Forward

Stig Nordbeck was Visiting Professor of Geography at Wayne State University during the spring of 1964. His series of lectures at the Michigan Inter-University Community of Mathematical Geographers Seminar in Brighton, Michigan was most stimulating and we encouraged him to expand this portion of his remarks so that the world community of mathematical geographers would also have their benefit.

Those who had the good fortune to enjoy the Nordbeck lectures first hand had the opportunity to be struck anew with the benefits of international exchange. Here was a man from the Swedish school who thought on parallel yet separate tracks to the Americans at Brighton. The process was mutually stimulating.

The flavor of the Brighton Seminars is that the work itself dominates. The production of this series of discussion papers has no schedule whatsoever, as those who receive it are amply aware. We produce a paper when we feel we have something to say of interest to you. Our seminars are most informal and student and faculty distinctions are kept at a minimum. Intellectual criticism can come strongly from any quarter.

This paper by Stig Nordbeck is in this intellectual tradition. Nothing is sacred to Nordbeck's mind. If he finds his geographic law fits rivers and cities and volcanoes, he says so; indeed, he proves so, even in the face of the fact that every university catalogue "proves" one cannot mix the physical and social sciences. What are the pedagogical and philosophical implications of Nordbeck's work? We do not fully know; but they will be disruptive and, therefore, criticized. If Galileo cried to his critics, "But it moves!" Nordbeck might cry to the critics of the spatial patterns he has discovered on diverse maps, "But they fit!"

William Bunge
Wayne State University
June, 1965

# The Law of Allometric Growth

by

Stig Nordbeck

The law of allometric growth was originally discovered by biologists. It states that the rate of relative growth of an organ is a constant fraction of the rate of relative growth of the total organism. Assume that y is the size of the organ and x that of the organism. Then the allometric growth law can be written as (1):

$$y = ax^b \qquad (1)$$

where a and b are constants (a is always positive). The formula (1) is derived by means of calculus as follows: It is known that the rate of growth of an organ at time t is $\frac{dy}{dt}$, where y is the size of the organ. The rate of relative growth thus is $\frac{dy}{dt}/y$ for the organ and $\frac{dx}{dt}/x$ for the organism. The principle of allometric growth now gives formula (2):

$$\frac{dy}{dt}/y = b\frac{dx}{dt}/x \qquad (2)$$

That formula (2) is equal to (3) is easily seen by substituting $\frac{dx}{dt}\cdot\frac{dt}{dy}$ in (d) for $\frac{dx}{dy}$,

$$\frac{1}{y} = b\frac{dx}{dy}/x \qquad (3)$$

Integrating (3) gives equation (4), where log a is a constant integration.

$$\log y = \log a + b \cdot \log x \qquad (4)$$

That this equation (4) is identical to formula (1) may be proven by taking logarithms of (1).

The law of allometric growth is valid for phenomena other than organs and organisms. Beckman (1958), for instance, assumed that the total urban population in a country was an "organism" and the cities its "organs." These assumptions allowed him to verify the "rank size rule" for cities. This rule states that the size of city number n is approximately one nth of the size of the largest city. In other words: The rank-number n of the city times its size (population) is constant. This empirical law was first observed by Auerbach (1913) and Zipf (1941).

Horton (1945) and Strahler (1954) introduced another such system in their analysis of drainage basins. The smallest "finger-tip" tributaries are designated as order 1. Where two channels of order 1 join, a stream segment of order 2 is formed, where two of order 2 join, a segment of order 3 is formed, and so on (See figure 1). Horton's law of stream numbers in a drainage basin states that the number of stream segments of each order forms a geometric series. There is 1 segment of the highest order, s of the next highest order, $s^2$ of the next order and so on. This bifurcation ratio rule for streams is similar to the rank size rule for cities. According to Beckman's model, built upon (among others) Lösch's theory of location, cities (when divided into different size classes) show the following pattern: There is one largest city, there are s cities of the next largest size, $s^2$ cities in next class, and so on. The bifurcation ratio rule can also be derived by the law of allometric growth as the rank size rule for cities.

In order to make it more general and easier to use, the law of allometric growth will here be formulated in two other ways. The first new formulation is the following. If the growth of an individual follows the law of allometric growth according to formula (1), it is possible to

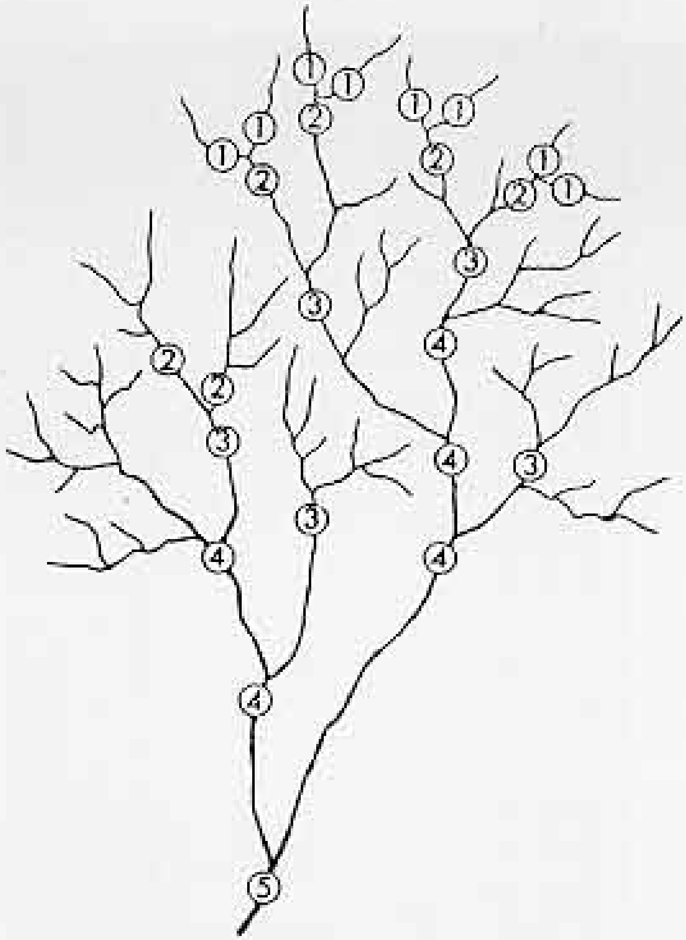


Figure 1
An example of stream branch orders as defined by Strahler and Horton.

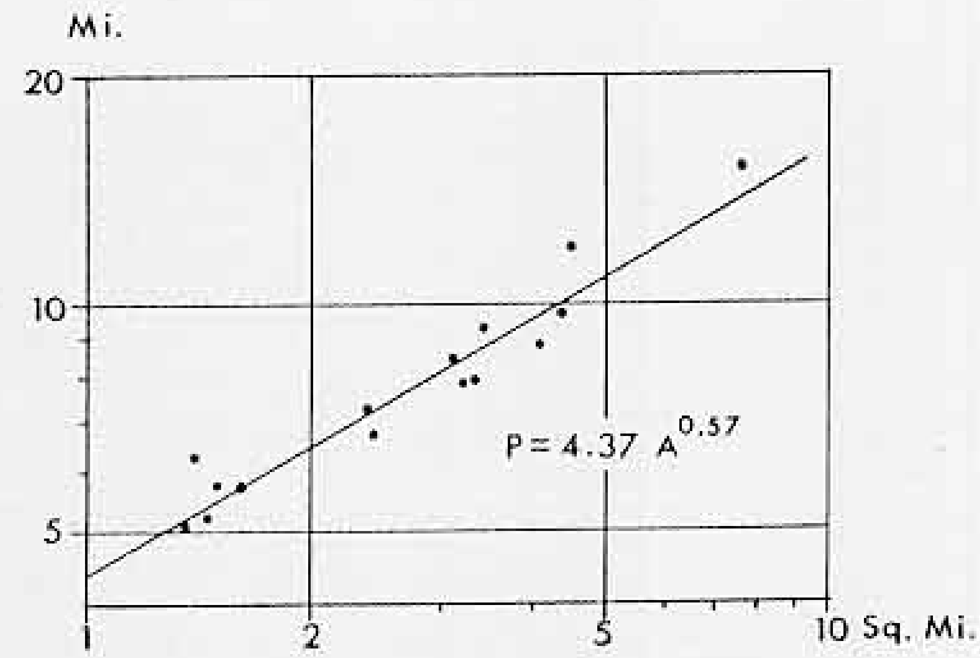


Figure 2
The relationship between perimeter and river length of drainage areas in fourth order basins of the Allegheny River.

measure the value of one variable at different times and to estimate the values of another variable by means of the measured values. This rule is used by parents when they measure their two-year old child's length and say that the length of the grown-up child shall be twice this length. The old Swedish farmers use this rule too, when they measure the body perimeters of their pigs or cattle. Knowing the length of the perimeter, a farmer can estimate the slaughter-weight of a pig with extreme exactitude. As a matter of fact such an estimation, based on measuring the perimeter of an animal, is much more accurate than an estimation based on the weight of the pig (cow) alive. A pig (cow) has the same (breast) perimeter when it is hungry and thirsty as when it is satisfied. Since the price of pork is highest for a special weight class, farmers try to slaughter their pigs when they can get the highest profits possible. It is therefore very important for them to calculate the slaughter-weight correctly. This estimation must be done 2-3 weeks before the slaughtering. The highest paid weight class varies in such a way that it is greater when there is a shortage of pork in Sweden than when there is a surplus. But the farmers, knowing the law of allometric growth, are able to allow for this changing weight standard.

The second new formulation of the law of allometric growth is the following: Instead of assuming the measurement of a growing individual at different times, it is assumed that a series of individuals all have the same shape but are of different size. In this case the law of allometric growth states that it is possible to estimate the values of a variable by means of the measured values of another variable. This rule makes it possible to measure a variable which is very difficult to measure. It is used by foresters when they first classify the trees they are going to cut down as very high trees, high trees, mean high trees etc. and then measure the diameter of each tree and use these values to estimate the total volume of the trees. It is also used by the aforementioned farmers. All animals do not have exactly the same shape. A pig can be very short compared with the mean pig. In this case the farmer takes this into consideration and allows the perimeters to become a little larger than in the mean case.

The law of allometric growth, and especially the version of it which is based on the same shape of a series of individuals, will here be applied to some geographical objects. Some simple examples and illustrations of this principle are given in table 1. In this table S means the shape of the individuals, M is the measured variable, DM is the dimension of M, L is a length variable, A is an area variable and V is a volume variable.

Rivers and the law of allometric growth.

