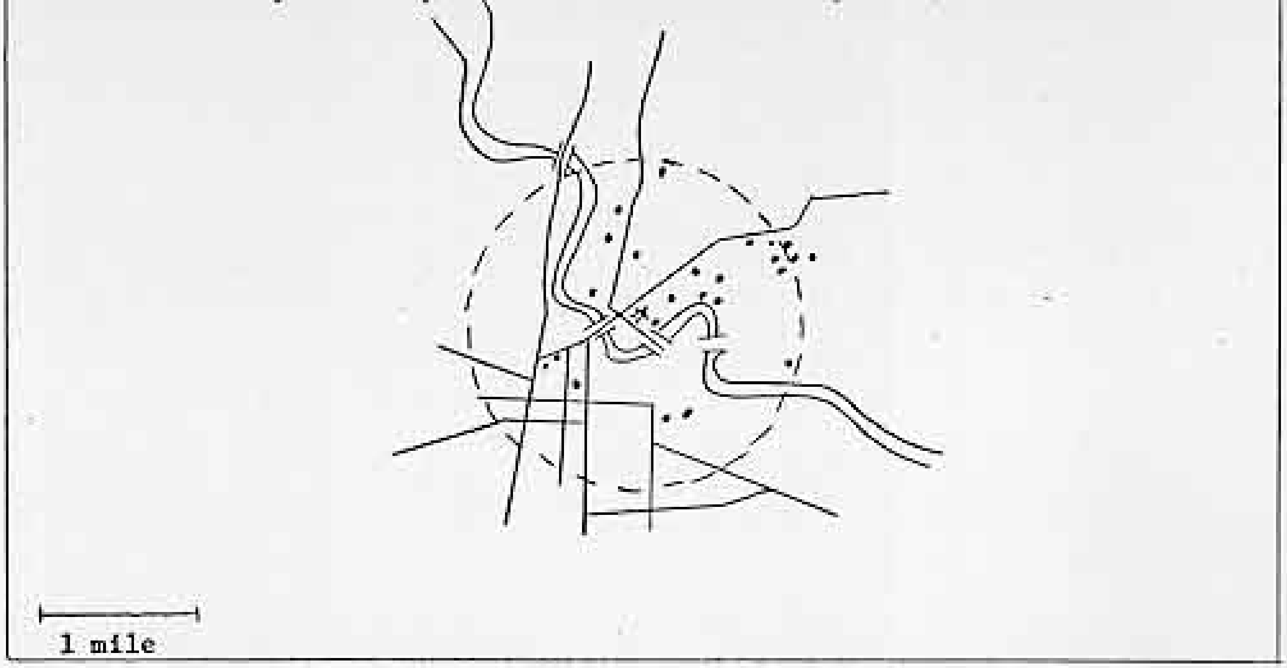
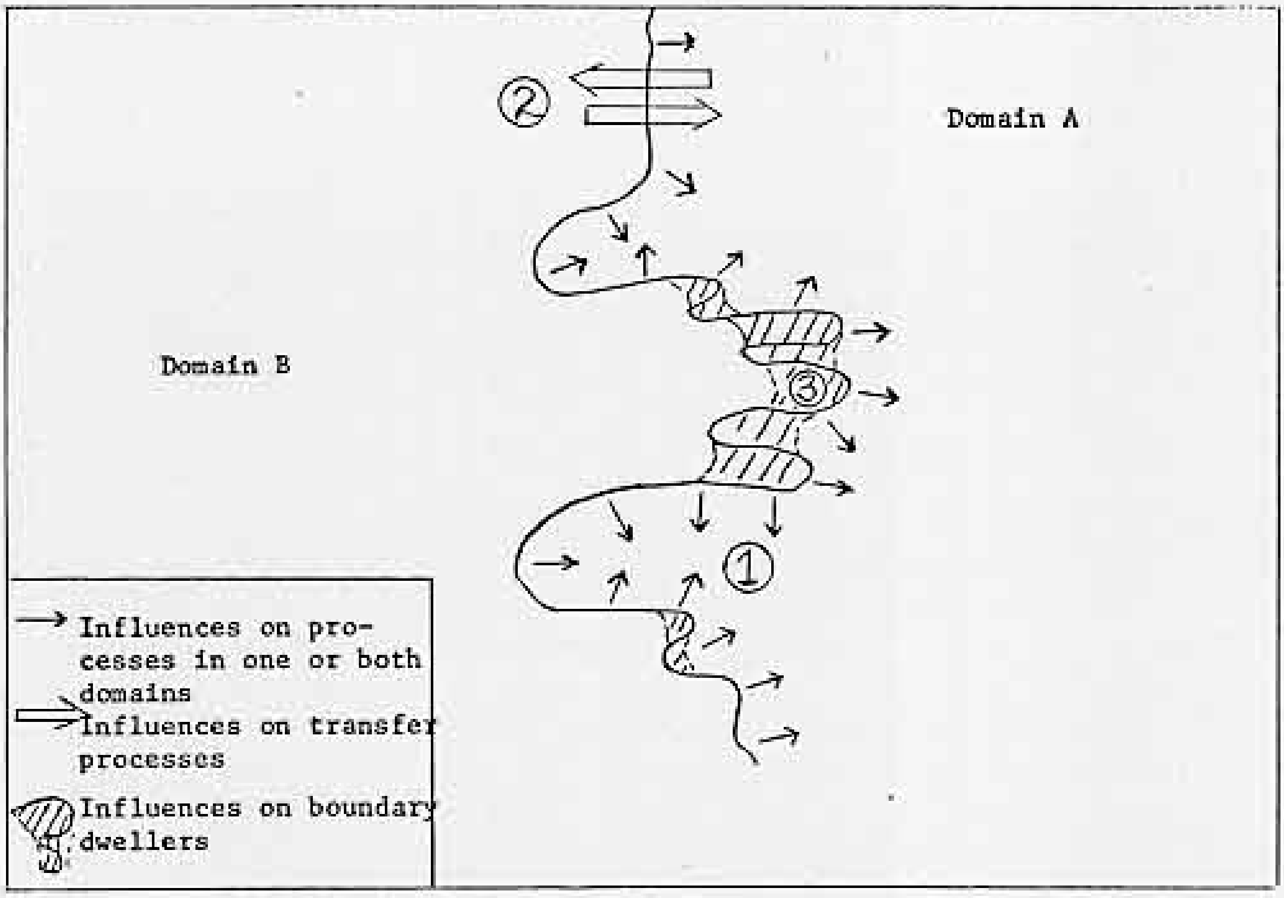


