Unlikely Intersections and Multi-Function Portraits

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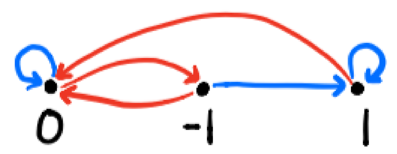
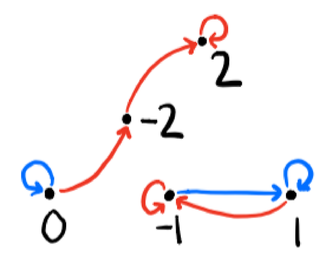
Introduction to Dynamics: Portraits

A realizable classical (single-function) portrait: $f(x) = x^2$



A realizable multi-function portrait: $f(x) = x^2$, $g(x) = x^2 - 1$

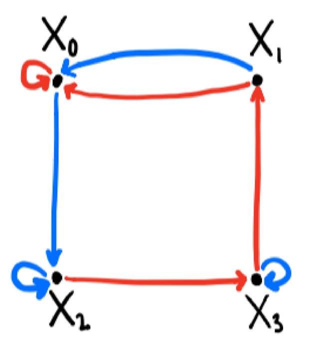


Not a portrait. $f(x) = x^2$, $g(x) = x^2 - 2$

Central Question

Given a multi-function portrait, can we find points $\{x_0, x_1, ..., x_m\} \subset \mathbb{C}$ and polynomials $\{f_1, f_2, ..., f_n\} \subset \mathbb{C}[x]$ of specified degrees that **realize** the portrait?

Realization Spaces and Dimension



$f(x_0) = x_2$ $\quad g(x_0) = x_0$

$\vdots \qquad \vdots$

$f(x_3) = x_3$ $\quad g(x_3) = x_1$

$f = a_2x^2 + a_1x + a_0$

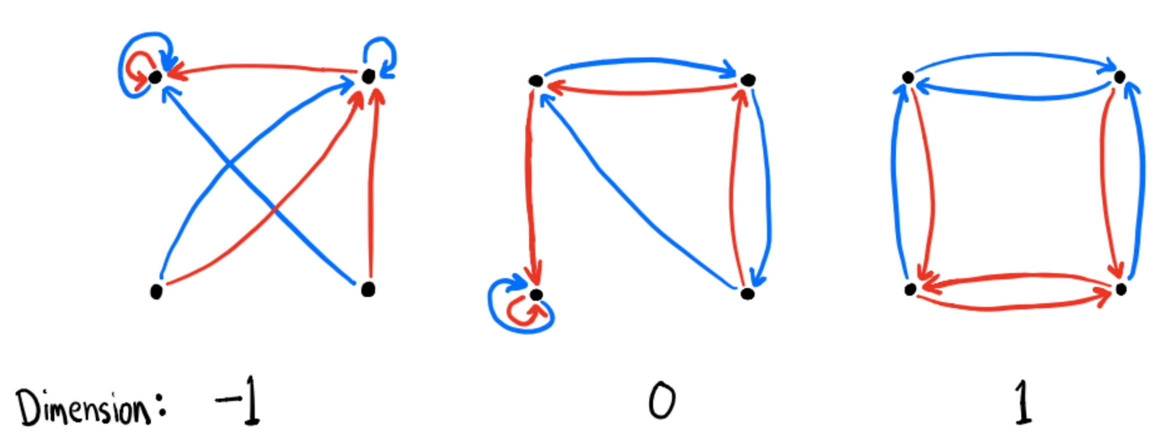
$g = b_2x^2 + b_1x + b_0$

A multi-function portrait and its system of equations, whose solution set is the portrait's *realization space*.

Dimension-Counting Heuristic

For a portrait's system of equations:

$$\#(\textit{variables}) - \#(\textit{equations}) - 2 \textit{ symmetries of } \mathbb{C} = \textit{expected dimension of realization space}$$



Two quadratics acting on four points: zero-dimensional realization space expected.

