Name:

Honor Code: KEY

Instructions:

- Use the space on the accompanying pages to work the problems. Do not use a bluebook. Attach additional worksheets if necessary.
- If you wish to have partial credit awarded for any of your incorrect answers **you must write clearly and legibly.** Explain your work in words, if necessary.
- Read the instructions provided with each problem.
- Don't Panic.

1. [15 points total] *State Space Problem.*

(a) [7] Determine the stability of the system given by

$$\dot{x} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} x + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u$$
$$y = \begin{bmatrix} 2 & -173 \end{bmatrix} x + 5u$$

(b) [4] Write the state space equivalent of the transfer function $G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+3}$, where *Y(s)* represents the output and *U(s)* represents the input. Use the state vector $x=[y]$.

(c) [4] Repeat (b) using the state vector $x=[2y]$.

a. The poles of the system can be found using

$$(sI - A)^{-1} = \begin{bmatrix} s-1 & -3 \\ -2 & s-4 \end{bmatrix}^{-1}$$

$$= \frac{1}{(s-1)(s-4) - 6} \begin{bmatrix} s-4 & 3 \\ 2 & s-1 \end{bmatrix}$$

$$= \frac{1}{s^2 - 5s - 2} \begin{bmatrix} s-4 & 3 \\ 2 & s-1 \end{bmatrix}$$

poles at $\frac{5 \pm \sqrt{25+8}}{2}$ which are 5.37 & $-.37$

$\therefore$ NOT STABLE

(b) $\dfrac{Y(D)}{U(D)} = \dfrac{1}{D+3} \Rightarrow (D+3)\,Y(D) = U(D)$

$$\dot{y} + 3y = u$$
$$\dot{y} = u - 3y$$

using $x = [y] = [x_1]$

$\dot{x} = [\dot{y}] = [u - 3x_1]$

write

$\dot{x} = Ax + Bu$

$y = Cx + Du$

$$\boxed{\begin{aligned} \dot{x} &= -3x + 1u \\ y &= 1x + 0u \end{aligned}}$$

(c) using $x = [2y] = x_1$

$\dot{x} = [2\dot{y}] = [2u - 6y] = [2u - 6(\tfrac{x_1}{2})]$

since $x_1 = 2y$, $y = x_1/2$

write

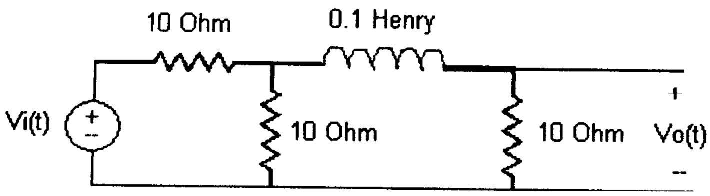
$\dot{x} = Ax + Bu$

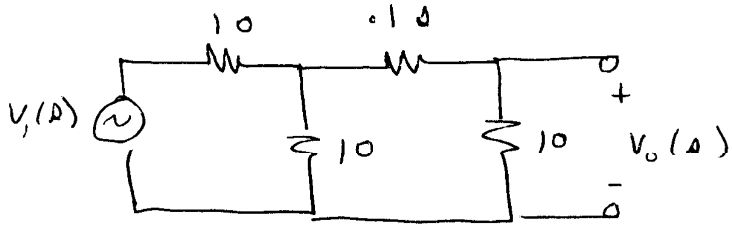
$y = Cx + Du$

$$\boxed{\begin{aligned} \dot{x} &= -3x + 2u \\ y &= \tfrac{1}{2}x + 0u \end{aligned}}$$

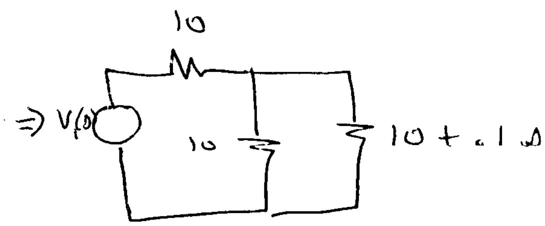
2. **[10 points]** *Systems Problem.*

Write the transfer function *Vo(s)/Vin(s)* for the following problem.