FIGURE 2

Three Classes of Boundary Influences:



deeply embayed and assume a sinuous form. This difference in shape makes for a difference in width of the boundary zone. The smooth coast has a narrow boundary zone and there is little room for boundary dwellers. The sinuous coast defines a boundary zone which is broader and many boundary creatures and elements exist. Clams are boundary dwellers. They are found most often in embayed areas where they are protected from the effects of the open sea. They also benefit by nearness to the domains of air and land even though they cannot exist in those domains. I propose to define boundary zones as a first step in analyzing those activities which are found confined to these zones. On the seacoast neither the sea nor the land dominates; each has its influence modified by the presence of the other.

I have purposely considered several types of phenomena at different geographic scales. I want to emphasize the unity of these topics insofar as they deal with boundary functions and shapes. I am convinced that there is an underlying unity in the spatial analysis. In order to make any progress in the analysis, specific abstract properties of boundary shapes and functions must be defined and measured. Analogy between widely different real world subject matters is a very dangerous procedure unless the phenomena are reduced in a crucible of abstract reasoning to the essential properties only. No unwanted transfer of meaning can be allowed. The essential properties we seek here are the ones with implications for spatial processes. We need clear definitions and measures of these properties.

Boundary Properties.

Several properties of boundaries may be specified. Clearly the dimensions of boundaries and the domains they contain are important characteristics. Boundaries have functions as well as dimensions. Some boundaries turn back elements that reach them. These are reflecting barriers. Their effect is to change the direction of movement but not to appreciably diminish the energy involved in the movement. Other barriers may absorb the elements that touch them thereby reducing the energy level of the process they confine. These are absorbing barriers. Such barriers reduce the level of energy of the process they bound. The Berlin Wall is an absorbing barrier. A third class of barrier is permeable. Part of the process passes through the boundary and part is either turned back or absorbed. These are permeable barriers. Obviously, a single boundary may involve to some degree all three of these functions. Subclasses of these three functions are also useful to consider.

It is clearly meaningless to talk about a boundary without specifying the activities involved in interaction with it. The same boundary will have different functions for different activities.

A major problem in analysis of the effects of boundaries is to devise operational definitions of boundaries which will allow systematic measure of the effects of a variety of boundary shapes and functions.

A reasonable assumption with regard to transfers from one domain to another is that transfers will depend on the permeability per unit length of a linear boundary or per unit area for a two-dimensional boundary. In cases with constant unit permeability the total transfer potential will be directly proportional to the length of the boundary line or, in the two-dimensional boundary case, to the total surface area.

Given the assumption above, the two ways to affect total transfers are to change permeability or to change the length of the boundary. If permeability is constant, different shapes of the same sized domains will have different transfer potentials. For minimum transfer, the circle is the optimum shape under these assumptions. The optimum shape for maximum interaction is not obvious. A shape could be compact with a very frilly edge resulting in a long perimeter. Another could be deeply indented such that all points within the domain are within some minimum distance to points outside the domain and yet the perimeter need not be exceedingly long.

Intuitively, measures of the length of a boundary and possibly its curvature would seem to be useful in analyzing effectiveness of boundary shapes. For example, the ratio of the length of a boundary to the area of the domain it encloses might be one such measure. The problem reduces to one of measuring length and possibly average curvature.