Data Collection

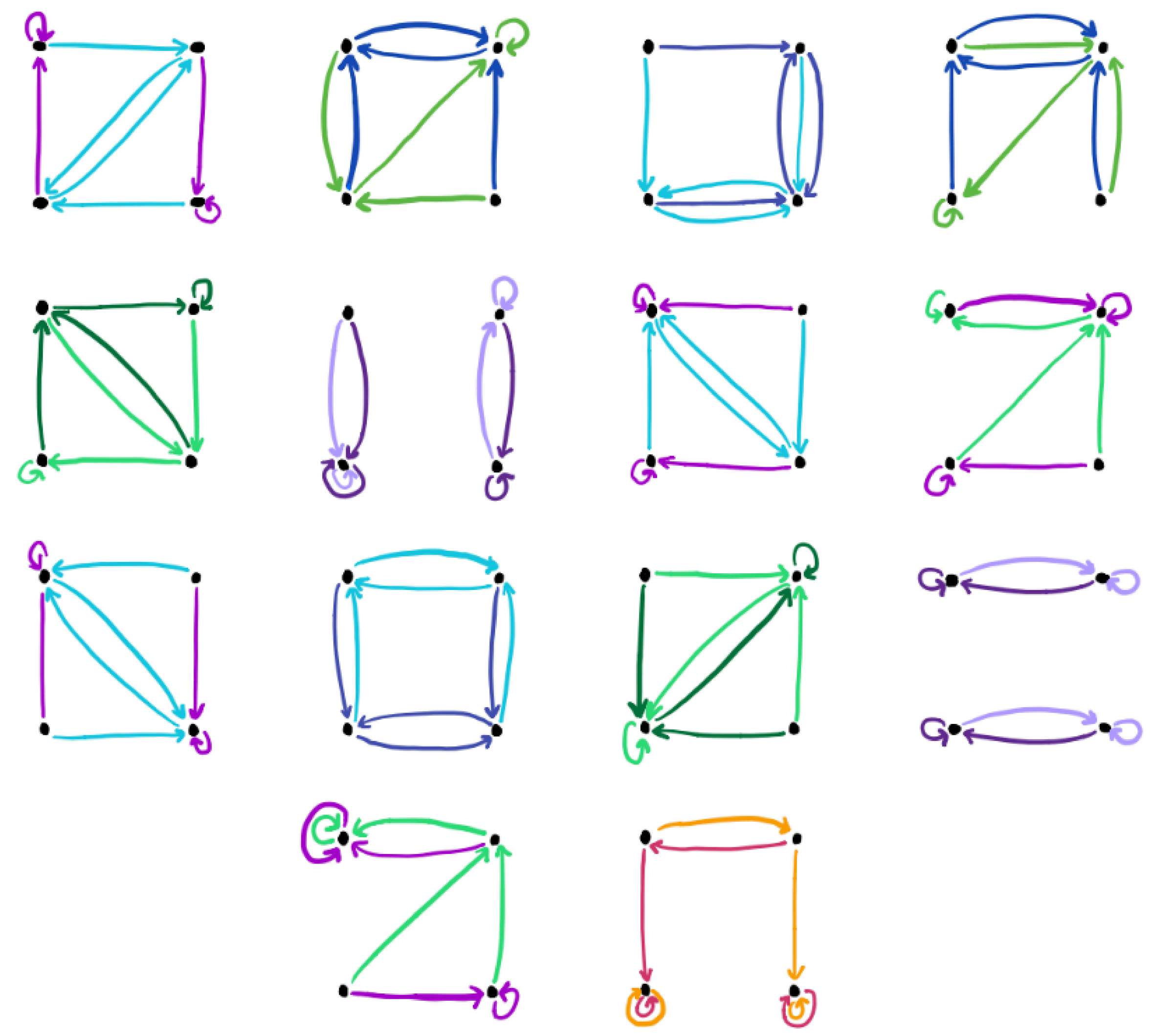
Two quadratics acting on four points	
Dimension	#(Portraits)
-1	206
0	560
1	14

Two cubics acting on six points	
Dimension	#(Portraits)[a]
-1	52,238
0	1,251,585
1	1,009
2	16