All drainage areas ought to have the same "shape." The same forces are always at work when water runs from the land to the oceans. These forces are independent of the ground over which the water passes, but the drainage area will become larger if the ground is soft and smaller if it is hard. The "shape" of the drainage basins is, however, the same. Hence the law of allometric growth is valid for drainage basins. Strahler (1957) writes, that studies of actual drainage basins in homogeneous rock masses show that geometrical similarity is closely approximated when mean values are considered. He gives the term "geometrical similarity" the same meaning which here has been given to the expression "same shape." If the ground is geologically not homogeneous, the shape of drainage areas probably will be deformed. But this deformation follows allometric growth

| S | M | DM | L | A | V |
|---|---|---|---|---|---|
| square | side = x | 1 | diagonal = d<br>$d = \sqrt{2} \cdot x$ | area = A<br>$A = x^2$ | - |
| cube | area = x | 2 | side = s<br>$s = \sqrt{\frac{x}{6}}$ | $A = x$ | volume = V<br>$V = \frac{x\sqrt{x}}{6\sqrt{6}}$ |
| circle | radius = x | 1 | perimeter = p<br>$p = 2\Pi x$ | $A = \Pi \cdot x^2$ | - |
| sphere | radius = x | 1 | $p = 2\Pi x$ | $A = 4\Pi \cdot x^2$ | $V = \frac{4}{3}\Pi x^3$ |
| sphere | volume = x | 3 | radius = r<br>$r = \sqrt[3]{\frac{3x}{4\Pi}}$ | $A = \sqrt[3]{36\Pi x^{\frac{2}{3}}}$ | $V = x$ |
| hexagon | side = x | 1 | $p = 6x$ | $A = \frac{3\sqrt{3}x^2}{2}$ | - |
| cube | volume = x | 3 | $s = \sqrt[3]{x} = x^{0.33}$ | $A = \sqrt[3]{x^2} = x^{0.67}$ | $V = x$ |

Table 1. If the dimension of the measured value x is 1, the length variable $L(x) = a \cdot x$, the area variable $A(x) = a \cdot x^2$ and the volume variable $V(x) = a \cdot x^3$ where a is different constants. If the dimension of x is 2, $L(x) = a \cdot x^{0.5}$, $A(x) = a \cdot x$ and $V(x) = a \cdot x^{1.5}$. If the dimension of x is 3, $L(x) = a \cdot x^{0.333...}$, $A(x) = a \cdot x^{0.667}$ and $V(x) = a \cdot x$.

rules too, and the law of allometric growth is still valid. This is illustrated by the following examples.

The perimeter P of a drainage area is an one-dimensional variable, and its area A is a two-dimensional one. Thus the equation $P = aA^b$ is valid. This equation is equal to the linear equation $\log P = \log a + b \log A$, where b ought to be very close to 0.5. Morisawa (1959) gives values of area and perimeter of 16 fourth-order basins in the Allegheny River watershed. These data are plotted on a log-log diagram (figure 2), and the line $\log P = \log a + b \log A$ is calculated by means of the least squares smoothing technique. The resulting line $P = 4.37A^{0.57}$ is drawn in the diagram. The b-value was expected to be very close to 0.5, and the received b-value 0.57 is therefore too high. This can be explained in the following way: The short perimeters are too short, probably due to the method of measurement. They were all measured on maps of scale 1:24 000 and these maps showed, of course, the same degree of generalization for all basins, independent of their size, which caused the perimeters of the small basins to appear too short. More precise measurements would probably give a somewhat lower b-value (closer to 0.5). If the two largest basins and the second smallest one are excluded the b-value also is quite close to the theoretical value 0.5. These three basins give also a b-value equal to 0.5 if their data are smoothed by the least squares method. Hence, the data in figure 2 can belong to two different sets of basins, but the very low number of observations makes it impossible to find out if there are one or two sets of basin types. The correlation between log P and log A is very high. The correlation coefficient r is equal to 0.9642.

The length L of a drainage area can be defined as the length of the main river (the longest river) from its smallest source-stream to its mouth. It is easier to measure this length L correctly than to measure the perimeter P of a drainage basin. It follows then that the correlation between log L and log A will be higher than the correlation between log P and log A. The calculated b-value will not differ so much from the theoretical b-value 0.5 as it did in the example in figure 2.

In figure 3 the lengths of 91 rivers were plotted against the areas of their drainage basins. These rivers are of quite different size from the Amazon river which has the largest drainage area (7 050 000 sq.km) to a small Swedish stream (470 sq.km) flowing from the South Swedish highland (a tributary of the Lagan river). All Swedish rivers which were recognized on a map of scale 1: 2 000 000 were used in the diagram if it was possible to find data for them, a total of 49 rivers. The remaining 42 rivers are all from other continents. They were chosen by a method similar to that used for the Swedish rivers. No classification of the rivers was made before they were plotted in the diagram, and no river was excluded because it did not fit the curve.

The least squares method applied to data in figure 3 gives the equation $A = 0.1039L^{2.0009}$. The area A is given in sq.km. and the length L in km. The computed b-value (2.0009) is as close to the theoretical b-value (2) as could be expected. Consequently the correlation between log A and log L is very high. The correlation coefficient r is 0.9905.

By means of the residual method, the rivers of the diagram in figure 3 can be divided into three groups. A plain river has a large drainage area compared with its length. The Amazon river and the Swedish rivers Lidan and Fyrisån belong to this group. A valley river has a small drainage area compared with its length. The Mekong river and the Dnjestr river belong to this group. Most of the rivers in the example, figure 3, belong to the

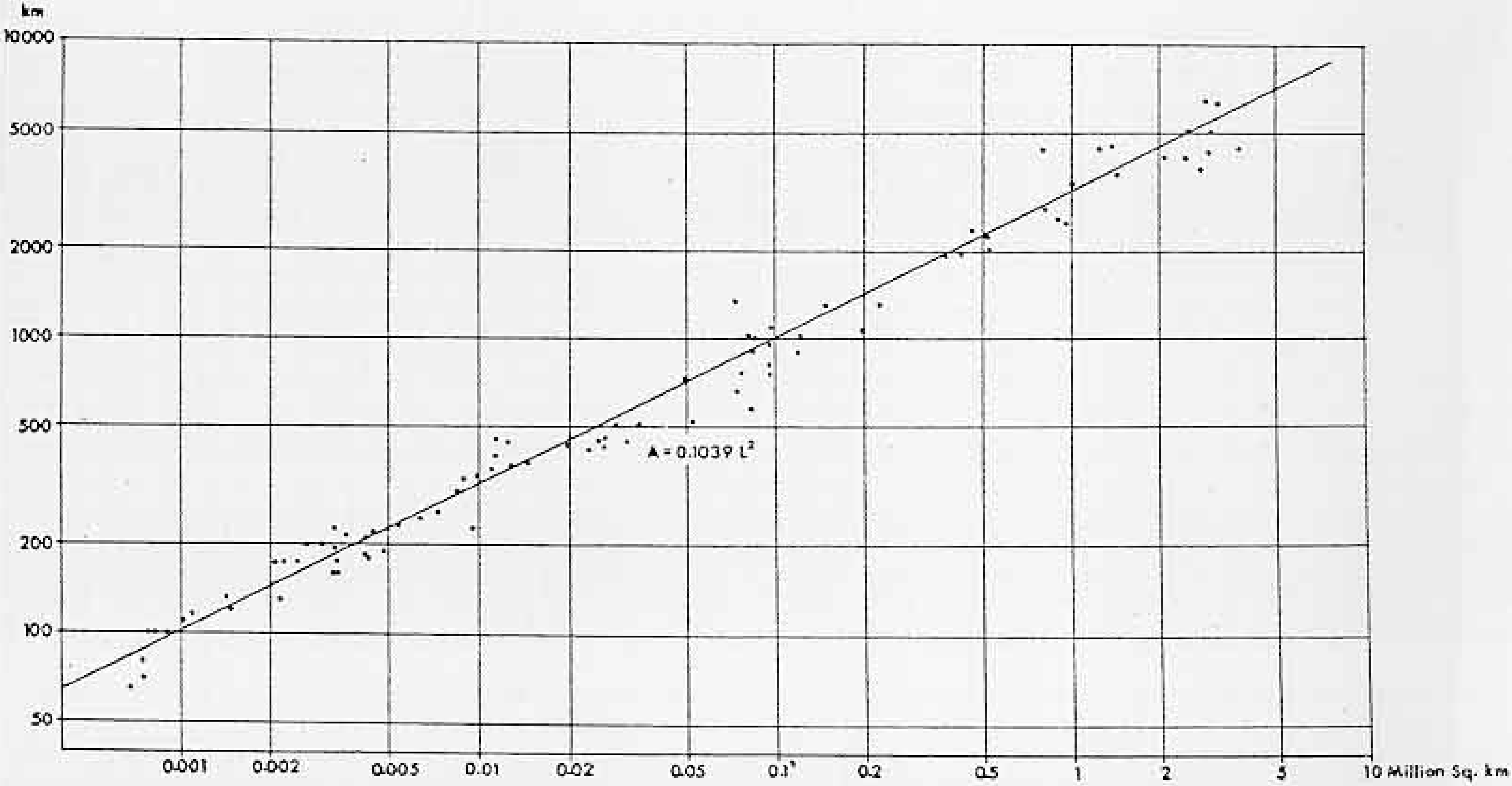


Figure 3
The relationship between river length, L, and drainage area ($km^2$).

third group, the mean rivers.

In the example above, the area of the drainage basins has been written as a function of the basin length. It is also possible to write the length as a function of the area. In this case the least squares method and the data in figure 3 give the formula $L = 3.44A^{0.49}$.

Allometric growth and meanders.

Leopold (1964) writes that nearly all natural channels are sinous to some extent. Not only do they exhibit a more or less regular aspect of sinousity, but the size of the curves assumed by a channel bears a constant relationship to the channel itself. Small channels wind in small curves and large channels in large curves. The meander length or wave length L is generally proportional to channel width W and so is the radius ($r_m$) of the curves which the channel exhibits.

All flowing water meanders. The melt-water running on glacier ice, and the water in the Gulf Stream both show the same meandering pattern as the water in rivers, in spite of the absence of sediment load and river banks. The nature of meanders and meandering forces make it very likely that all meanders have the same shape. Consequently, the law of allometric growth is valid for meanders. Figure 4 is a revised version of a figure in Leopold (1964). In this figure the meander length L in feet is plotted on a log-log diagram against the channel width W, in feet. Both L and W are one-dimensional variables. It follows then that the theoretical b-value in formula $L = aW^b$ is equal to 1. Leopold (1964) found the b-value to be 1.01 and the a-value 10.9, as is seen in figure 4. If the mean radius of curvature $r_m$ is plotted against the meander length L, the b-value is 0.98(figure 5). In both these cases the calculated b-value is so close to the theoretical b-value as to be considered equal. The line

$L = 10.9W^{1.01}$ is drawn in figure 4 and the line $L = 4.7r_m^{0.98}$ in figure 5.