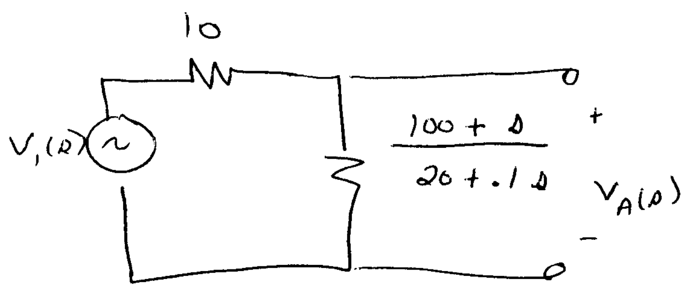




$\Rightarrow$



$\Rightarrow$



$$V_A(s) = V_1(s)\left[\frac{\dfrac{100+s}{20+.1s}}{10+\dfrac{100+s}{20+.1s}}\right]$$

$$V_A(s) = V_1(s)\left[\frac{100+s}{300+2s}\right]$$

$$V_o(s) = \frac{10}{10+.1s}V_A(s) = \frac{1000+10s}{(300+2s)(10+.1s)}V_1(s)$$

$$\boxed{\frac{V_o(s)}{V_1(s)} = \frac{1000+10s}{(300+2s)(10+.1s)}} = \frac{50}{150+s}$$

3. [25 points] *Root Locus Problem.*

Draw the root locus of the unity feedback system with

$$G(s) = \frac{s-5}{s^2+5s+10}$$

(i) [1] Location of pole(s) and zero(s) : z at 5, poles at $-2.5 \pm 1.94j$

(ii) [1] The locus is on the axis between : $(-\infty, +5)$

(iii) [2] The locus has asymptotes defined by :

$$\sigma = \frac{\sum p - \sum z}{n_p - n_z} = \frac{-10}{1} = -10 \qquad \theta = \frac{\pm k\,180^\circ}{n_p - n_z} = 180^\circ$$

(iv) [5] The *jw-axis* crossing(s) are at (give value(s) of *K* and *s*):

$$\frac{G}{1+GH} = \frac{s-5}{s^2+5s+10+K(s-5)} = \frac{s-5}{s^2+s(5+5K)+(10-5K)}$$

s^2	1	$10-5K$
s^1	$5+5K$	
s^0	$10-5K$	

$K=-1$ (NOT VALID)

$K=2$

at $K=2$, Denominator is $s^2+10s=0$, $s(s-10)=0$

axis crossing at $s=0$

(other pole at $s=10$ then)

(v) [5] The break-in and/or break-away point(s) are at:

$$-\frac{d}{ds}\,\frac{s^2+5s+10}{s-5} = -\left[(s^2+5s+10)(-1)\frac{1}{(s-5)^2} + \frac{(2s+5)}{s-5}\right]$$

$$= -\left[\frac{-s^2-5s-10+(2s+5)(s-5)}{(s-5)^2}\right]$$

$$= -\left[\frac{-s^2-5s-10+2s^2-10s+5s-25}{(s-5)^2}\right]$$

$$= -\left[\frac{s^2-10s-35}{(s-5)^2}\right]$$

$s = 12.74$ & -2.74

(vi) [5] The angle(s) of departure and/or arrival from all pole(s) and zero(s) are:

From $-2.5+1.94j$; $\measuredangle$ From $-2.5-1.94j$ $= \underline{90^\circ}$

$\measuredangle$ From 5 $= 180 - \tan^{-1}\left(\frac{1.94}{7.5}\right) = \underline{165.5^\circ}$