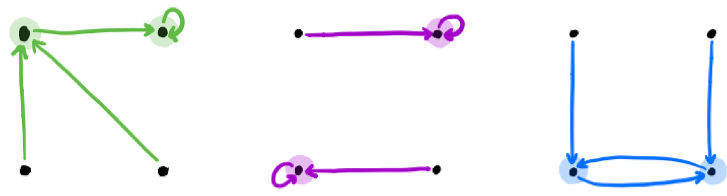
[a]for the computed 97% of data

Two quadratics acting on five points	
Dimension	#(Portraits)
-1	16590
0	246
1	3

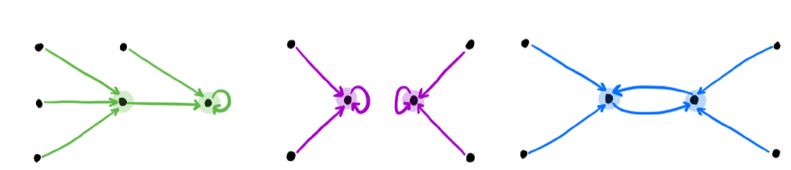
Portraits Comprised of "Two-Image" Maps



One-dimensional portraits: 2 quadratics, 4 points



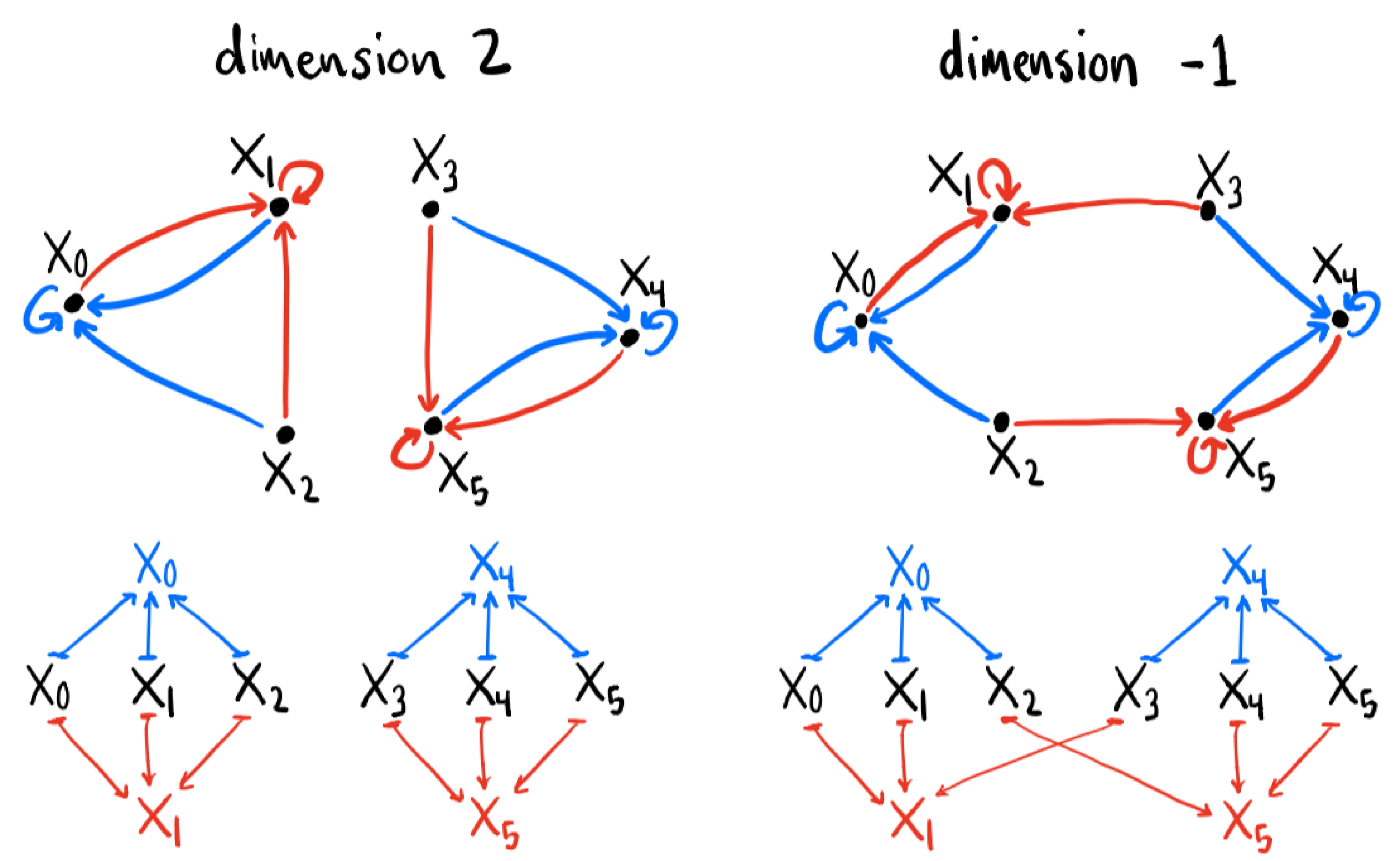
Quadratic two-image single-map portraits



Cubic two-image single-map portraits

Theorem: Classifying Unlikely Intersections

Given a portrait of degree d on $2d$ points, if each polynomial has two images, then the realization space has dimension $d - 1$ or is empty.