Volcanoes and allometric growth.

The shape of a volcano depends on the materials of which it consists. A viscous lava forms very high cone volcanoes because the slope angle of such a lava is very high. The angle of slope of a recently formed cinder cone ranges between 26° and 30°. Strato-volcanoes are more extensive than cinder cones and may have slope angles as high as 35°, the natural angle of slope of volcanic ash, but their slopes generally range between 20° and 30°. The lava of shield volcanoes is highly fluid and travels far down the low slopes, which do not usually exceed 4° or 5° (Strahler 1960). Hence, the shape of volcanoes can be considered a transportation problem. This can be compared with an economic application: The rings constructed by von Thunen. The shape of a volcano can be derived in the same way, using the same technique that he did when he got his rings.

All volcanoes belonging to the same class can be assumed to have the same shape. Consequently the law of allometric growth is valid for such volcanoes. In figure 6 the height H in meters of six volcanoes is plotted against their volume V in cubic meters. The least squares method gives the formula $H = 0.102V^{0.3946}$. The height H is an one-dimensional variable and the volume V a three-dimensional one. It follows then that the expected b-value is 0.3333... There are at least three different explanations for the fact that the calculated b-value for volcanoes differs so much from the theoretical b-value.

1. It is here assumed that the length of a length variable $L(x)$ can be calculated by means of the formula $L(x) = ax$ and an area variable by aid of $A(x) = ax^3$. This is true only in such cases where the height $H(x)$ is a length variable and thus the formula $H(x) = ax$ is valid. If the

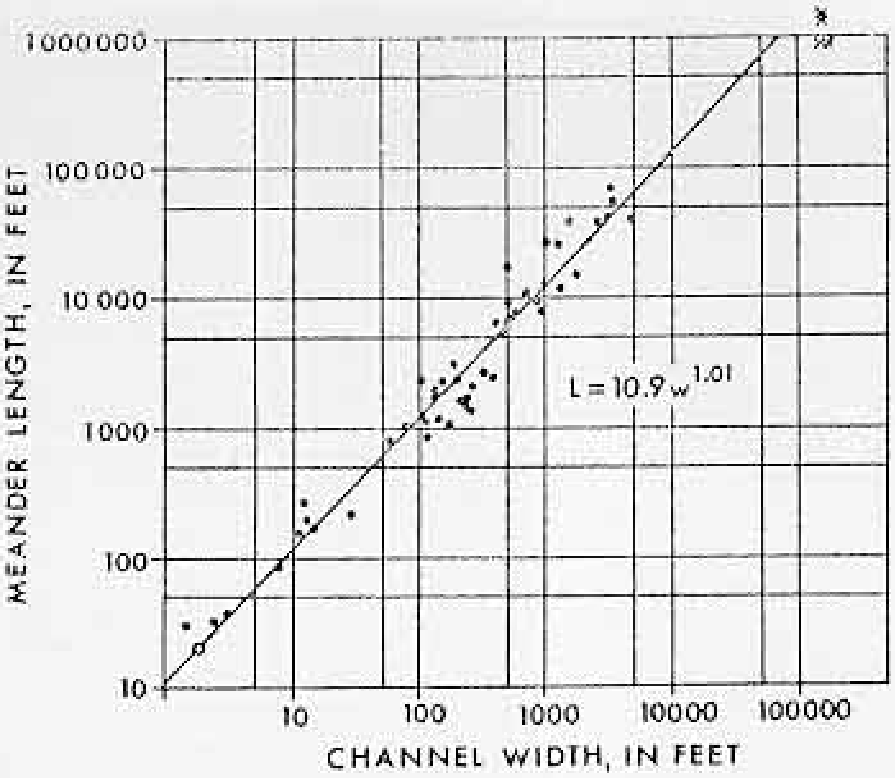


Figure 4

The relationship between meander length, L, and meander width, w.

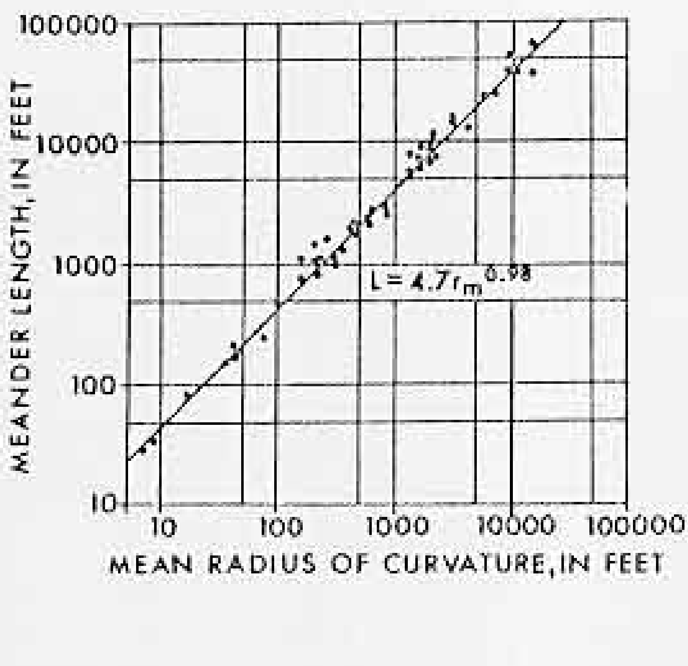


Figure 5

The relationship between meander length and the radius of curvature in channels.

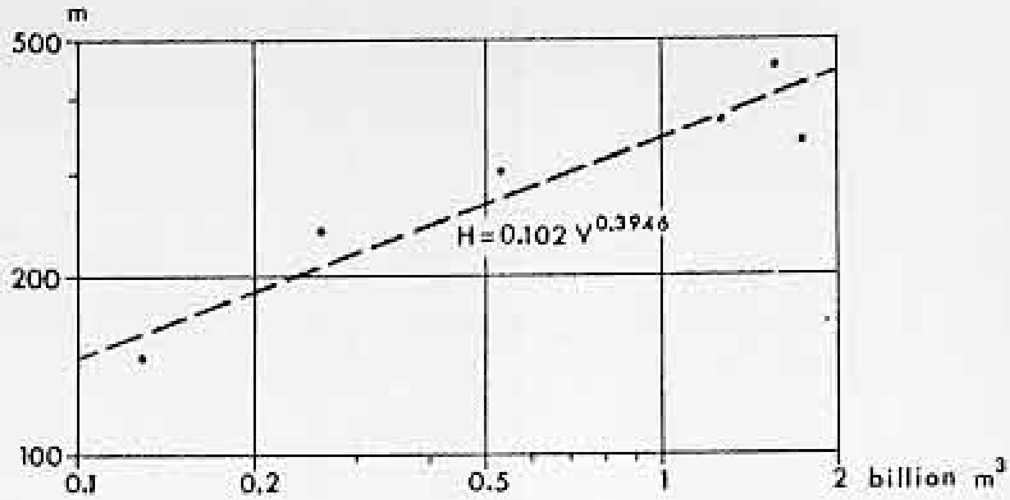


Figure 6
The relationship between vulcano height (meters) and vulcano volume (meters$^3$).

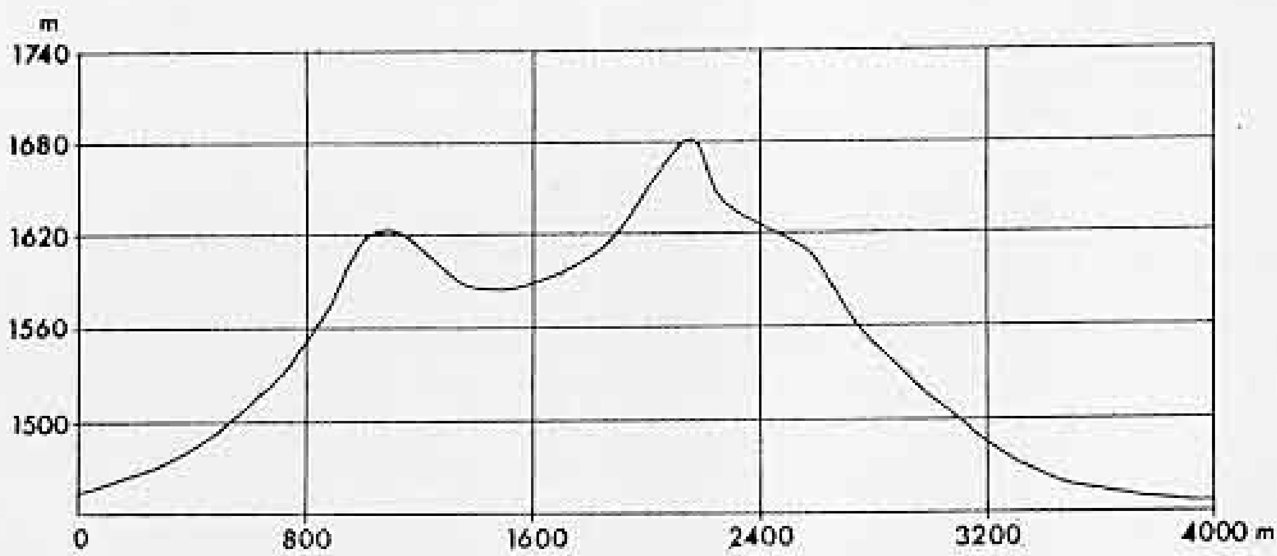


Figure 7
A section through Butte I, a vulcano in Idaho.

equation $H(x) = ax^{0.5}$ is valid the volume formula is changed to $V(x) = ax^{2.5}$. Hence, $L(x) = aV^{0.4}$, where V is the volume of the volcano. This b-value is equal to the calculated b-value in the example (figure 6), but there are no reasons why the height of a volcano should not be a length variable. Thus it is not true that the height formula $H(x) = ax^{0.5}$ is valid. There must be other reasons by which the over-estimated b-value can be explained.

2. Errors in the measurement of the height, area, or volume of the volcanoes. It is impossible to define a volcano correctly. Where does the volcano start? These errors are of the same kind, and independent of the size of the measured volcano. All measured values are probably too low but the errors of the lowest values are greater than those of the higher ones, if these errors are measured in per cent of the actual value.