The length of a rectifiable arc and curvature at a point are familiar mathematical concepts. A rectifiable arc is one in which the tangent to the curve is continuous at all points on the segment. The following theorem emphasizes this points: (Courant, p.277)

> Every curve $y = f(x)$ for which the derivative $f'(x)$ is continuous, is a rectifiable curve, and its length between $x = a$ and $x = b$ $(b \geq a)$ is given by the formula
>
> $$S(a,b) = \int_a^b (1-y'^2)^{1/2}\, dx$$
>
> where $$y'^2 = \left(\frac{dy}{dx}\right)^2$$

This definition introduces a serious measurement problem. If the length of boundary is to be used in an index of effectiveness of shape, then the length of the line should be at least measurable in theory. Unfortunately most empirical lines have many points of discontinuity on them. The first derivative does not exist at all points on the curve. On deductive grounds it is not at all clear that the length of a boundary can be measured with theoretical rigor.

The Paradox of Length.

Steinhaus (1954) has pointed out the paradox of the length of empirical lines. Empirical lines are generally not rectifiable. The more accurately an empirical line is measured the longer it gets. The series of lengths obtained by repeated measures with finer and finer instruments does not ordinarily converge to a finite value. In general, the magnitude of the increments of change between successive measures does not vary systematically.

The length of a river or ridge line or other empirical line may be approximated by summing straight line segments between points on the line. Longer and longer lengths are obtained as the points are chosen closer and closer together. The same is true of the perimeter of a leaf. The finely serrated edge may be approximated at various scales but the finely measured lengths are always longer than those that are roughly measured. Even if a microscope were employed the length would continue to grow until at molecular level the length would approach the infinite.

The paradox is not to be confused with the fact that all physical quantities are subject to errors of measure. The problem remains regardless of the level of accuracy. A finer measure will always be longer. Nor should this problem be thought of as a theoretical curiosity. No agreement as to the length of an empirical line can be expected unless the scale of measurement is given. This is rarely done. One often hears, for example, recreation promoters who claim their region has hundreds of miles of lake front or fishing streams. No mention of how such values were arrived at is given. Nor can diplomats agree that the Polish-German border is so many hundred kilometers long, and so forth. Table 1. shows the extent of disagreement regarding the length of a mutual boundary between various nations. All such measures have little meaning.

Table 1. *

Disagreement on Length of Land
Frontier between Selected Nations

Land-frontier between	Kilometers as stated by	
	the former country	the latter country
Spain and Portugal	987	1214
Netherlands and Belgium	380	449.5
USSR and Finland	1590	1566
USSR and Romania	742	812
USSR and Latvia	269	351
Estonia and Latvia	356	375
Yugoslavia and Greece	262.1	236.6

* From Lewis F. Richardson, "The Problem of Contiguity" General Systems Yearbook v.6 (1961) p.169.

On the Epsilon Length.

Julian Perkal (1956b) has suggested a method of measuring the length of an empirical line. He proposes to define empirical length as a new concept which is analogous but not identical to the abstract length of a rectifiable arc. The notion depends on the idea of an ε-neighborhood of a curve. The ε-neighborhood may be defined as follows. Let capital letters represent sets of points and small letters represent individual points. The ε-neighborhood of a curve X is the set of all points on the plane for which the distance from the curve is not greater than ε, where ε is a fixed small distance. In set notation:

$$(1) \qquad A_{\varepsilon}(X) = E_{X}[d(p,X) \leq \varepsilon]$$

where $d(p,X)$ is the distance from a point p to the nearest element of the set X.

The set $A_{\varepsilon}(X)$ may be regarded as a function of ε and X and is monotonic increasing and continuous with respect to both arguments. The set $A_{\varepsilon}(X)$ may be regarded as defining the area $a_{\varepsilon}(X)$ which is therefore also a continuous, monotonic increasing function of ε and X.

Perkal (1956b) modified a definition of length of a rectifiable curve X given by H. Minkowski which is:

$$(2) \qquad L(X) = \lim_{\varepsilon \to 0} \frac{a_{\varepsilon}(X)}{2\varepsilon}$$

When this definition is used without passing to the limit, the ε-neighborhood of X includes a strip along the curve and two semi-circular areas around the end point of the curve to be measured. Not passing to the limit, the area required for definition of length is the strip along both sides of the line less the end areas and the definition becomes

$$(3) \qquad L_{\varepsilon}(X) = \frac{a_{\varepsilon}(X) - \pi\varepsilon^2}{2\varepsilon}$$

This is analogous to finding the length of a rectangle by dividing the area by its width. It may be seen by expression (3) that the length of a line is directly proportional to the area of the ϵ-neighborhood and inversely proportional to the size of the ϵ chosen. (See Figure III.)

Epsilon - Convex Curves.

Let the curve X be called ϵ-convex if we can draw at any point on the curve tangent circles with diameter ϵ, on both sides of the curve, having the point of contact as the only point in common with the curve. The epsilon length of a rectifiable curve is always less than the abstract length and approaches that length from below as ϵ gets smaller. Thus, using this definition we avoid the run-away length with closer and closer approximations.