Maximal-dimension cubic portrait (top left) and an un-realizable cubic portrait (top right). Respective partitions of the point sets based off of preimages below.

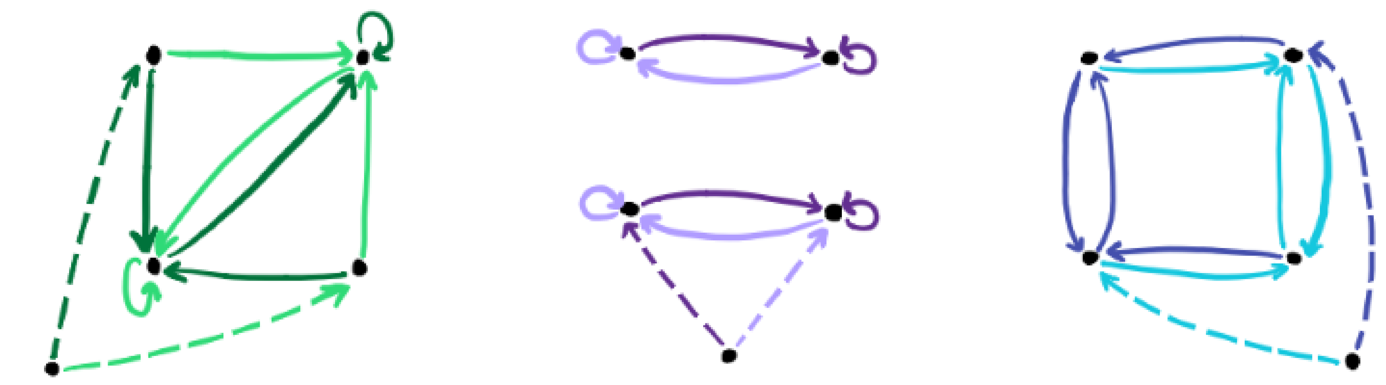
Bounding #(Realizations) in Zero-dimensional Cases

Theorem: Bound on #(realizations)

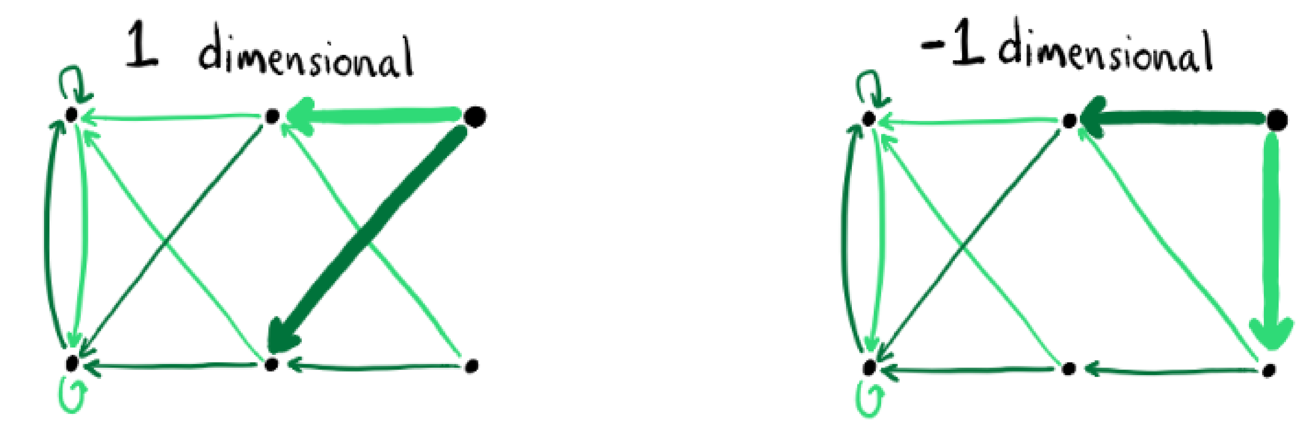
Consider a portrait of two degree d polynomials on $2d$ points. If the realization space is finite, then it contains at most $\left(\binom{d+1}{2} + 1\right)^{2d-2}$ points.

A future goal is to sharpen this bound.