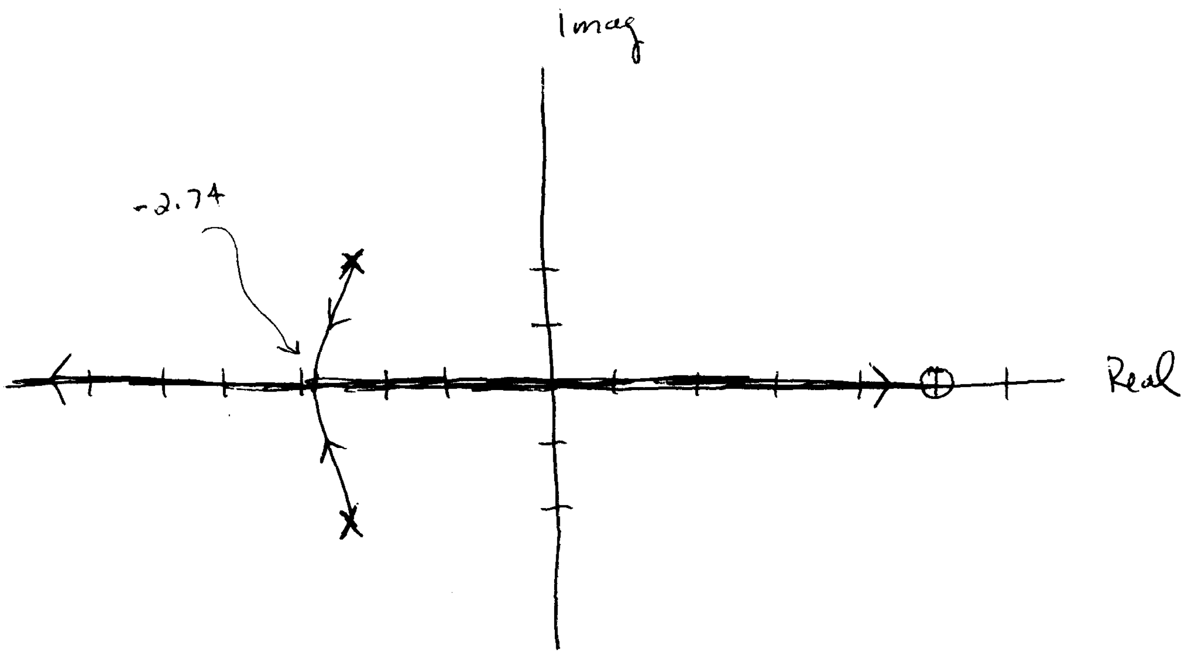
angle of departure $= 180 \mp 90 + 165.5$

$= 255.5^\circ$

From $\underline{-2.5-1.94j}$ (by symmetry) $= 104.5^\circ$

From 5 $=$ 180° (angle of arrival)

(vii) [6] Draw the locus as accurately as possible



4. [25 points total] *Controller Problem.*

Given the unity feedback system $G(s) = \dfrac{s+20}{(s+2)(s+8)}$

(a) [2 points] Calculate the desired system poles if the system is to operate with 10% overshoot and 1.0 seconds settling time.

(b) [10 points] Show that you *cannot* design a PD controller to make the system meet these specifications.

(c) [10 points] Design a lead compensator that exploits pole-zero cançellation to meet the specifications.

(d) [3 points] Comment on the validity of this second order approximation.

(a) 10% OS ⇒ $\zeta = 0.5901$ $\quad T_s = 1.0 \Rightarrow \dfrac{4}{\zeta\omega_n} = 1$ or $\omega_n = 4/\zeta = 6.778$

desired poles = $\underline{-4 \pm j5.47}$

(b) PD ⇒ place a zero somewhere. to be on the locus $\sum \angle p - \sum \angle z = k\,180^\circ$

angle from zero at -20 is $\tan^{-1}(5.47/16) = 18.87^\circ$

pole at -2 is $180 - \tan^{-1}(5.47/2) = 110.08^\circ$

-8 is $\tan^{-1}(5.47/4) = 53.82^\circ$

$$\sum \angle p - \sum \angle z = 145.03$$

in order to get an odd multiple of 180°, new zero must be placed at -34.97° (325.03°) which is not possible.

(c) Cancel the pole at $(s+2)$

design a zero; using angle criterion

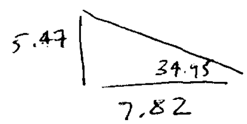
angle from zero at -20 : $18.87°$

pole at -8 = $53.82°$

$\sum p - \sum z = 34.95°$

new pole must have angle $145.05°$

must be at



$p = 3.82$
unstable!

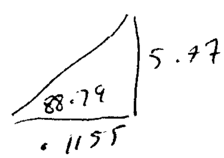
Try cancelling pole at $(s+8)$

angle from zero at -20 $18.87°$

pole at -2 110.08

$\sum p - \sum z = 91.21°$

new pole must have angle $88.79°$



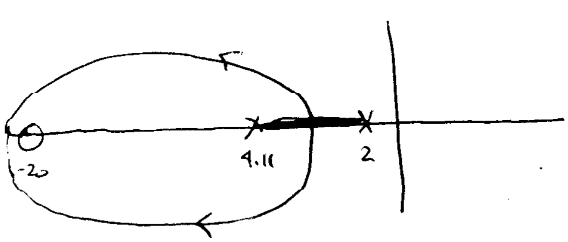
$p = -4.1155$ (stable)

Lead compensator $K = \dfrac{\Pi|s-p|}{\Pi|s-z|} = \dfrac{|-4+j5.47+4.11||-4+j5.47+2|}{|-4+j5.47+20|}$

$= \dfrac{(5.4711)(5.824)}{16.909} = 1.8844$

Lead: $1.8844\left(\dfrac{s+8}{s+4.1155}\right)$

(d) Root Locus



2nd approx ok - dominant poles at $-4 \pm j5.47$ extra zero 5 times as far away; extra zero may cause slight change in OS.