3. The volcanoes examined were old volcanoes. Figure 7 shows a profile of one of them: Butte I of Menon Buttes, Idaho, U.S.A. Figure 8 is a profile of Butte II. These two old volcanoes are considered as having the same shape. This was true when they were young. But erosion has changed their shape. This erosional wane is not allometric but arithmetical. That means that the correct value can be calculated by the formula $y = e + cx$, where both y and x are one-dimensional variables and e is the erosion constant. The height of one volcano is $H_1$ and of the other $H_2$. The two volcanoes are of the same age and the erosion has been the same in both cases. Thus the correct heights are $H_1 + e$ and $H_2 + e$, where e is the erosion constant. Assume that $H_1$ is x and $H_2$ is cx, where c is greater than 1. Both $H_1$ and $H_2$ are too low, but the relative error of $H_1$ is greater than the relative error of $H_2$. Erosion has also made the volumes of the volcanoes too low. However, the relative error of the volume is not as

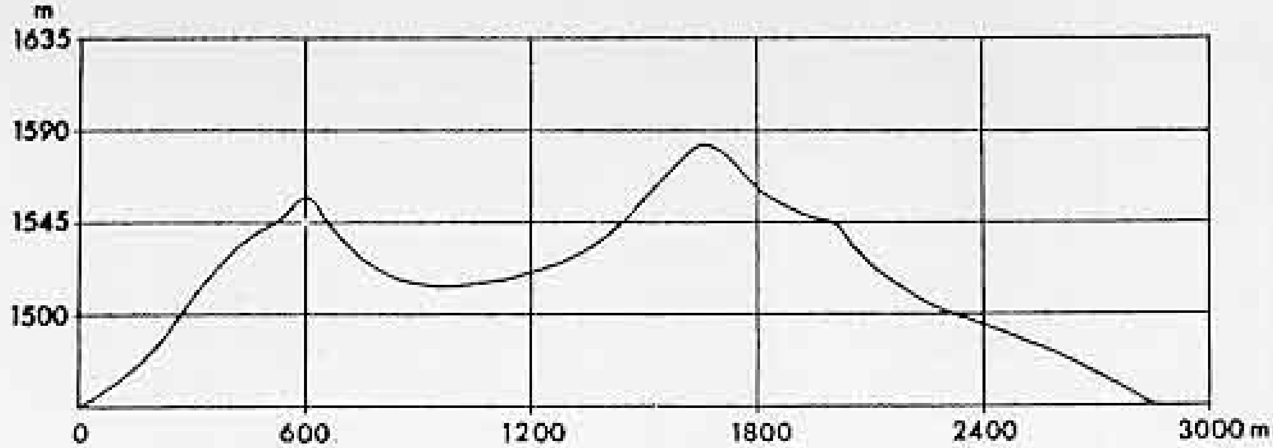


Figure 8

A section through Butte II, a vulcano in Idaho.

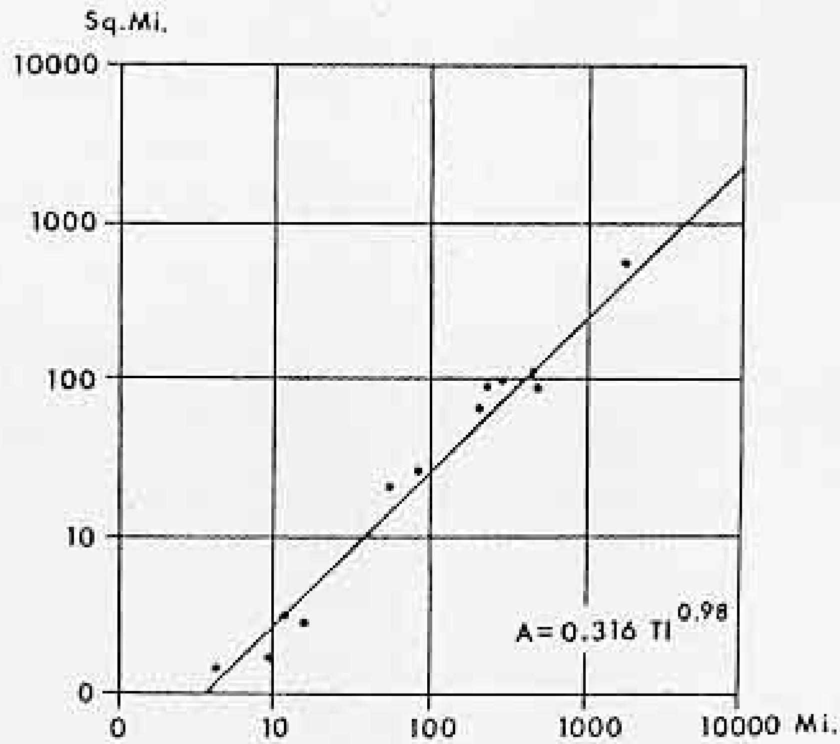


Figure 9

The total stream length, $T_1$, is a two-dimensional variable. Hence, the equation $A=aT_1$, where A is the area of a drainage basin, is valid. The least square method gives the equation $A=0.316T_1^{0.98}$ for this example.

great as the relative error of the height. It is only the transportation of materials away from the volcano that changes its volume. The transportation of materials from the top of the volcano to its base does not change its volume, but changes its height. Compared with the relative denudation of heights, the relative denudation of the volumes is low and can be considered equal to zero. Hence, the effects of erosion are that both the height and the volume of an old volcano are too low and that the relative error of the height of a small volcano is much larger than that of a bigger one. Consequently the b-value of the formula $H = ay^b$ will be higher than 0.333... which is the theoretical b-value if H and V are correctly measured and if the erosion has not changed these variables H and V. The correlation between log H and log V in the example, figure 6, is quite high. Thus the correlation coefficient r is equal to 0.9717. The line $H = 0.102V^{0.3946}$ is drawn in figure 6.

Recognition of dimensionality.

It is sometimes difficult to recognize dimensionality of the measured or of the estimated variable. The perimeter of a drainage basin was an one-dimensional variable (figure 2) and so was the length of the main river (figure 3). The total stream length of drainage areas is often erroneously recognized as an one-dimensional variable, too. An area has the dimensionality 2, a line 1 and a point 0. The total length is a sum of lines. However, it is not a length variable but an area variable as can be seen by the following examples. The size of an area A can be measured by aid of points arranged systematically or at random and by aid of a reference area B. The size of B is known, and so is the number of points which would belong to B if these points were arranged in exact same way as those belonging to A are. This number is here called $N_B$. The number of points belonging to A is $N_A$. It is possible to estimate the size of area A by means of the formula $A/B = N_A/N_B$ where A is the unknown size of A and B the known size of B. The error of this estimation depends upon $N_A$ in such a way that increasing $N_A$ will give a decreasing error. Consequently, the error of the estimation of the size of an area A by the aid of the formula $A = BN_A/N_B$ is very small if $N_A$ is large enough. Hence, $N_A$ is a two-dimensional variable. The number of street intersections in a city as a measurement of the area of that city is one application of this statement. Another one is the number of points in a drainage basin at which two streams join.

The size of an area A can also be measured by means of randomly or systematically arranged lines. These lines cover a reference area B, the size of which is known (area A usually belongs to area B). The total length of the lines which belong to A is called $T_A$ and the total length of the lines belonging to B is named $T_B$. If $T_A$ is large enough, the area of A can be calculated by aid of the formula $A = BT_A/T_B$. Consequently, the total length $T_A$ is a two-dimensional variable. The total length of the streets in a city can be used as measurements of its area, since they are organized in a regular system. The rivers and streams of a drainage area also form a regular system. Hence, the total stream length is a two-dimensional variable.

In figure 9 the areas of some drainage areas are plotted against the total length of corresponding streams. The sizes of the areas are called A and the total lengths of the streams are called $T_1$. The least squares smoothing technique gives the equation $A = 0.316T_1^{0.98}$; the line is drawn in figure 9. This figure is a revised version of a figure in Morisawa (1959). Since A and $T_1$ both are two-dimensional variables the expected b-value in the formula $A = aT_1^b$ is equal to 1. The calculated b-value is equal to 0.98 and this is so close to 1 that the two b-values can be considered equal.

Densely populated areas and allometric growth.

It is very easy to get data about the population of built-up areas in Sweden. These data are given in the official statistics of Sweden if the population of the built-up area is larger or equal to 200 persons. But it is very difficult or impossible to get data about the area of the built-up areas. It would therefore be very excellent if the unknown size of a built-up area could be estimated by aid of the known population.

The population of a built-up area depends on three variables: the size of the area which is a two-dimensional variable (length and width) and the population density which is defined in the following way: A reference area of specified shape and size is moved over the built-up area and the population density of a point (x,y) is the number of persons belonging to the reference area that has this point as its central point (Nordbeck, 1964). The distribution of population is a two-dimensional distribution and the density function is the frequency function of this distribution. It is very easy to show that all density functions thusly defined are continuous by integration (double integrals) (Nordbeck, 1965). This verification of the continuity of the density functions is easiest to do if the reference area has a regular shape and if its size is constant. In this sense a square with its sides parallel to the axes of the coordinate system is the best reference area.

A three-dimensional diagram such as an isarithmic map of a built-up area is quite small compared with the built-up area. Such a three-dimensional diagram corresponds closely to a contour map of a volcano.

There is always a downtown with quite low population densities. This is a result of the very high rents in the downtown area. Only offices, stores, etc., can afford these high rents. In very small built-up areas the "downtown" area consists of a town square, a church, and some very small stores, or of a railway-station and stores, etc. Churches, squares, and so on, belong to these downtown areas because they were there first and not because they can pay high rents. The "uptown" has the highest population densities. These are the slum or apartment areas as in Sweden. The rents in such areas are lower than in the downtown area. The buildings can therefore be used as dwellings. Increasing transportation costs cause the population densities to decrease with increasing distance from the uptown area until the suburban areas grade into rural land use. If a profile is drawn along a line through a built-up area using the population density as dependent variable this profile will be quite similar to those of the volcanoes in figures 5 and 6.

The population of an administrative unit consisting of an old city decreases as the population of the total built-up area grows (Kant 1962, Clark 1954, Berry 1958). This was observed for Amsterdam as early as the 16th century (Dickinson 1961). This decreasing population is explained by the fact that the growing downtown with low population density has pushed the uptown with high density to the outside of the boundaries of the administrative unit.

The law of allometric growth is valid for built-up areas if it can be assumed that they all have the same "shape." Consequently it would be possible to estimate the size of a built-up area of known population using the well-known technique of measuring one variable and estimating another variable.