The ratio ϵ/L characterizes the approximation that ϵ-length holds to ordinary length. For a suitable approximation ϵ should be chosen to make this ratio small, for example, one-tenth or less.

Measures of the Epsilon - Length of a Line.

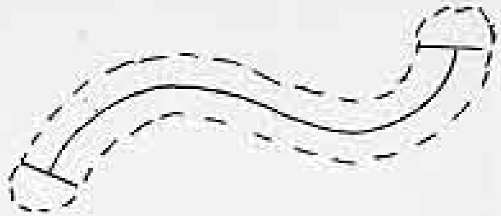
Steinhaus (1960) describes a method of measuring an area by using a lattice of points contained in the area as an approximation. For a square lattice in which a is the horizontal and vertical distance between points, the area of a region is proportional to $n/k\ (a^2)$ where n is the number of points in the region, and k is the number of trials or placing of the lattice on the region. For a triangular lattice in which b is the distance between points, the formula for the approximation to an area is $\frac{n\sqrt{3}}{2k} b^2$

Substituting these equations for the area value in equation (3) yields

$$(4) \qquad L_\epsilon(X) = \begin{cases} \dfrac{na^2}{2k\epsilon} - \dfrac{\pi}{2}\epsilon & \text{for a square lattice} \\[2ex] \dfrac{nb^2\sqrt{3}}{4k\epsilon} - \dfrac{\pi}{2}\epsilon & \text{for a triangular lattice} \end{cases}$$

The size of the lattice used for measuring the area is arbitrary and may be

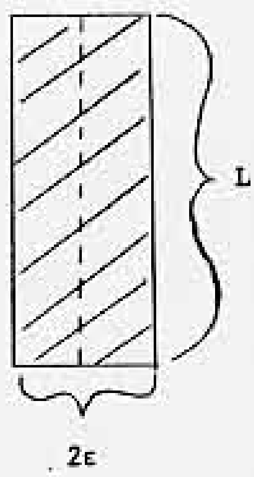
FIGURE 3



$a_\varepsilon(x)$ equals the area within ε distance of the curve. The ε-length of the curve is this area less the two half circles at the endpoints divided by the width of the strip, that is, by 2ε.

(3) $$L_\varepsilon(x) = \frac{a_\varepsilon(x) - \pi\varepsilon^2}{2\varepsilon}$$

This is analagous to finding the length of a rectangle by dividing its area by its width. $L = \frac{A}{W}$ where L = length, A is area, W is width (2ε).



chosen to be a function of the ε. Perkal was very clever at adjusting these constants so that they cancelled one another out and yielded a very simple formula for the measure of the ε-length of a line using the lattice point approximation. By setting $a = 2\varepsilon = k$ for the square lattice; and by setting $b = 2\varepsilon$ and $k = \varepsilon\sqrt{3}$ for the triangular lattice (k is rounded to the nearest integer) the following formula results

$$L_{\varepsilon}(X) = n - \frac{\pi}{2}\varepsilon \begin{cases} a = 2\varepsilon = k & \text{for square lattice} \\ b = 2\varepsilon,\ k = \varepsilon\sqrt{3} & \text{for triangular lattice} \end{cases} \tag{5}$$

The ε-length of a line may be measured directly using this formula and the appropriate-sized lattice with ε-neighborhood circles drawn around each lattice point. A template with the lattice of circles may be used. The lattice is placed on the curve to be measured and each circle containing any segment of the curve is counted. Repeat the process <u>k</u> times with random placement of the lattice. If the curve is closed the correction factor $\pi/2$ (ε) is left off. If ε is in millimeters, the ε-length is in millimeters. Convert this value to actual ε-length by the scale of the map or photograph used.