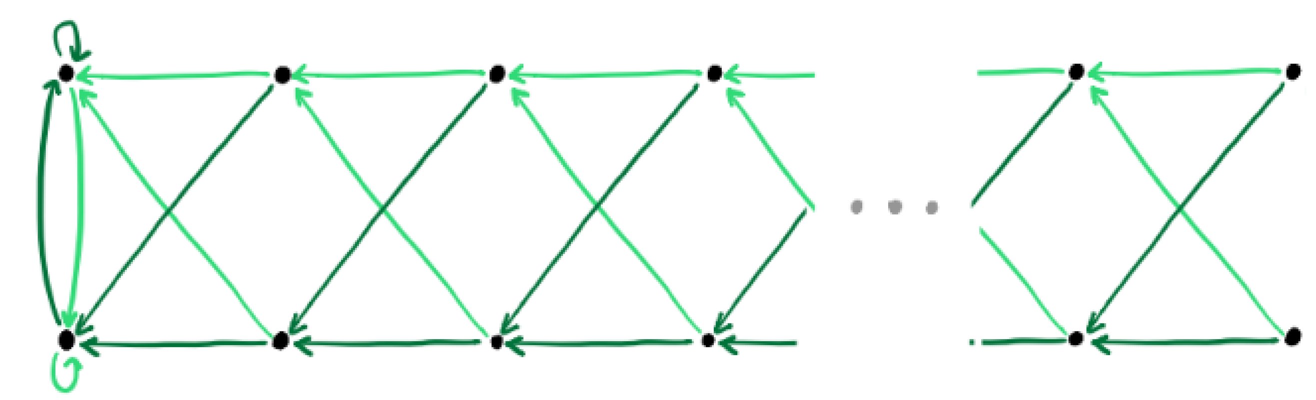
Realizable Portraits with Many Points



One-dimensional portraits for 2 quadratics acting on 5 points.



How the top-right point is added determines the dimension of the resulting portrait.

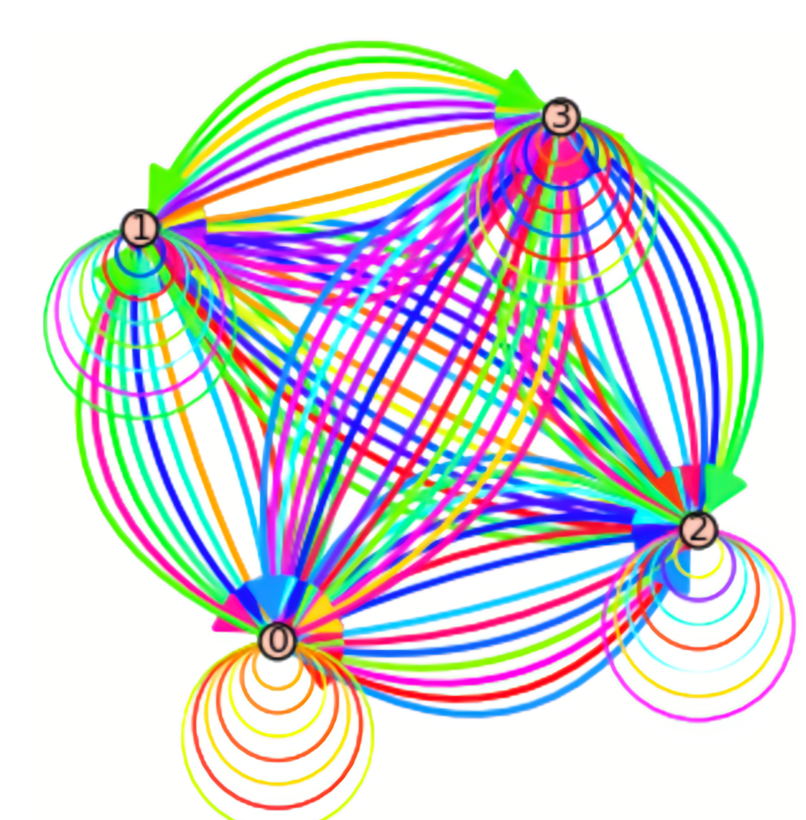


An arbitrarily large 1-dimensional portrait.

Theorem: Constructing Large Portraits of Positive Dimension

Let $f \in \mathbb{C}(x)$, and let S be a set such that $f(S) \subset S$ and for $y \in f(S)$, $f^{-1}(y) \subset S$. If there exists a degree 1 rational function $\ell(x)$ such that $f \circ \ell = f$, then $(\ell \circ f)(S) \subseteq S$.

Realizable Portraits with Many Maps



A realizable portrait with 28 quadratics acting on 4 points.

Acknowledgements

Thank you to our mentors: Trevor Hyde, John Doyle, and Max Weinreich. This research was supported by NSF Grant No. DMS-1439786 for Summer@ICERM.