The population of Swedish towns are shown plotted against their areas (figure 10). These data refer to administrative units. In most cases both the population and the size of such a unit are different from the population and the size of the corresponding built-up area. This deviation between the data used in figure 10 and the correct data for built-up areas is so large that it is impossible to use them in the least squares method or in any other smoothing technique to determine the a- and b-values of the formula $A = aP^b$, where A is the size of the built-up area and P its population. Most of the areas of the administrative units are too high compared with the size of the built-up areas. The biggest town in the world is Kiruna (13,181 sq.km.) but the built-up area of Kiruna is less than 20 sq.km. On the other hand, the area of the administrative unit can be smaller than the built-up area. This is valid for the Stockholm area, Gothenburg area, and for the surroundings of Malmoe. Consequently, the data of figure 10 can not be used when the line $A = aP^b$ is calculated by means of an objective smoothing method. However, this line $A = aP^b$ can be determined by the aid of a subjective method. Such a subjective method uses only the data for those administrative units which coincide with their built-up areas. This method gives the line $A = 0.03P^{2/3}$. It can be assumed that all built-up areas are circles or can be approximated by circles. It follows then that the radius of such a circle can be estimated by means of the known population, since the area $A = \pi r^2$ where r is the length of the radius. The line $r = 0.1P^{0.333...}$ is drawn in figure 10. Both of these equations $A = 0.03P^{0.666...}$ and $r = 0.1P^{0.333...}$ can be used as an approximation of the area A (in sq.km.) amd the radius r (in km.) of the built-up areas in Sweden. These estimations are not exact but they are much better than estimations based upon the areas of administrative units.

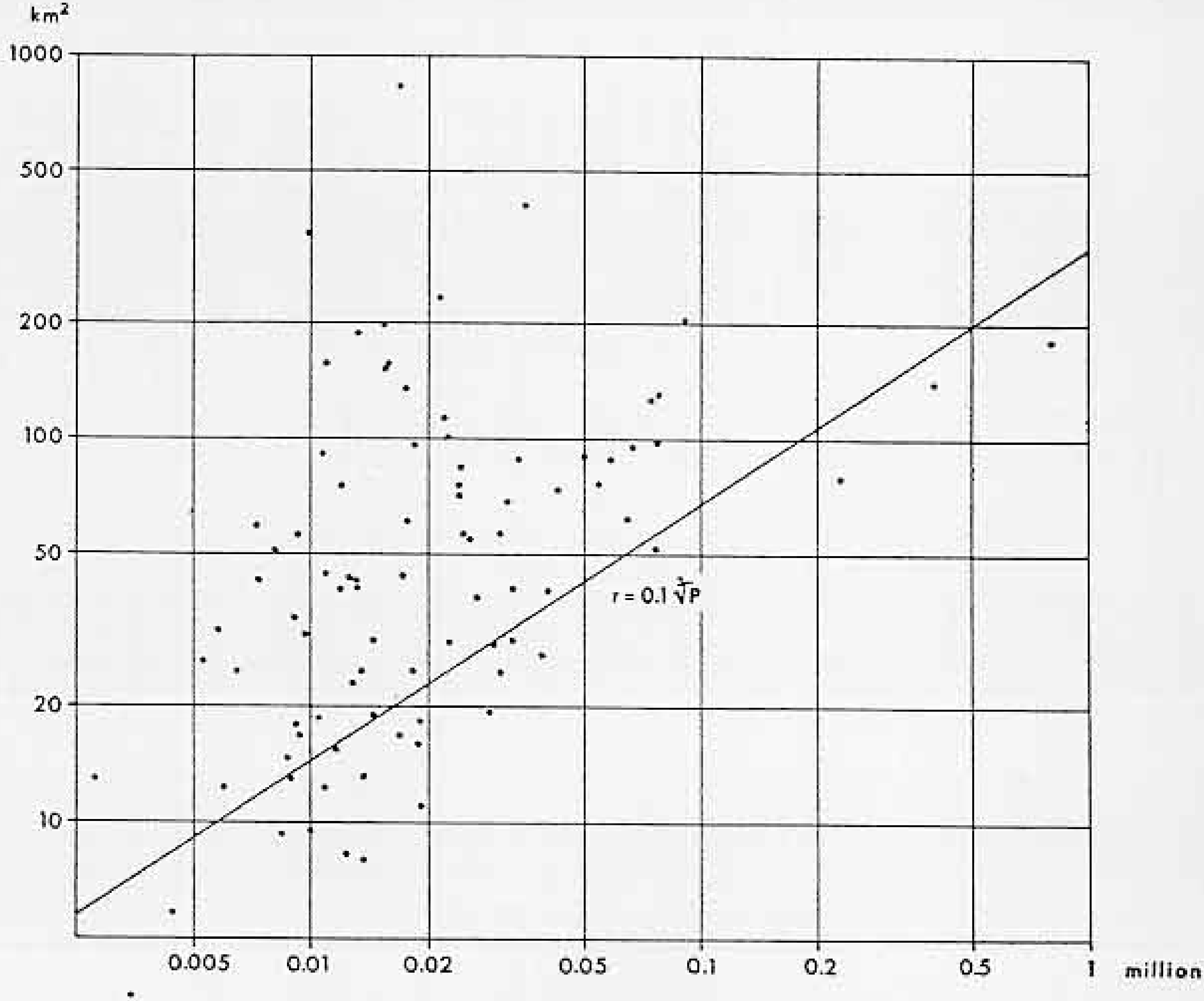


Figure 10
The relationship between area ($km^2$) and population of Swedish cities (administrative areas) in 1960.

The built-up area in Sweden (Swedish "tätort") is defined as an agglomeration of dwelling houses which are supposed to be connected by links. Each house of the "tätort" must have at least 1 such link the length of which is less than 200 meters. The population of all Swedish "tätorter" with more than 200 inhabitants is given in the Swedish census.

In the American statistics (Census, 1960) an urbanized area is defined as an area that contains at least one city of 50,000 or more inhabitants and the surrounding closely-settled incorporated places with more than 2 500 inhabitants, or with less than 2 500 inhabitants if such a place has a closely-settled area of 100 dwelling units or more. Enumeration districts in unincorporated territory with a population density of 1 000 inhabitants or more per square mile are also included in the urbanized area. Other enumeration districts with lower population density are also recognized as urbanized areas if including them eliminates enclaves or close indentations in the urbanized area (of 1 mile or less across the open end) or links outlying enumeration districts of qualifying density that were no more than 1.5 miles from the main body of the urbanized area. Counties which are classified as urban, towns in the New England States, and the townships in New Jersey and Pennsylvania are always included in the urbanized areas. This is a short summary of the definition of urbanized areas in the census in U.S.A. for 1960. In 1950 blocks were the smallest units used, instead of the larger enumeration districts. The 1950 criteria also permitted exclusion of portions of counties classified as urban, etc.

The definition of urbanized area in 1950 more closely approximated the actual area than did that of 1960 but there were significant differences between the urbanized area and the built-up area. The error resulting from the poor definition of urbanized area is a function of the perimeter of the area. Hence, it is an one-dimensional variable and the formula $e = ax$ is valid where e is the error and x a length variable. If the population P is a three dimensional variable the formula $P = cx^3$ is valid (c in the formula is a constant). In that case the relative error is equal to $(ax)/(cx^3) = dx^{-2}$; if x is very large the relative error is very small. The definition of urbanized area excludes some of the built-up area. The given data for the size of the urbanized areas is therefore a poor approximation of the size of the real built-up area. If the area is small the relative error will be too high and the area too low.

The population of built-up areas and the size of these areas for 1950 in the U.S.A. were plotted in a diagram with logarithmic scales (See figure 11). The least squares method gives the curve $A = 0.00126P^{0.86}$ where A is the area in square miles and P the population. If these areas are approximated with circles having the radius r in miles the following formula is received: $r = 0.020P^{0.43}$. The line $r = 0.020P^{43}$ is the dashed line in figure 11. The correlation between log A and log P is very high; the correlation coefficient r is equal to 0.9279.

Figure 12 shows the correlation between the area and the population of built-up areas in the U.S.A. for 1960. The correlation coefficient r in this case is equal to 0.9224. The least squares method gives the line $A = 0.00151P^{0.8757}$ or the line $r = 0.0219P^{0.44}$, which is the dashed line in figure 12 (r is the radius of a circle having the area A).

The solid line in figures 11 and 12 is the same line as was drawn in figure 10, that is the line $r = 0.1P^{0.333...}$ where r is in kilometers. The dashed line in figure 11 is $r = 0.030P^{0.43}$ km. and that of figure 12 is $r = 0.033P^{0.44}$ km. The dashed and the solid lines are not parallel. The b-values obtained are about 0.10 higher than the theoretical b-value 0.333... if the population density is a one-dimensional variable. This deviation can be explained in more than one way. The population density can be a variable with lower dimensionality than 1 and the poor definition of urbanized areas can give a higher b-value. It was pointed out when the volcanoes were discussed that a 0.5-dimensional height variable gave the theoretical b-value equal to 0.40. The population density ought to be a zero-variable. A man living in a big town needs the same area as a man living in a small town. However, figures 11 and 12 show that the dimensionality of the population density is closer to 1 than it is to 0.

The poor definition of urbanized areas has had the effect that the relative error of the smaller built-up areas was quite large, in such a way that the size of the area was too low. In 1950 the definition of built-up areas worked with blocks and in 1960 it used enumeration districts. This change in the definition of the urbanized areas had the effect of excluding some real built-up areas from the urbanized areas. In other words, the 1950 definition of an urbanized area more closely approximated the actual built-up areas than that of 1960. Consequently the b-value of 1950 is 0.01 lower than that of 1960. It is most likely that a perfect definition of urbanized areas would give a b-value which would be close to 0.333.... The population density of enumeration districts belonging to urbanized areas was 1000 inhabitants per square mile.

The b-value in the formula $r = aP^b$ will probably be higher than 0.44 if the definition of urbanized areas is changed in such a way that the lower limit of the population density of the enumeration districts is higher than 1000. Data for units designated "Densely-Inhabited Districts" are given in the 1960 census statistics of Japan. A densely-inhabited district is defined as an area consisting of contiguous enumeration districts with a population density of 4000 inhabitants or more per square kilometer. It is delineated within the boundary of city, town or village constituting an agglomeration of 5000 or more population as of October 1, 1959. The area and population of densely-inhabited districts in 1960 in Japan were plotted against each other in figure 13. The least squares method gives the equation $A = 0.000281P^{0.9146}$ sq.km., or $r = 0.00946P^{0.46}$ km. The correlation coefficient r is equal to 0.9665.