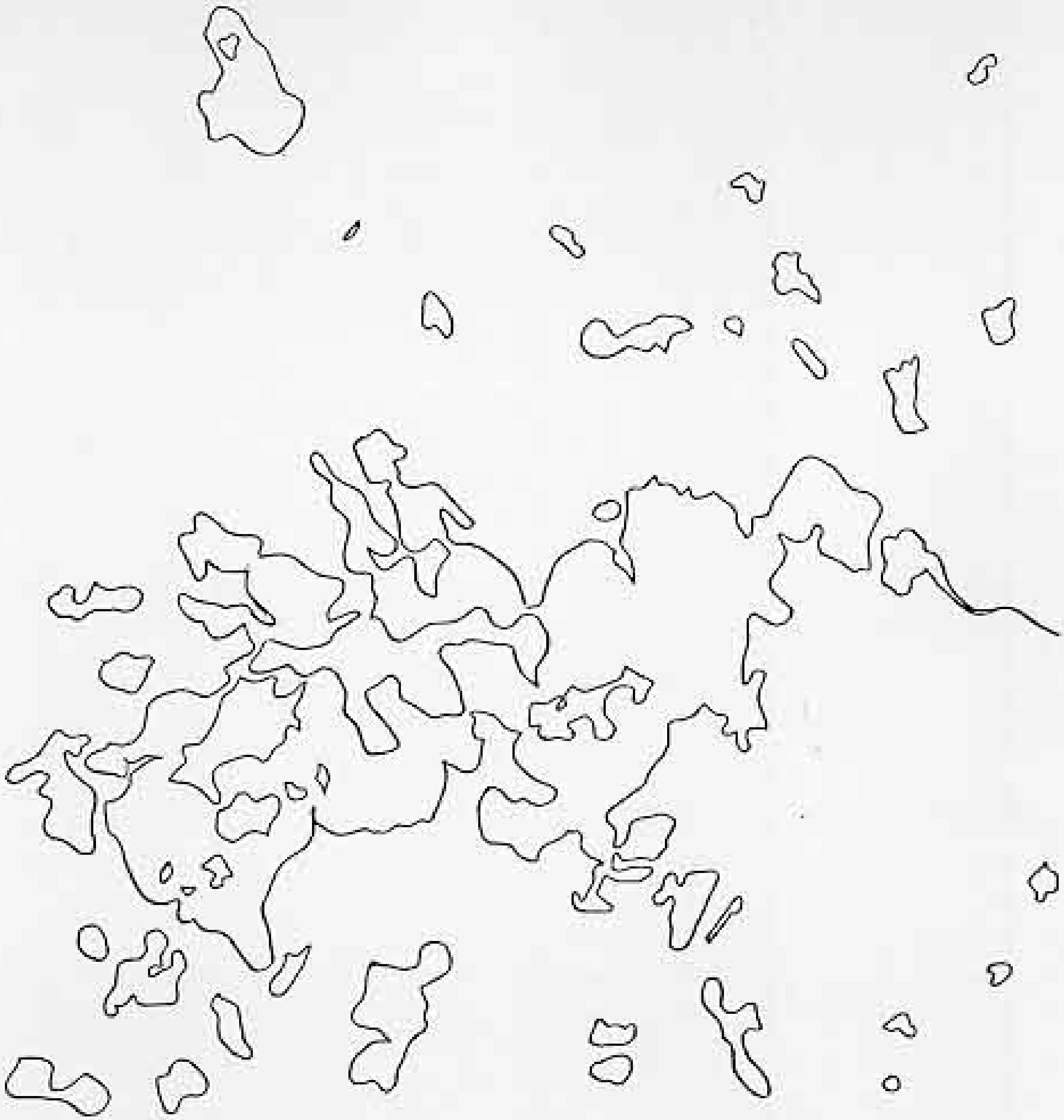
Figures IV and V and Tables 2 and 3 show the results of an application of these measures for Lake Minnetonka and a portion of the East coast of the United States. As expected, the smaller the ε, the longer the ε-length.

<u>Mechanically Defined Epsilon-Values.</u>

Some immediately practical consequences result from adoption of the notion of the ε-length of a line. Consistent and comparable measures of length can be made for line phenomena by suitable choice of ε.

Each map measuring device, such as a hand held map measurer, the plotting head of an automatic plotter, and so forth, have a minimum turning radius as a consequence of the mechanical properties of the measuring device. This turning radius is the ε-value of the instrument. Lengths measured by the device depend upon this value. Naturally different lengths of the same boundary will result from using different