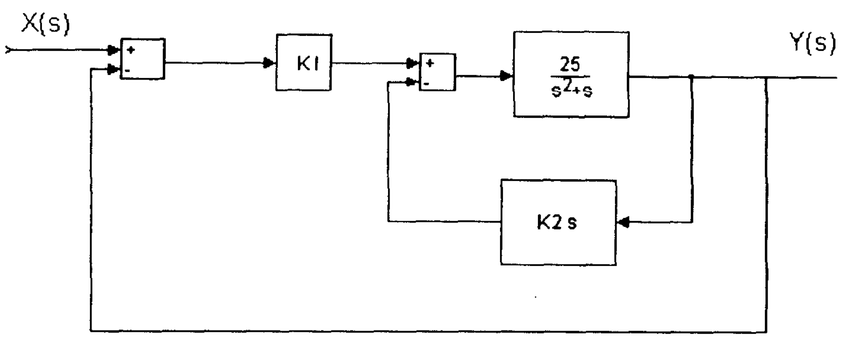
5. [25 points total]

(a) [8] Find the values of K1 and K2 in the following that will yield 25% overshoot and a settling time of 0.2 seconds.

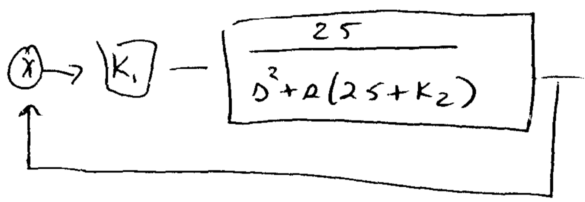
(b) [3] Find the steady state error of your system due to a step input and draw the step response.

(c) [3] Find the steady state error of your system due to a ramp input and draw the ramp response.

(d) [11] Design a controller that will reduce the steady state error due to a ramp input to zero, without appreciably changing the transient response. Explain your reasoning.



(a)



$$\frac{G}{1+GH} = \frac{\frac{25}{s^2+s}}{1 + K_2 s \frac{25}{s^2+s}} = \frac{25}{s^2+s(1+25K_2)}$$

$$\boxed{\frac{25\ K_1}{s^2+s(1+25*K_2)+25K_1}}$$

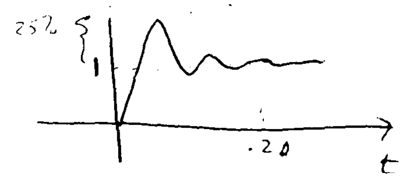
25% OS $\Rightarrow$ $\zeta = 0.4037$, $4/\zeta\omega_n = 0.2$ $\Rightarrow$ $\omega_n = \frac{4}{.2\zeta} = 49.54$

so $s^2 + s(1+25*K_2) + 25K_1 = s^2 + 40s + 2454.2116$

or $\underline{K_2 = 1.56}$ $\underline{K_1 = 98.168}$

(b.) $SSE = \lim_{s \to 0} \dfrac{s R(s)}{1 + G(s)} = \lim_{s \to 0} \dfrac{1}{1 + \dfrac{25 k_1}{s^2 + s(1 + 25 k_2)}}$

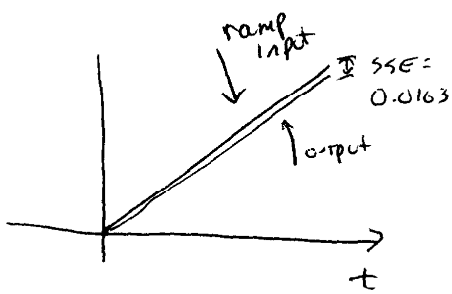
$= \underline{0}$



(c) to a ramp: $\lim_{s \to 0} \dfrac{s R(s)}{1 + G(s)} = \lim_{s \to 0} \dfrac{\frac{1}{s}}{1 + \dfrac{25 k_1}{s^2 + s(1 + 25 k_2)}}$

$= \lim_{s \to 0} \dfrac{1}{s + \dfrac{25 k_1}{s + 1 + 25 k_2}}$

$= \dfrac{1 + 25 k_2}{25 k_1} = 0.0163$



(d) To drive SSE to zero, we need to increase the system type by 1. A PI Controller will work, e.g.

$$\frac{s + 0.1}{s}$$

this is sufficient since the dominant poles $(-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -20 \pm j\,45.32)$ are very far away.