This definition of densely-inhabited areas means that large parts of the built-up areas are excluded. Consequently, the b-value will be higher than in the examples in figures 11 and 12. But the a-value is too low, too. However, as is seen of figures 11, 12, and 13, the cities in the U.S.A. are more spacious than the Japanese ones. This is explained by the existence of poor public transportation systems in the U.S.A. The American man has no choice. He must use a car in order to go from his home to downtown. The higher living standard in the U.S.A. can also explain this deviation. More people in the U.S.A. than in Japan can afford to live in suburbs and to have a car. It follows then that the population density of an American town is lower than that of a Japanese town.

The disadvantages of the approximation of built-up areas with urbanized areas and densely-inhabited districts can be eliminated in the following way. The size of most of these areas is too low. The areas are divided into different size classes. Let us now suppose that one area of each size class is equal to the corresponding built-up area. This area is the largest one of the size class compared with the population. In other words: The chosen area has the lowest population density of the areas

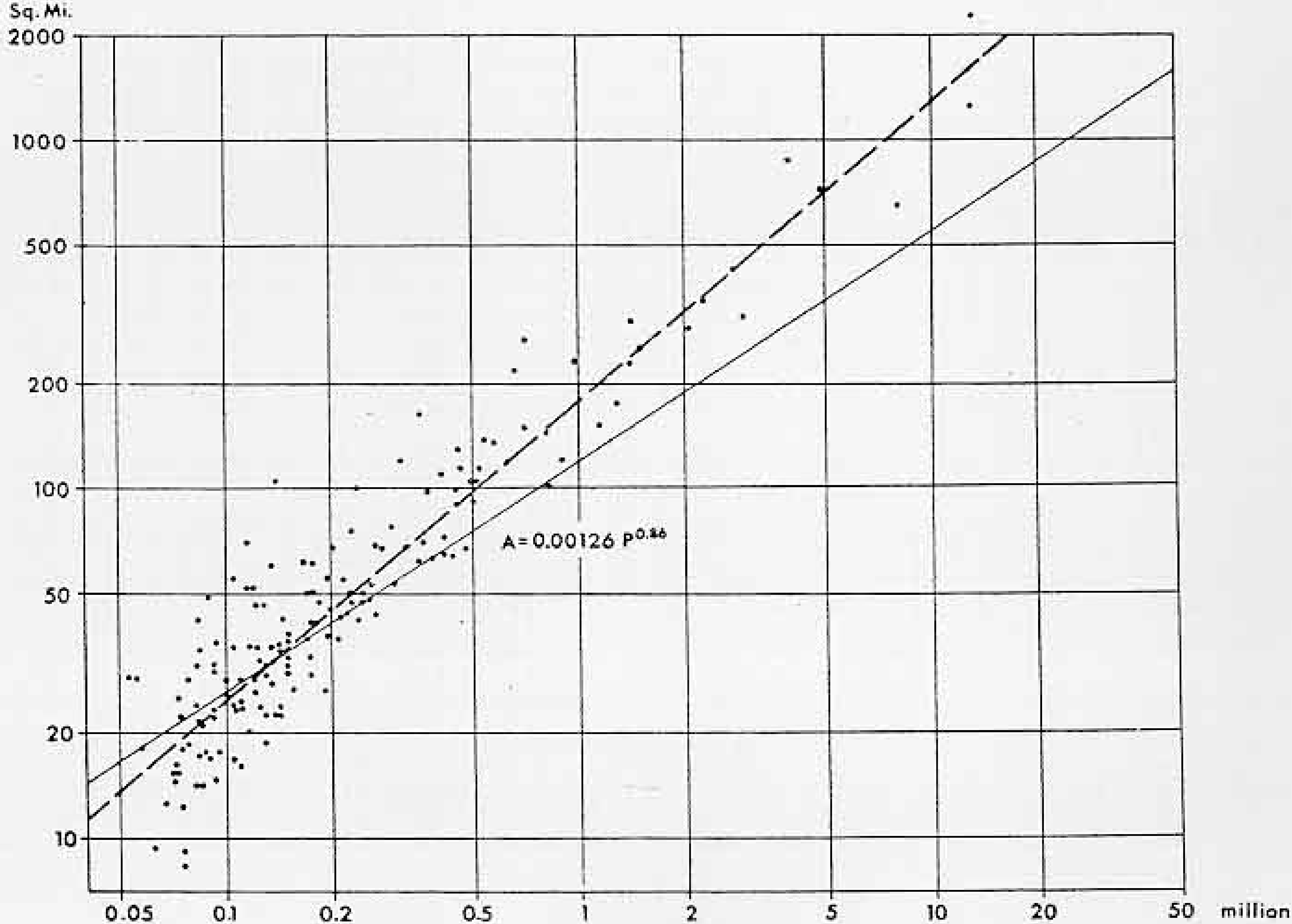


Figure 11

The relationship between area (miles$^2$) and population

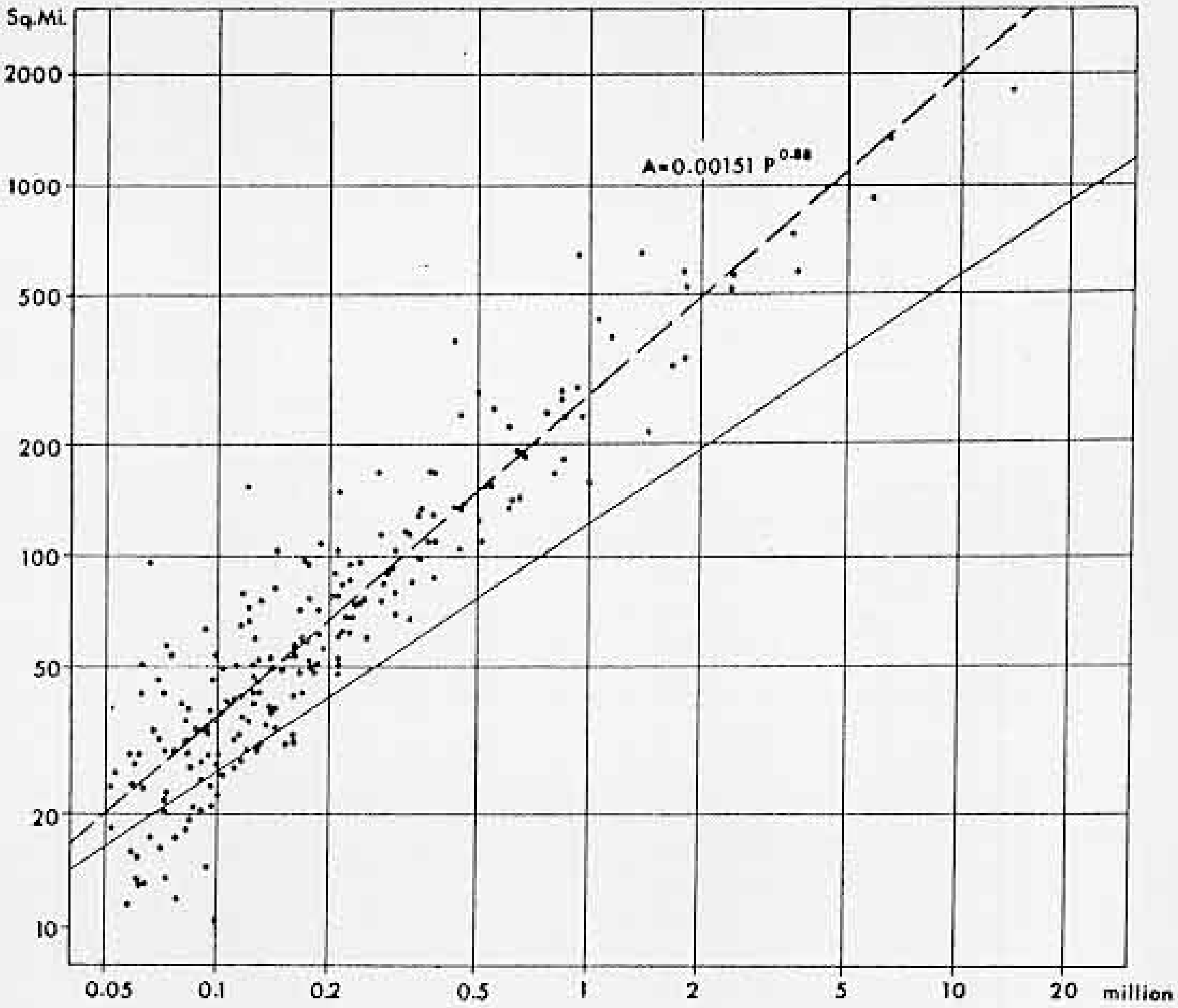


Figure 12.
The relationship between area (miles$^2$) and population of Urbanized Areas in U.S. in 1960.