FIGURE 4



Lake Minnetonka

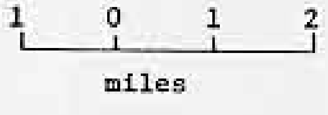


FIGURE 5

Portion of the East Coast of the United States

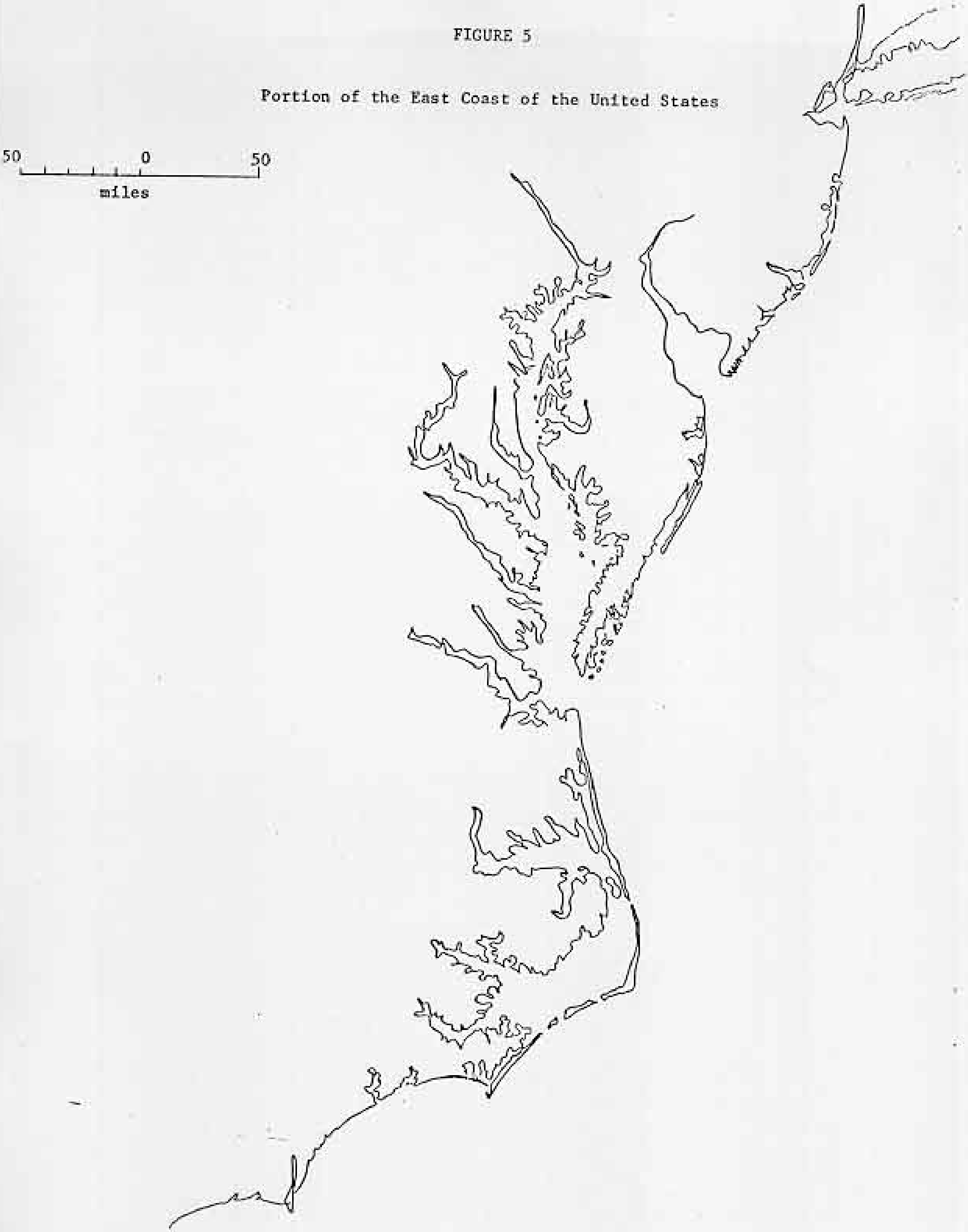


TABLE II

Measure of the ε-length of a Line Using Lattice Point Approximations

Lake Minnetonka
map scale 1:126,720

Triangular Lattice

trials

K	n
1	92
2	100
3	104
4	103
5	96
6	97
7	100

$\sum n = 692 - 6 = 686$ mm

$\varepsilon = 4$ mm
4x 126,720 .507 km
or .32 miles
~ 1700 feet

Total length of lake boundary (main lake only)

686 x 126,720 86.93 km
or 54.0 miles

Square Lattice

trials

K	n
1	62
2	63
3	63
4	58
5	62
6	64
7	65
8	61
9	63
10	59

$\sum n = 619 - 8 = 610$ mm

$\varepsilon = 5$ mm
5 x 126,720 .634 km
or .39 miles
~ 2060 feet

Total length of lake boundary (main lake only)
610 x 126,720 77.3 km
or 48.0 miles

TABLE III

Atlantic Coast Approximate
ε-length Using Triangular Lattice

map scale 1:3,168,000

trials

K	n
1	163
2	153
3	160
4	164
5	155
6	168
7	170

$\sum n = 1123 - 6 = 1117$ mm

$\varepsilon = 4$ mm
4 x 3,168,000 = 12.67 km
or 7.9 miles

Total length of portion of coast shown,
(islands not included)

1117 x 3,168,000 3538×10^6 mm
3538.7 km
or 2199±8=2200 miles

instruments or different scales of map or photographs of the object to be measured. No true length exists and if each measure by several methods are carefully done, there is no basis for choosing the best value independent of the subject matter under study. That is, the researcher must decide if there is a functionally significant minimum radius involved in the objects under observation. If such a radius can be identified the ϵ may be consciously specified by the researcher rather than being arbitrarily established by mechanical limits of the methods of observation.

Other means of empirical observation are subject to the same restraints. The grain size of photographs, the resolving power of lens and other remote sensing devices such as infra-red sensors establish minimum ϵ for measurement and all length depends upon this ϵ. Clearly, if length is an important variable in the analysis, the researcher, not his machines, must specify the most suitable ϵ to be used.

For some purposes a consistent ϵ is all that is needed. Perkal (1958a) suggests a method using the ϵ length concept in which consistent generalization of a map can be achieved when constructing a small scale map. What is required in this case is not the absolute ϵ-radius of the objects studied but rather a technique for consistent generalization of all parts of the map. Tobler (1964) has found using the mechanical ϵ of graphic plotters yields satisfactory results in map generalization.