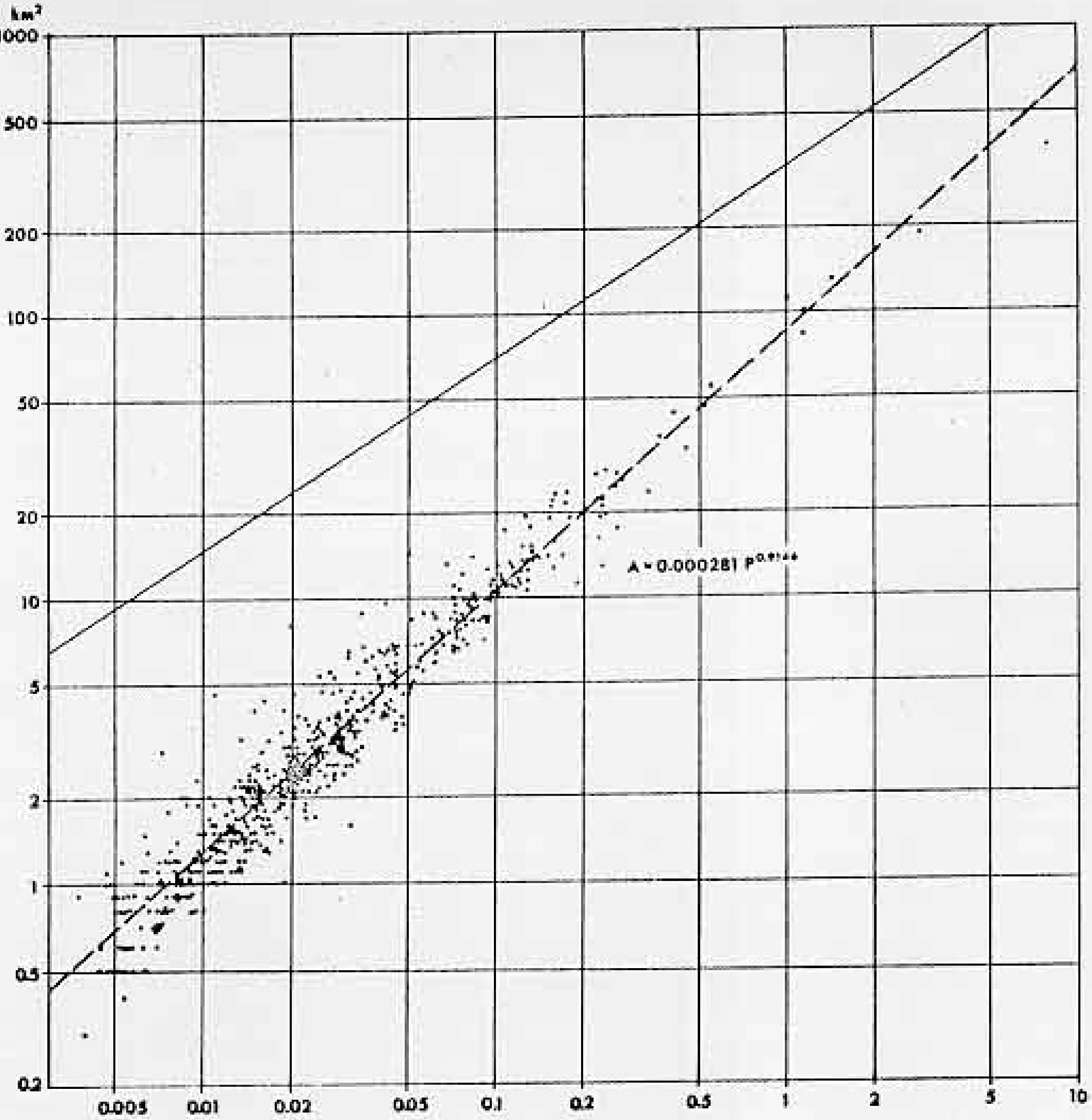


Figure 13

The relationship between area ($km^2$) and population of Densely Inhabitated Districts of Japan 1960.

belonging to the same size class. This can be done since there is so much data given (1960: U.S.A., 213; Japan, 518; 1950: U.S.A., 155 observations). It is seen in figures 11, 12 and 13 that these selected areas will give a b-value close to 0.33... The upper limitations of the dots in the figures are lines parallel to the line $r = 0.1P^{0.33...}$ which line is drawn in all three figures.

Built-up areas and allometric growth.

The data used in figures 10-13 are taken from the official statistics of Sweden, U.S.A. and Japan. The urbanized areas of U.S.A. and the densely-inhabited districts of Japan are poor approximations of the corresponding built-up areas. The b-values obtained were too high in spite of the high correlation between the log A (area) and log P (population). It was also pointed out that correct values of the built-up areas and of the population probably would give a lower b-value. This, however, is only a conjecture and is not very satisfactory. The areas of about 70 Swedish built-up areas (tätorter) are situated in South Sweden (Skåne and Blekinge). This part of the country was mapped in 1959-1961. Hence, the topographical map sheets show the situation of about 1960; the same time for which population data are available.

The area A and the population P of the measured built-up areas were plotted in figure 14. The correlation coefficient r is equal to 0.9762. Regression of log A on log P gives the line $A = 0.0085P^{0.664}$. If the area A is a circle with the radius r the length of r is given by the following formula: $r = 0.053P^{0.332}$. A is in square kilometers and r is in kilometers. The line $r = 0.053P^{0.332}$ can be compared with the line $r = 0.1P^{0.333...}$ drawn in figure 10 which gives too high an approximation of r. This approximation is quite better than such an one built on the administrative units.

The line $d = 0.1P^{0.333...}$ can be used as an approximation of the diameter d. The regression of population P on area A gives the line $A = 0.00687P^{0.696}$.

According to the diagram (figure 14), the built-up areas can be divided into three or more main groups:

1. Spacious built-up areas. Such areas have large areas compared with their populations. There are some towns belonging to this group which have many summer houses which have been erroneously classified as permanent dwelling houses. However, these areas can in most cases be recognized as the areas having the smallest population compared with size. There are only a few of the areas in figure 14 which belong to this group.

2. Mean built-up areas. This group includes most of the areas in figure 14. The area of a mean built-up area is less than $0.01P^{0.666...}$ and greater than $0.007P^{0.666...}$

3. Compact built-up areas. These have high populations compared with their areas. Old fishing villages belong to this group as do some railroad-station villages.

In the example in figure 14 the unknown size of the built-up area was estimated by means of the known size of the population. Of course this technique also permits estimation of the population based upon the known size of the built-up area. If the shape of a series of built-up areas is known,the formula for estimating the population also is known. The size of the areas can be measured from air photos. Using this method two men can estimate the population of 200-300 built-up areas in one day.

Population potentials.

The population potential $V_i$ for place number i is defined as follows.

$$V_i = \sum_{j \neq i}^{n} (P_j/d_{ij}^b) + Oi \qquad (5)$$

where $O_i$ is the contribution to the potential at i from the place i. $P_j$ is

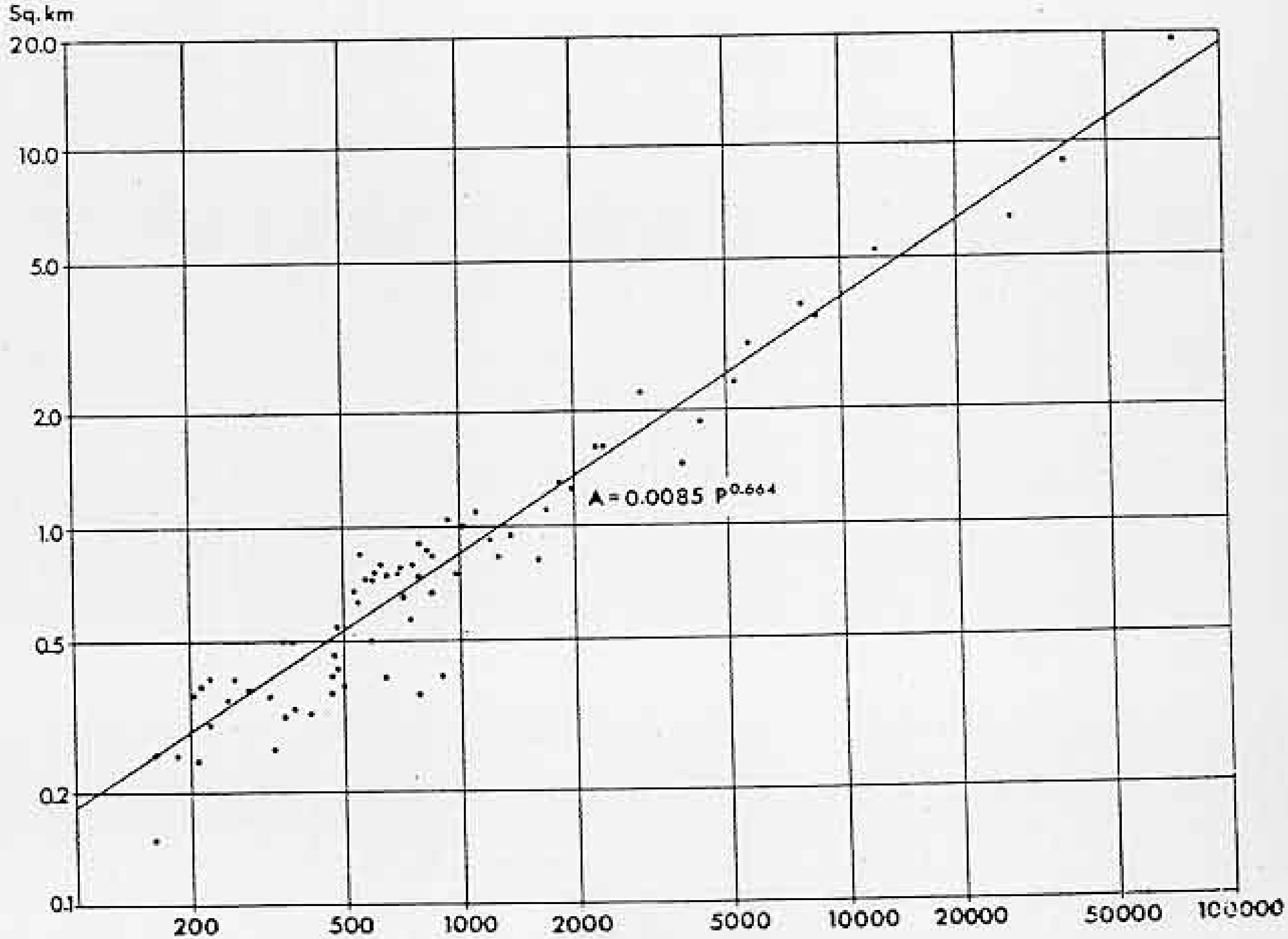


**Figure 14**
The relationship between area (km$^2$) and population of built up areas in Southern Sweden in 1960.

the population of place j, and $d_{ij}$ is the distance between place i and place j, n is the total number of places, and b is a constant. If it can be assumed that place number i is close to a circle with the radius R and that the population is uniformly distributed over this circle, $O_i$ is calculated by formula (6).

$$O_i = \frac{2P_i}{R^2} \int_0^R r^{1-b}\, dr \qquad (6)$$

If the constant b in formula (6) is equal to 1, $O_i$ is equal to $2P_i/R$ which value usually is used as an approximation of $O_i$ independent of the value of the constant b used in the summation of the first part of the potential formula (5). A b-value equal to zero makes the $O_i$ equal to $P_i$. It must also be observed that b is not allowed to be equal to 2 or greater than 2, because the value of $O_i$ in both these cases becomes infinite. The b-value in the sum of formula (5) has no such liminations.

The calculation of the population potentials involves a great number of numerical operations. Consequently, it is convenient to let a computer do these operations. Since the population of each place is known the computer can also determine the radius of each area by means of the allometric growth formula. Such an estimation of the radius is much better than one based upon administrative units. An administrative unit can be divided into two parts; A core equal to its central built-up area with the radius $R_c$, and one ring with the width $R_d$-$R_c$ where $R_d$ is the radius of the administrative unit. The population of the core is $P_{ci}$ and that of the ring is $P_{ri}$. The contribution to the population potential coming from the place itself is determined by formula (7).

$$O_i = \frac{2P_{ci}}{R_c^2} \int_0^{R_c} r^{1-b}\, dr + \frac{2P_{ri}}{(R_d - R_c)^2} \int_{R_c}^{R_d} r^{1-b}\, dr \qquad (7)$$

It is here assumed that $R_c$ is less than $R_d$.

This example shows how the radius of a built-up area can be determined by means of the allometric growth formula and how the size of this radius is used in population potential formulas. Other potentials, such as income potential are calculated in the same way as the population potential. In all these cases the radius of the place itself must be estimated as correct as possible since the greatest contribution to the potential generally comes from the place itself.

Deformation of shape.

There are no anthropoid giants over 4 meters tall. The length of the biggest mosquito is less than 2 inches. This depends on the fact that the constitution of a mosquito body does not allow it to grow too much. The wings and the legs of an insect cannot bear a too heavy a body. The tower of Babel could not have been as high as it is depicted. It is impossible to build a wooden house more than a few dozen meters high. Iron or concrete buildings can be many hundreds of meters high. The highest mountain of the world is less than 10 000 meters high. A mountain much more than 10 000 meters in height must change shape due to structural failure of its base.

The above examples indicate that the size of a growing individual having a special shape always has an upper limit. If the individual exceeds this limit its shape must be deformed. This deformation of the shape can easily be recognized in a diagram with logarithmic scales because there is a break in the point distribution corresponding to the change from one shape to another.

It is most likely that the cities and built-up areas cannot grow without any limitation. It is impossible to say the value of this limitation but as is seen in figures 11, 12 and 13 the large built-up areas such as New York, Los Angeles and Chicago have not yet passed this limitation. They all fit the curves perfectly. This is also true for Tokio and other large towns in Japan. Perhaps the shape of a built-up area is not deformed until its population is more than 100 million inhabitants.

Summary.

There are three main groups of growth:

1. Arithmetical growth which means that a growing individual increases with the same absolute value during each of a series of time periods all having the same length. Such a growth curve follows the formula $y = a + bx$. This is a straight line in a diagram with arithmetical scales.

2. Geometrical growth. A growing individual increases with the same relative value during the time periods. The formula $y = ab^x$ is valid for this kind of growth. In a diagram having the y-scale logarithmic and the x-scale arithmetic this equation is represented by a straight line.

3. Allometric growth. The relative growth of an organ y is equal to a constant fraction of the relative growth of the body x. This growth curve follows the formula $y = ax^b$ which is a straight line on a diagram with double logarithmic scales.

The streets of a town can be looked upon as organs and the town as the organism, etc. Thus the allometric growth formula has some very interesting geographical applications. It is formulated in such a way that a variable x can be estimated by means of an other variable y which is much easier to measure than x. This formulation of the law of allometric growth is valid for a series of individuals all having the same shape but of different sizes.

The rank-size rule for cities can be verified by means of this law (Beckman, 1958). The corresponding rule for incomes (Pareto, 1896-97) can also be derived by aid of a technique similar to Beckman's. The bifurcation ratio rule by Horton (1945) and Strahler (1952) is a rank-size rule valid for river channels and it can be verified in the same way as other rank-size rules.

The law of allometric growth is applied in this paper to rivers, meanders, volcanoes, drumlins, urbanized areas (U.S.A., 1950 and 1960), densely-inhabited areas (Japan, 1960) and built-up areas. In all these cases the correlation between log y and log x is extremely high. The correlation coefficients are all greater than 0.9. The constant b in the formula $y = ax^b$ shows the dimensionalities of the variables x and y. If y is an one-dimensional variable and x a two-dimensional one the theoretical b-value is equal to 0.5. The lengths of some rivers of widely varying sizes were compared with their corresponding drainage areas. The b-value obtained was very close to 0.5. This value can be compared with the value 0.6 which Leopold-Wolman-Miller (1964) refer to. The deviation between these two b-values can be explained in the following way: The b-value 0.6 was calculated by means of a few rivers and the difference in size between the largest and the smallest river was not great. A wider sampling of river and basin sizes would probably give a b-value closer to 0.5.

The a-value in the formula $y = ax^b$ can be used when classifying the given y-values. The rivers were divided into three groups: 1. Plain rivers with a short main river and a large drainage area. 2. Mean rivers. 3. Valley rivers with long main rivers compared with their drainage areas. The built-up areas of Southern Sweden were also divided into three types:

Compact built-up areas, Mean areas and Spacious built-up areas.

The formula $y = ax^b$ was used when the areas of the built-up areas in Sweden were estimated on the basis of calculated population and income potentials. It can also be used when the area of a built-up area is known and the population of this area to be estimated. In this case the area can be measured using large-scale air photos.

This paper has introduced some geographical applications of the allometric growth formula. This formula undoubtedly has many geographical applications which have not been mentioned here. The formula is always valid for an individual belonging to a series of individuals if all these have the same shape or almost the same shape. Consequently, if some individuals which are supposed to have the same shape do not fit the allometric growth formula they do not have the same shape. However, it must also be observed that there can be individuals which fit the formula in spite of having different shapes.

Acknowledgements

The volume of the volcanoes, figure 6, was measured by Mr. John Haake, Department of Geography, Wayne State University. The size of the built-up areas, figure 14, was measured by amanuens Thomas Hjortsjö, Department of Geography, University of Lund. The figures were drawn by Mr. Rezsö László. The calculation of the correlation coefficients and the a- and b-values in figures 11-13 were done by the Computer Department, Wayne State University, and by the Computing Center, University of Michigan. The English was corrected by Mr. Daniel Snyder, Department of Geography, Wayne State University. I thank all and especially professor William Bunge, Wayne State University, who gave me so much help.

Stig Nordbeck
University of Lund
December, 1964

Literature

Atlas of Sweden: Hydrology, Watercourses and Drainage-Basins, Territorial Waters. Stockholm 1957.

Auerbach, F., Das Gesetz der Bevölkerungskonzentration. Petermann's Geographische Mitteilungen 59. 1913.

Bagnold, R.S., Some aspects of river meanders. U.S. Geol. Survey Prof. Paper 282-E.

Beckman, M.J., City Hierarchies and the Distribution of City Size. Economic Development and Cultural Change. Vol. VI, No. 3, April 1958.

Bengtsson, B.-E. and Nordbeck, S., Construction of Isarithms and Isarithmic Maps by Computers. Nordisk Tidskrift for Informationsbehandling. (Bit). Hefte 2. 1964.

Berry, B. and Garrison, W.L., Alternate Explanations of Urban Rank-Size Relationships. Annals Association of American Geographers. Vol. 48. 1958.

Berry, B.J.L., Simmons, J.W. and Tennant, R.J., Urban Population Densities: Structure and Change. The Geographical Review. Vol. LIII, 1963.

von Bertalanffy and Pirozynsky, Ontogenetic and Evolutionary Allometry, Evolution, Vol. VI, No. 4.

Census of Population: 1960, Volume I, Characteristics of Population, Part 1, United States Summary.

Chorley, R.J., The Shape of Drumlins. Journal of Glaciology. 1958.

Clark, C., The location of industries and population. The Town Planning Review. Vol. XXXV. No. 3. 1964.

- Urban Population Densities, Journ. Royal Statist. Soc. Ser. A. Bd 114. 1951.

Dickinson, R.E., The West European City. London 1961.

Essays on Growth and Form Presented to D'Arcy Wentworth Thompson. Edited by W.E. le Gros Clark and P.B. Medawar. Oxford University Press. First edition 1945. Reprinted 1947.
Medawar, P.B., Size, Shape and Age.
Reeve, E.C.R. and Huxley, J.S., Some problems in the study of Allometric Growth.
Richards, O.W. and Kavanagh, A.J., The Analysis of Growing Form.

Förteckning över Sveriges vattenfall I-III. Statens meteorologisk-hydrologiska anstalt och Kungl. vattenfallsstyrelsen. Stockholm 1930, 1932, 1945.

Horton, R.E., Drainage-basin characteristics. Am. Geophys. Union Tranv. No. 13. 1932.

- Erosional development of streams and their drainage basins. Geol. Soc. Am. Built 56. 1945.

Huxley, J., Problems of Relative Growth. London 1932.

Kant, Edg., Zur Frage der inneren Gliederung der Stadt. Proceedings of the IGU Symposium in Urban Geography. Lund 1960. Lund Studies in Geography. Ser. B. No. 24.

Leopold, L.B., Rivers. American Scientist.

Leopold, L.B. and Langbein, W.B., The concept of entropy in landscape evolution. U.S. Geol. Survey. Prof. Paper 500-A.

Leopold, L.B. and Wolman, M.G., River Meanders. Geol. Soc. Am. Built. Vol. 71.

Leopold, L.B., Wolman, M.G. and Miller, J.P., Fluvial Processes in Geomorphology, San Francisco 1964.

Lösch, A., The Economics of Location. New Haven 1954.

Morisawa, M.E., Relation of Quantitative Geomorphology to Stream Flow in Representative Water-sheds of the Applachian Plateau Province. Technical Report No. 20. Project NR 389-042. Office of Naval Research, Department of Geology, Columbia University. New York 1959.

Nordbeck, S., Framställning av kartor med hjälp av siffermaskiner. Lund 1964.

Nordström-Johnsson-Norborg-Fourén, Geografiska tabeller med arbetsuppgifter. Stockholm 1964.

Norling, G., Folkräkningar och geografi. Ymer. Häfte 3. 1964.

1960 Population Census, Densely-Inhabited Districts. Bureau of Statistics, Office of the Prime Minister. Tokio 1961.

Pareto, V., Cours d'Economie Politique, Lausanne 1896-97.

Rapoport, A., Some theoretical consequences of the allometric growth equations. Bulletin of Mathematical Biophysics. Vol. 17. 1955.

Stahl, W.R., Similarity and Dimensional Methods in Biology. Science, Vol. 137, 1962.

Strahler, A.N., Dimensional analysis in geomorphology, Off. Nav. Res. Proj. NR 389-042, Tech. Rep. 7, Dept. Geol., Columbia Univ., N.Y. 1957.

- Quantitative Analysis of Watershed Geomorphology. Am. Geophys. Union Trans. 1957.

- Quantitative geomorphology of erosional landscapes. C.-R. 19th Intern. Geol. Cong., Algiers, 1952. 1954.

- Physical Geography. New York 1960.

Thompson, D.W., On growth and Form. Cambridge 1959.

von Thünen, J.H., Der isolierte Staat in Beziehung auf Landwirtschaft und Nationalökonomie. Hamburg 1826.

Zipf, G.K., National Unity and Disunity. Bloomington Ind., 1941.