Functionally Defined Epsilon-Neighborhoods.

The concept of the ϵ-length of a line provides an operational definition which may be used to measure empirical lines such as seacoasts and rivers. The length of a seacoast varies depending upon the purposes involved. For example, for any stretch of the Atlantic seacoast, the length is longer from the point of view of somebody in a rowboat than for somebody in an ocean liner. The difference depends upon the turning radius of the vehicles.

The length of this coast is also different from the point of view of railroads. Another factor is involved here, namely, the boundary of the ϵ-neighborhood may be thought of as the trace of points nearest the coastline left by the edge of a circle of ϵ-radius which is rolled along the coastline on both sides. Because of the shape of the capes and bays, the length of this trace will not be the same on each side of the seacoast. Ships swing wide to avoid the capes but trains swing inland to avoid bays and estuaries. The coastline, then, appears to have an inside length and an outside length for purposes of movement along its perimeter depending upon the technical requirements of the vehicle used. The statement may be made more general. The boundaries of the ϵ-neighborhood of a line are ordinarily not equal; the one-side length equals the other-side length only if the line is ϵ-convex. I will return to this peculiar point later.

Clearly, one method of defining a functionally significant ϵ-radius has to do with the turning radius of vehicles. This radius depends upon technical consideration of mass and energy and of uncertainties in steering including both navigational needs and shoulder and head room requirements at operational speeds. The length of barriers, therefore, are different for purposes of an expressway as compared to a highway; for very high speed ground vehicles, such as proposed for the Northeast Corridor route, as compared with a typical railroad; for vehicular travel as compared with walking and so forth. Distances between points in a region vary by mode of travel, in part, as a consequence of these technical reasons.

Transportation characteristics are one way a functional ϵ-radius may be specified for a study. Another, somewhat more subtle but probably more important, functional meaning can be applied to the ϵ-neighborhood. The ϵ-neighborhood of a point is the area of a circle of ϵ-radius. Most objects under study in spatial systems have internal spatial requirements. Take, for example, various land uses in a metropolitan region. Farms, factories, residential sub-divisions, individual residential lots, and so forth have typical internal dimensions within some range of variability. These site requirements must be taken into account in defining the spatial patterns of the land uses in the region. Suppose, for example, you are planning to identify the shape of the urbanized area of the region and to do this you are to employ a remote sensing device with a resolution power of one square foot. Let us say this instrument can identify soft ground from hard surfaces. The decision is made that hard-surfaces are urbanized area and soft ground are areas of vegetation and non-urban land. The mechanical ϵ-neighborhood of the instrument covers one square foot and it assigns each square foot to one of the two classes depending on the proportion of hard and soft within the unit area.

If the entire metropolitan region were then monitored and plotted on a map, the urbanized areas would not be found. The lawns of residential plots would have been classified as non-urbanized area, country roads and farm buildings would have been classified as urban. The problem is that the concept urbanized area contains an implied ϵ-neighborhood, which depends on the site requirements of the various activities found in metropolitan areas. If an ϵ-neighborhood of the instrument had been made larger so that it was equivalent to the average-sized residential lot and if the typical proportion of soft to hard surface for the residential site requirements were specified; the urbanized zone produced by these automatic means would be in much better correspondence to the theoretical concept of an urbanized area.

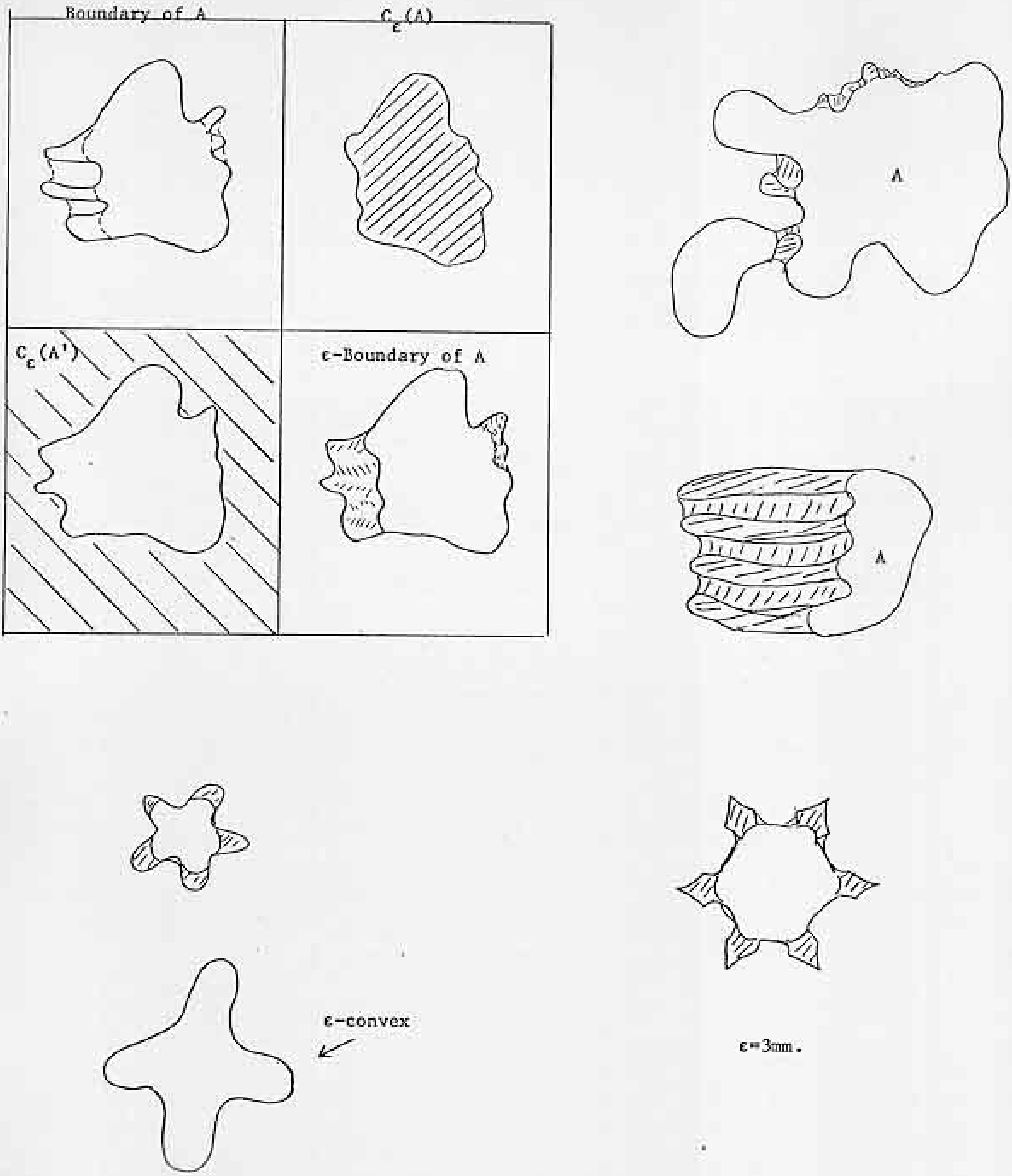
These observations provide insight into the dichotomy of site versus situation commonly found useful in spatial analysis. Variations within an ϵ-neighborhood are site variables and are specified by indices of texture, roughness, intensities, etc. Variations between ϵ-neighborhoods are situational or locational variables and these variables define shape, relative position, length, etc.

The Boundary Zone.

The idea of an ϵ-neighborhood of a point leads to an operational definition of boundary zone. Consider domain A surrounded by an irregular boundary. The ϵ-convex set of A is defined in the following manner. Let p be a member of a set of points contained within but not on a circle of radius ϵ, which in turn is wholly within the domain D. Then the points p, in the collection of all such circles define the open ϵ-convex set of A. Denote this set by $C_\epsilon(A)$. All the points of A need not be in $C_\epsilon(A)$. Those points are excluded that are in places where the boundary of A curves back on itself with less than 2ϵ distance across the loop. Because this set is open the boundary points of A are also excluded.

FIGURE 6

Boundary Zones Defined by ε-Convex Edge of Domain A and its Complement A'



The complement of A is A'. A $C_\varepsilon(A')$ exists but is equal to the complement of $C_\varepsilon(A)$, i.e., $C_\varepsilon(A)'$ only when the boundary is an ε-convex curve as defined previously. If the boundary is not an ε-convex curve, a space or collection of points is left between $C_\varepsilon(A)$ and $C_\varepsilon(A')$ which is the ε-edge or ε-boundary zone. This edge is reduced to the boundary line for ε-convex curves.

The boundary zone may be wide or narrow depending on the shape of the boundary and, of course, on the ε chosen. Figure 4 provides some examples. In some cases the ε-boundary zone contains more of one domain than its complement. Segments of either domain or both may be isolated as well by the boundary zone.

A boundary zone defined in this fashion suggests many ways in which the efficiency of a shape for various purposes might be analyzed. If the permeability of the boundary zone is less than the permeability of a unit area of the domain on either side, the most efficient locations for crossing the boundary would be in places where the boundary zone is the narrowest.

The relative strength of each domain in the boundary zone may be established by the proportion of each domain found there or perhaps by the relative difference in the length of the convex envelope on the one side of the edge zone compared to the other side.

Phenomena found to be confined to zones within a certain distance of a boundary may be designated as boundary dwellers and a functional ε for them would be indicated. It would remain, of course, to establish theories for each particular subject matter as for why certain activities are found at boundaries only.

Many other interesting topics are suggested by the ideas I have presented here. Central to Julian Perkal's ideas is the ε-neighborhood. It may be characterized as site convexity or convex in the small. I have great expectations for its applications to spatial analysis of all types. I think it will be particularly useful in studying the effects of boundary shapes on spatial processes, subjects which have received little attention to date, and which I have tried to introduce to you today.

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